

Modified Hopfield Neural Network Approach for Solving Nonlinear Algebraic Equations

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Abstract

In this paper, we present an neural network approach to solve a set of nonlinear equations. A modified Hopfield network has been developed to optimize a energy function. This approach provides faster convergence and extremely accurate solutions for all solvable systems. We solved and discussed several illustrative examples in order to depict the powerfulness of the proposed method.

Keywords: Modified Hopfield network, Optimization, Nonlinear algebraic equations, Newton's method, Energy function

1 Introduction

The conventional Hopfield model is the most commonly used model for auto-association and optimization. Hopfield networks are auto-associators in which node values are iteratively updated based on local computation principle: the new state of each node depends only on its net weighted input at a given time [1]. This network is fully connected network and the weight matrix determination is one of the important tasks while using it for any application. In [2], author had discussed the computational properties of biological organisms to the construction of computers. These properties can emerge as the collective properties of systems having large number of simple equivalent components. He elaborated the notion of content addressable memory. In this model, the processing devices are called neurons. Each neuron has two states similar to those of McCulloch and Pitts [4], i.e., firing (1) and not firing (0). Each neuron is connected to other neuron by the connectionist weight W_{ij} . The model proposed in [2], assumes that there is no self connection, i.e., $W_{ii} = 0$, and weights are symmetrical, i.e., $W_{ij} = W_{ji}$. Author demonstrated the stability of the proposed model by giving a valid Lyapunov function or energy function. The alteration in states of Hopfield model causes monotonic decrement in energy or Lyapunov function. In [3], author proposed and studied the architecture of the continuous model that can

be implemented using passive and active elements such as resistor, capacitor and Op-Amp. By applying the nonlinear characteristic of Op-Amp in terms of sigmoidal function author demonstrated the convergence of Lyapunov energy function. Soon after this work, author presented the application of Hopfield neural network in optimization problems by solving the well defined Traveling-Salesman Problem [5]. This application of Hopfield neural network made it a very popular model. Since then, this network has been applied to various applications. The Hopfield neural network has a well demonstrated capability of finding solutions to difficult optimization problems such as Traveling salesman problem (TSP), Analog-to-Digital conversion (A/D) [6, 7], Job scheduling problems [8], Quadratic assignment and other NP-complete problems [9] etc.

Other than the above applications, Hopfield neural networks have been used to several other application such as matrix inversion [11], parametric identification of dynamical systems [13], economic load dispatch [12] etc. Kanad et.al. [10] used Hopfield neural network approach for solving a set of simultaneous linear equations. They treated this as an optimization problem and hence applied Hopfield approach to find solution of linear equations.

In this paper, we proposed a modified Hopfield neural network approach to obtain the solutions from a set of nonlinear algebraic equations. A new energy function is formulated for this problem. Weights and bias values for the network are calculated with the help of derived energy function. The network results in desired solutions at the minimum of network's energy. Results obtained by proposed model are compared with solutions obtained from *Newton's method*.

This paper is organized as follows: Motivation of proposed work is explained in section 2. Architecture and stability analysis of the proposed network is reported in section 3. Simulation results are presented in section 4. We concluded our work with brief discussion in section 5.

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2 Motivation

Researchers have used quadratic form of energy function or cost function for solving neural-network based optimization problems. For problem having non-quadratic cost function or higher order cost function, quadratic-cost function based neural network architectures are not useful. To overcome this drawback a new approach is necessitated which can handle such problems efficiently and should results in desired solutions for such problems. As an example, the conventional Hopfield neural network is unable to provide the desired solutions for a set of nonlinear algebraic equations. In this paper, a new approach is proposed which incorporates higher order combinations of neuron states while formulating the energy function for the proposed model. These higher order terms in the proposed network are achieved by using the different combinations of states i.e. product, summation, or both. This energy function is used to derive the weights and bias values of the network. With the appropriate weights and bias values the total energy of the network decreases to its minimum.

3 Modified Hopfield Neural Network Architecture

The architecture for the proposed network is shown in Figure 1. It is a nonlinear interconnected system of n neurons. The nonlinear amplifier shown in figure has an input-output characteristics described by a activation function $\varphi(\cdot): \mathcal{R} \rightarrow [0 \ 1]$. This function is assumed to be continuously differentiable and strictly monotonically increasing. The most common activation function is the sigmoid function and is given by:

$$x = \varphi(u) = \frac{1}{1 + e^{-u/\theta}} \quad (1)$$

where the positive parameter θ controls the slope of activation function.

The proposed network of n neurons is shown in Figure 1. In this nonlinear aggregation at neuron units is considered, this nonlinear aggregation incorporates both multiplication and linear summation of states. It is observed from the figure that the product unit produces function $f_{ij}(\cdot)$, where $(i, j) \in [1 \dots n]$. The function $f(\cdot)$ can be any combination of states $x_1, x_2, \dots, x_i, i \in n$. The output obtained from the product units are linearly summed through the synaptic weights W_{ij} , $(i, j) \in n$. Applying Kirchhoff's current law at the input of each amplifier, we get

$$C_i \frac{du_i}{dt} + \frac{u_i}{R_i} = \sum_j W_{ij} f_{ij}(x_1, x_2, \dots, x_n) + I_i \quad (i, j) \in n \quad (2)$$

The output states of the network is given by $x = \varphi(u)$. The activation function $\varphi(\cdot)$ is strictly increasing, differentiable, and invertible i.e., $u = \varphi^{-1}(x)$. Rewriting Equation 2 as

$$C_i \frac{d\varphi^{-1}(x_i)}{dt} + \frac{\varphi^{-1}(x_i)}{R_i} = \sum_j W_{ij} f_{ij}(x_1, x_2, \dots, x_n) + I_i \quad (i, j) \in n \quad (3)$$

The energy function for the proposed model is given by:

$$E = \left(\sum_i \int_0^{x_i} \frac{\varphi^{-1}(s)}{R_i} ds - \sum_i \sum_j W_{ij} f_{ij}(x_1, x_2, \dots, x_n) x_i - \sum_i I_i x_i \right), \quad (i, j) \in n \quad (4)$$

By using conventional methodology we can show that the derivative for the proposed energy function is always less than or equal to zero, i.e. $(\frac{dE}{dt} \leq 0)$ and energy of the system is bounded.

In next section, the proposed methodology is demonstrated by considering two numerical examples. Although these examples are very simple examples but it demonstrate the capability of proposed technique. This approach can be easily extended for obtaining the solutions from complicated system of nonlinear algebraic equations of any order.

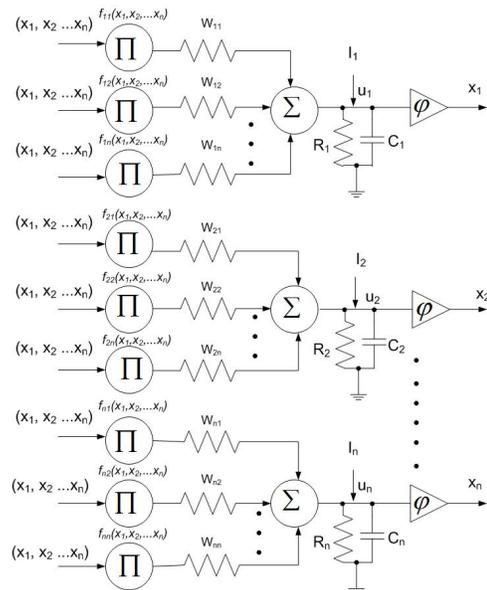


Figure 1: Proposed Hopfield type architecture for solving the nonlinear equation

3.1 Energy Function Based Approach for Optimization

Consider a set of algebraic equation given by following:

$$\begin{aligned} f_1(x_1, \dots, x_j, \dots, x_n) &= P_1 \\ &\vdots \\ f_i(x_1, \dots, x_j, \dots, x_n) &= P_i \\ &\vdots \\ f_m(x_1, \dots, x_j, \dots, x_n) &= P_m \end{aligned} \quad (5)$$

In the above equation $f_i(\cdot)$ is function of variables $x_1, \dots, x_j, \dots, x_n \in \mathbb{R}$ and $P_i \in \mathbb{R}$ is a real constant. Our objective is to find the values for variables $x_1, \dots, x_j, \dots, x_n$ such that it satisfies the Equation 5. In order to solve any optimization problem using proposed approach the problem has to be cast in the form of energy function. If formulation of energy function is appropriate then it can be used to derive weights and bias values of the network. Numerical simulation of dynamical equations of model with the appropriate weights and bias results in desired solution of the optimization problem. To obtain the solution using proposed approach an energy function has to be formulated. The energy function for the proposed model is derived as:

$$\begin{aligned} E &= \sum_{i=1}^m (g_i(\cdot))^2 \\ g_i(\cdot) &= (f_i(x_1, \dots, x_j, \dots, x_n) - P_i) \end{aligned} \quad (6)$$

Equation 6 has been used for designing the proposed network. The number of neurons in network is equal to the number of variables whose value is to be determined. In the given problem we have n number of variables hence the network have n number of neurons. The network dynamics are governed by following differential equation:

$$\begin{aligned} \frac{du_j}{dt} &= -\frac{\partial E}{\partial x_j} \\ x_i &= \varphi_j(u_j), j = 1, \dots, n \end{aligned} \quad (7)$$

where u_j is the net input to the j th neuron in the network and x_j is its output. In this application, the function $\varphi_j(\cdot)$ is a linear input-output transfer function for the j th neuron. Calculating the partial derivatives of Equation 6 with respect to unknown variables $x_1, \dots, x_j, \dots, x_n$ and collecting the terms of identical order will results in equation of Hopfield equations like form. The coefficients and constants in the available expression gives the weights and bias values for the network respectively.

3.2 Stability Analysis of the Proposed Network

In this section, the stability analysis for the proposed network has been carried out. In order to prove the proposed energy function is a valid energy function, the basic notions of Lyapunov stability theory are used. Lyapunov's method is a approach which provides us the qualitative aspects of system behavior.

Consider an autonomous system given in Equation 5. Let $E(\mathbf{X})$ be the total energy associated with the system, here \mathbf{X} represents the vector of variables $x_1, \dots, x_j, \dots, x_n$. If the derivative $\frac{dE(\mathbf{X})}{dt}$ is negative for all $\mathbf{X}(\mathbf{X}(\mathbf{t}_0), \mathbf{t})$ except the equilibrium point, then it follows that energy of the system decreases as t increases and finally system will reach the equilibrium point. This holds because energy is non-negative function of system state which reaches a minimum only if the system motion stops. In this case the system has ($\mathbf{X} \in x_1, \dots, x_j, \dots, x_n$) variables and hence the energy function $E(\mathbf{X})$ is given by:

$$\begin{aligned} E(\mathbf{X}) &= \sum_{i=1}^m (g_i(\cdot))^2 \\ g_i(\cdot) &= (f_i(x_1, \dots, x_j, \dots, x_n) - P_i), i = 1, \dots, m \end{aligned} \quad (8)$$

At any instant, the total energy $E(\mathbf{X})$ in the system is positive unless the system is at rest at the equilibrium state, where the energy is zero. This can be written as:

$$\begin{aligned} E(\mathbf{X}) &> 0 \quad \text{when } \mathbf{X} \neq \mathbf{0} \\ E(0) &= 0 \quad \text{when } \mathbf{X} = \mathbf{0} \end{aligned} \quad (9)$$

The rate of change of energy is given by

$$\frac{d}{dt}E(\mathbf{X}) = \frac{\partial E(\mathbf{X})}{\partial x_j} \frac{dx_j}{dt}, j = 1, \dots, n \quad (10)$$

The dynamics for the proposed Hopfield model is given by following:

$$\frac{du_j}{dt} = -\frac{\partial E(\mathbf{X})}{\partial x_j} \quad (11)$$

Using Equation 10 and Equation 11 we get:

$$\frac{d}{dt}E(x_j) = -\frac{du_j}{dt} \frac{dx_j}{dt} \quad (12)$$

State u_j is obtained by passing the state x_j through a non-linear activation function. This relation is represented by:

$$\begin{aligned} u_j &= \varphi(x_j) \\ \text{or} \\ x_j &= \varphi^{-1}(u_j) \end{aligned} \quad (13)$$

The nonlinear activation function $\varphi(\cdot)$ is chosen in such a way that, $\varphi(\cdot)$ is continuously differentiable and strictly monotonically increasing, that is, $\varphi(p) > \varphi(p')$ if $p > p'$. Considering the relation as shown in Equations 12 and 13 results in:

$$\frac{d}{dt}E(\mathbf{X}) = -\left(\frac{du_j}{dt}\right)^2 \frac{\partial \varphi^{-1}(\cdot)}{\partial(x_j)} \quad (14)$$

Thus $\frac{dE(\mathbf{X})}{dt}$ is negative at all points except the equilibrium state. This proves the stability in Lyapunov sense. The total energy of the system will decrease with time.

4 Application Examples

To illustrate the application and demonstrate the effectiveness of the proposed method, two example systems are employed.

4.1 Example 1

Consider the following nonlinear equation:

$$AX^3 + BX^2 + CX + D = 0 \quad (15)$$

Equation 15 is typical algebraic equation having cubic nonlinearity. X is the state variable. A, B, C and D are the constant coefficients. The energy function for this equation is given by:

$$E = -\frac{1}{2}(AX^3 + BX^2 + CX + D)^2 \quad (16)$$

The network dynamics is calculated by using Equation 7 and is given as:

$$\frac{du}{dt} = -\frac{1}{2}(2 \times (AX^3 + BX^2 + CX + D) \times (3AX^2 + 2BX + C)) \quad (17)$$

Rewriting Equation 17 we get:

$$\frac{du}{dt} = -\frac{1}{2}(6A^2X^5 + 10ABX^4 + (8AC + 4B^2)X^3 + (6AD + 6BC)X^2 + (4BD + 2C^2)X + 2CD) \quad (18)$$

The proposed Hopfield type architecture for this optimization problem is shown in Figure 2. To simplify the expression we neglected leakage term after the aggregation and consider only the dynamics of capacitance. Referring to Figure 2, we write the dynamical equation for the proposed model as:

$$\frac{du}{dt} = (W_1X^5 + W_2X^4 + W_3X^3 + W_4X^2 + W_5X + I_{bias}) \quad (19)$$

where W_1, W_2, W_3, W_4, W_5 are the associated weights and I_{bias} is the constant input to the network. The net value of aggregated input at the neuron incorporates the linear and nonlinear terms. This aggregation reflects the biological fact of nonlinear dendritic aggregation. Equation 20 presents the energy function for the pro-

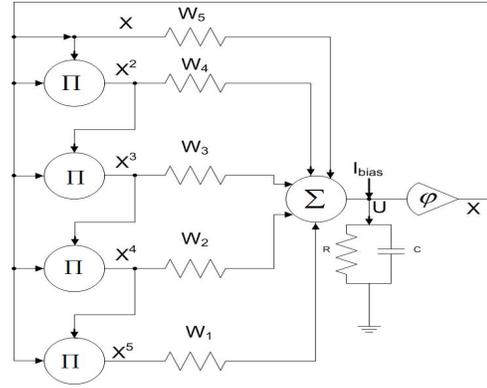


Figure 2: Proposed architecture for solving the nonlinear equation, $AX^3 + BX^2 + CX + D = 0$

posed Hopfield type architecture.

$$E = -X(W_1X^5 + W_2X^4 + W_3X^3 + W_4X^2 + W_5X + I_{bias}) \quad (20)$$

The weights and bias values are obtained by performing the comparison between Equation 19 and Equation 18. These weights and bias are given by:

$$\begin{aligned} W_1 &= -\frac{1}{2}(6A^2) \\ W_2 &= -\frac{1}{2}(10AB) \\ W_3 &= -\frac{1}{2}(8AC + 4B^2) \\ W_4 &= -\frac{1}{2}(6AD + 6BC) \\ W_5 &= -\frac{1}{2}(4BD + 2C^2) \\ I_{bias} &= -\frac{1}{2}(2CD) \end{aligned} \quad (21)$$

The input-output relation for the p th neuron of model is given by $X_p = \phi(u_p)$. For our application, the function ϕ_p is considered a linear input-output transfer function. The response for this transfer function is drawn in Figure 3. The network is simulated numerically using Euler's method after determining the weights and bias. The

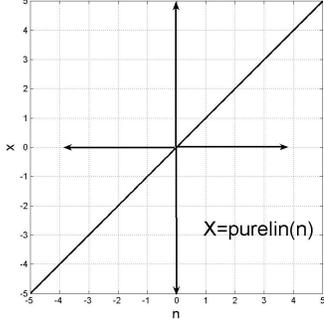


Figure 3: Plot of linear transfer function

Euler's formation of the network dynamics is given by:

$$\begin{aligned} u(t+1) = & u(t) + \Delta t(W_1X^5(t) \\ & + W_2X^4(t) + W_3X^3(t) \\ & + W_4X^2(t) + W_5X(t) + I_{bias}) \end{aligned} \quad (22)$$

Results with parameters $A = 1$, $B = -6$, $C = 11$ and $D = -2$ are shown in Table 1 at different initial values i.e. $u(0)$, $X(0)$. The energy profile of the network is drawn in Figure 4. It is evident here that the energy for the proposed model monotonically decreases to zero and model possesses stable dynamics. We compared our results with *Newton's method* (Table 1), which is conventionally used to calculate the solutions of equations. It is observed here that the solutions obtained from both the methods are same. The performance of proposed method will improve for higher dimensional systems because we need to determine the suitable weights and biases that minimizes the energy function. In *Newton's method* the calculation of Jacobian matrix and its inverse for finding the solutions of equations is one of the prime limitation and this can be overcome by the proposed method.

X_0	u_0	X_{final}	Value of Equation 15
<i>Proposed Method</i>			
0.001	0.0	0.2037	-6.6×10^{-16}
0.31	0.5	0.2037	0.0
1.9	-0.09	0.2037	8.8×10^{-16}
<i>Newton's Method</i>			
0.001	-	0.2037	0.0
0.31	-	0.2037	0.0
1.9	-	0.2037	0.0

Table 1: Solution of the Equation 15 at different initial conditions with the *proposed method* and *Newton's method*. Parameters during simulations are: $A = 1$, $B = -6$, $C = 11$ and $D = -2$.

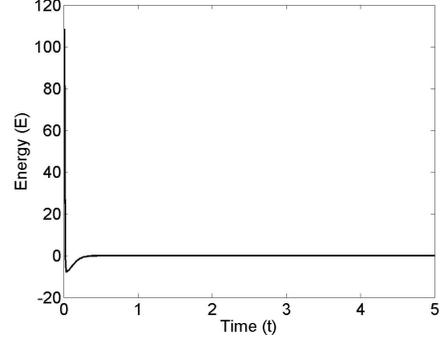


Figure 4: Energy function profile for the model

4.2 Example 2

This example demonstrates the application of proposed method for solving a pair of nonlinear equations, given by:

$$A_1X_1 + A_2X_2 + A_3X_1X_2 + A_4 = 0 \quad (23)$$

$$B_1X_1 + B_2X_2 + B_3X_1X_2 + B_4 = 0 \quad (24)$$

where X_1 and X_2 are the state variables whose solutions for some particular set of constant coefficients have to be find out. $A_1, A_2, A_3, A_4, B_1, B_2, B_3,$ and B_4 are the constant coefficeint of these equations. The energy function with the given nonlinear equations is formulated as:

$$\begin{aligned} E = & -\frac{1}{2} \left((A_1X_1 + A_2X_2 + A_3X_1X_2 + A_4)^2 \right. \\ & \left. + (B_1X_1 + B_2X_2 + B_3X_1X_2 + B_4)^2 \right) \end{aligned} \quad (25)$$

Using Equation 7 the time derivatives $\frac{du_1}{dt}$ and $\frac{du_2}{dt}$ are calculated as:

$$\begin{aligned} \frac{du_1}{dt} &= -\frac{\partial E}{\partial X_1} \\ \frac{du_2}{dt} &= -\frac{\partial E}{\partial X_2} \end{aligned} \quad (26)$$

These dynamical equations for proposed network is rewritten in its expanded form by:

$$\begin{aligned} \frac{du_1}{dt} &= -\frac{1}{2} \{ (2(A_1X_1 + A_2X_2 + A_3X_1X_2 + A_4) \\ & (A_1 + A_3X_2) \\ & + 2(B_1X_1 + B_2X_2 + B_3X_1X_2 + B_4) \\ & (B_1 + B_3X_2)) \} \\ \frac{du_2}{dt} &= -\frac{1}{2} \{ 2(A_1X_1 + A_2X_2 + A_3X_1X_2 + A_4) \\ & (A_2 + A_3X_1) \\ & + 2(B_1X_1 + B_2X_2 + B_3X_1X_2 + B_4) \\ & (B_2 + B_3X_1) \} \end{aligned} \quad (27)$$

which is further simplified as

$$\begin{aligned}\frac{du_1}{dt} &= -\frac{1}{2}\{P_1X_1 + P_2X_2 + P_3X_1X_2 \\ &\quad + P_4X_2^2 + P_5X_1X_2^2 + P_6\} \\ \frac{du_2}{dt} &= -\frac{1}{2}\{M_1X_1 + M_2X_2 + M_3X_1X_2 \\ &\quad + M_4X_1^2 + M_5X_1^2X_2 + M_6\}\end{aligned}\quad (28)$$

where

$$\begin{aligned}P_1 &= -2(A_1^2 + B_1^2) \\ P_2 &= -2(A_2A_1 + A_3A_4 + B_1B_2 + B_3B_4) \\ P_3 &= -2(2A_1A_3 + 2B_1B_3) \\ P_4 &= -2(A_3A_2 + B_3B_2) \\ P_5 &= -2(B_3^2 + A_3^2) \\ P_6 &= -2(A_1A_4 + B_1B_4)\end{aligned}\quad (29)$$

$$\begin{aligned}M_1 &= -2(A_1A_2 + A_3A_4 + B_1B_2 + B_3B_4) \\ M_2 &= -2(A_2^2 + B_2^2) \\ M_3 &= -2(2A_2A_3 + 2B_2B_3) \\ M_4 &= -2(A_3A_1 + B_3B_1) \\ M_5 &= -2(A_3^2 + B_3^2) \\ M_6 &= -2(A_2A_4 + B_2B_4)\end{aligned}\quad (30)$$

The architecture for the proposed network for this optimization problem is shown in Figure 5. Figure 5 is used to develop the dynamical equations and energy function for the proposed network. The dynamic equations for the proposed network and corresponding energy function with these dynamic equations are given in Equation 31 and Equation 32 respectively. We derived following dynamic equations (equation 31) for the model depicted in Figure 5.

$$\begin{aligned}\frac{du_1}{dt} &= (W_1X_1 + W_2X_2 + W_3X_1X_2 \\ &\quad + W_4X_2^2 + W_5X_1X_2^2 + I_{bias1}) \\ \frac{du_2}{dt} &= (V_1X_1 + V_2X_2 + V_3X_1X_2 \\ &\quad + V_4X_1^2 + V_5X_2X_1^2 + I_{bias2})\end{aligned}\quad (31)$$

$$\begin{aligned}E &= -\frac{1}{2}\{X_1(W_1X_1 + W_2X_2 + W_3X_1X_2 \\ &\quad + W_4X_2^2 + W_5X_1X_2^2 + I_{bias1}) \\ &\quad + X_2(V_1X_1 + V_2X_2 + V_3X_1X_2 \\ &\quad + V_4X_1^2 + V_5X_2X_1^2 + I_{bias2})\}\end{aligned}\quad (32)$$

The weights and biases for the network shown in Figure 5 are calculated by comparing coefficients of Equation 31

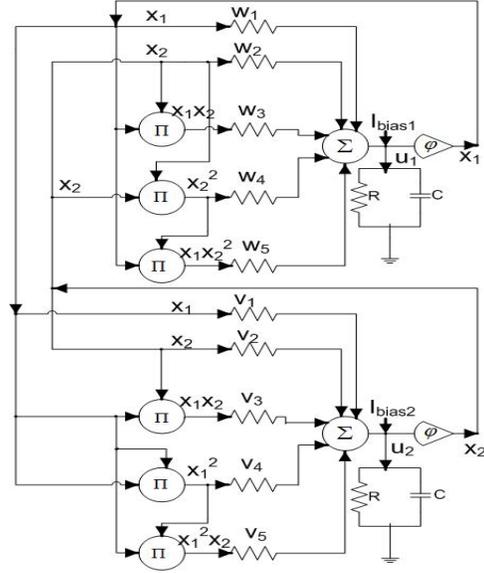


Figure 5: Proposed Hopfield neural network like architecture for finding the solutions for the set of nonlinear equations given in Equation 23 and 24

with those of Equation 28. The calculated weights and bias values in terms of coefficients of the equations are given as:

$$\begin{aligned}W_1 &= -2(A_1^2 + B_1^2) \\ W_2 &= -2(A_2A_1 + A_3A_4 + B_1B_2 + B_3B_4) \\ W_3 &= -2(2A_1A_3 + 2B_1B_3) \\ W_4 &= -2(A_3A_2 + B_3B_2) \\ W_5 &= -2(B_3^2 + A_3^2) \\ I_{bias1} &= -2(A_1A_4 + B_1B_4)\end{aligned}\quad (33)$$

$$\begin{aligned}V_1 &= -2(A_1A_2 + A_3A_4 + B_1B_2 + B_3B_4) \\ V_2 &= -2(A_2^2 + B_2^2) \\ V_3 &= -2(2A_2A_3 + 2B_2B_3) \\ V_4 &= -2(A_3A_1 + B_3B_1) \\ V_5 &= -2(A_3^2 + B_3^2) \\ I_{bias2} &= -2(A_2A_4 + B_2B_4)\end{aligned}\quad (34)$$

Solutions for the given nonlinear equations are obtained after the numerical simulations at different initial conditions. These solutions are compared to the solutions obtained from conventional *Newton's method*. Table 2 shows the simulation results. The total energy profile for the proposed network during the simulation is drawn in Figure 6. It is evident that the total energy of the system monotonically decreases to zero and attains its global minima.

It is found from simulation results (Table 2) and energy function plot (Figure 6) that the proposed Hopfield type model owns stable dynamics and the energy of the system decreases monotonically.

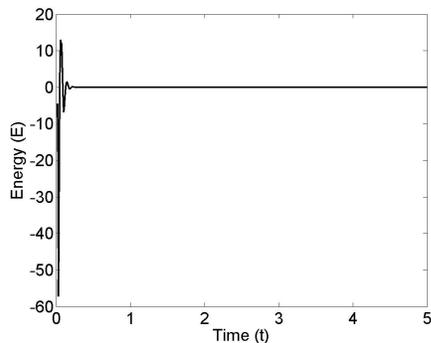


Figure 6: Energy function profile for the model

5 Conclusion

In this paper, a novel approach for solving a set of non-linear algebraic equations using a Hopfield neural network type architecture has been proposed. Energy function formulation and stability analysis for the network has been carried out. The capability of the proposed network is tested by solving two difficult examples. The total energy profile of the system during simulation is sketched for both cases. It is observed that the total energy of system decreases monotonically and finally settles to its minimum. The network provides the desired solutions at its energy minima. Results for the nonlinear equations used in the numerical examples are tabulated in Table 1 and Table 2 at three different initial conditions. The performance of the proposed network is efficient in terms of the computational cost and total time require to obtain final results. Results are compared with *Newton's method*. It has been observed that results obtained with both of these methods are same, but the proposed methods avoids the inherent limitations of *Newton's method* i.e. calculations of the Jacobian matrix and its inverse.

Hence, it can be concluded, the proposed approach avoids the limitation of conventional Hopfield neural network and it can be applied to non-quadratic or higher order optimization problems efficiently. When compared to conventional *Newton's method* in finding the solutions for nonlinear set of algebraic equations it avoids the calculations of the Jacobian matrix and its inverse. The proposed network is a fixed weight, it can solve any order nonlinear algebraic set of equations with lesser computational cost.

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Table 2: Solution of the Equation 31 at different initial conditions with the proposed method and Newton's method. Parameters during simulations are: $A_1 = 1, A_2 = 2, A_3 = 3, A_4 = 4, B_1 = 5, B_2 = 3, B_3 = 1,$ and $B_4 = 2$.

$X_1(0)$	$X_2(0)$	$u_1(0)$	$u_2(0)$	$X_1(Final)$	$X_2(Final)$	Value of Equation 23	Value of Equation 24
<i>Proposed Method</i>							
0.0	-0.3179	1.095	-1.8740	0.5	-1.2857	0.0	-8.8×10^{-16}
0.4282	0.8956	0.7310	0.5779	0.5	-1.2857	0.0	-8.8×10^{-16}
0.0403	0.6771	0.5689	-0.2556	0.5	-1.2857	0.0	-8.8×10^{-16}
<i>Newton's Method</i>							
0.0	-0.3179	-	-	0.5	-1.2857	0.0	4.4×10^{-16}
0.4282	0.8956	-	-	0.5	-1.2857	0.0	-4.4×10^{-16}
0.0403	0.6771	-	-	0.5	-1.2857	4.4×10^{-16}	4.4×10^{-16}