Generation of Equations of Motion in Reference Frame Formulation for FEM Models

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Abstract: **This paper presents a method of the reformulation of equations in the inertial frame formulation, used for FEM models of flexible multibody systems, to the floating frame formulation. Results of a simulation example, performed using the modal transformation, show the good agreement with the standard inertial frame methods.**

Index Terms: **Flexible multibody systems, floating frame of reference formulation, inertial frame formulation, ANSYS.**

I. Introduction

In recent years, considerable effort has been devoted to modeling, design and control of flexible multibody systems (FMS).

The *inertial frame formulation* (*IFF*) and the *floating frame of reference formulation* (*FFRF*) are nowadays two popular techniques for the simulation of dynamics of FMS, which are widely implemented in many multibody simulation software (e.g. ABAQUS/explicit, Simpack, etc.) and in engineering simulation software (e.g. ANSYS) [1], [2].

Nowadays the FFRF offers the most efficient method for the simulation of FMS undergoing small elastic deformations and slow elastic speed. In the FFRF the motion of elements is considered as a combination of the rigid body motion and deformations. This approach allows using the modal reduction technique, which greatly decreases the simulation time.

Because of the wide use of both simulation approaches it seems very promising to develop a method of the transformation of equations of motion, obtained using IFF, to the equations in the FFRF. This method can be used as a basement for an interface between various simulation tools, using different formulations.

Let us consider a finite-element ANSYS model of a flexible body, consisting on continuum elements. Let n denote the number of nodes in the simulated body. Nowadays the nodes in all continuum elements in ANSYS have no rotation degrees freedom (DOFs), but only three translation DOFs (e.g. SOLID45, SOLID95, SOLID46, etc) [4]. Using IFF, implemented in ANSYS, we can write the following equations of body motion in the absence of constraints:

$$\mathbf{M_A}\ddot{\mathbf{u}}+\mathbf{K_A}\mathbf{u}=\boldsymbol{\tau} \qquad (1)$$

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where $\mathbf{u}$ denote the $3{\cdot}n$ vector of nodal displacements, $\boldsymbol{\tau}$ is the $3{\cdot}n$ length vector of nodal point loads, $\mathbf{M_A}$ is a mass matrix, and $\mathbf{K_A}$ is a stiffness matrix.

Below we show how we can reformulate the equations of motion of the model in the FFRF, using the initial values $\mathbf{M_A}|_{t=0}$, $\mathbf{K_A}|_{t=0}$ generated by ANSYS. The method was implemented for the simulation of the dynamics of a flexible pendulum example.

II. Kinematics of a Model

Let us consider an orthonormal body coordinate system $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$. Let $\mathbf{x}_0$ denote the absolute coordinates of the body reference with respect to the global coordinate system $\{O\ \mathbf{e}^1\mathbf{e}^2\mathbf{e}^3\}$, as it is shown in Fig. 1.

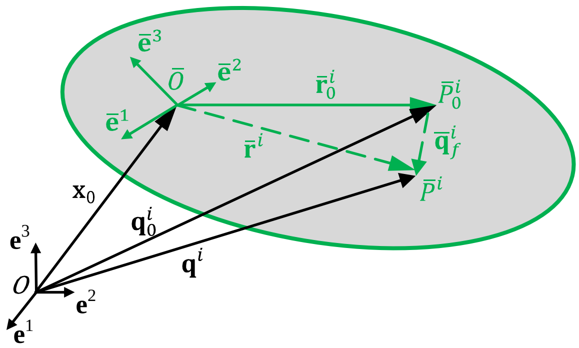

Fig. 1 General model of a flexible body

Let $\mathbf{q}_0^i$ be the absolute position of the i-th node in the undeformed state:

$$\mathbf{q}_0^i = \mathbf{x}_0 + \mathbf{A}(\boldsymbol{\theta}_0) \cdot \bar{\mathbf{r}}_0^i \qquad i = 1..n \qquad (2)$$

where $\bar{\mathbf{r}}_0^i$ is position of the i-th node in the undeformed state in body frame, $\mathbf{A}(\boldsymbol{\theta}_0)$ is the transformation matrix that defines the orientation of $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$ with respect to $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$, $\boldsymbol{\theta}_0$ is the vector of Euler parameters.

In vector form we can write (2) as

$$\mathbf{q}_0 = \mathbf{x}_0^{\mathbf{n}} + \mathbf{A}^{\mathbf{n}}\,\bar{\mathbf{r}}_0 \qquad (3)$$

where $\mathbf{q}_0 = \left(\mathbf{q}_0^{1\,T} \ \ \dots \ \ \mathbf{q}_0^{n\,T}\right)^T$, $\bar{\mathbf{r}}_0 = \left(\bar{\mathbf{r}}_0^{1\,T} \ \ \dots \ \ \bar{\mathbf{r}}_0^{n\,T}\right)^T$, $\mathbf{x}_0^{\mathbf{n}}$ is a $3n$-lentgth vector, consisting of $\mathbf{x}_0$:

$$\mathbf{x}_0^{\mathbf{n}} = (\mathbf{x}_0^T \ \ \dots \ \ \mathbf{x}_0^T)^T \qquad (4)$$

$\mathbf{A}^{\mathbf{n}}$ is a $(3n, 3n)$ block diagonal matrix, consisting of $\mathbf{A}$:

$$\mathbf{A^n} = diag_i(\mathbf{A}) \quad i = 1..n \tag{5}$$

Let $\mathbf{q}^i$ denote the vector of absolute position of the *i*-th node. Then the vector of the deformation of the *i*-th node in inertial frame can be defined as:

$$\mathbf{q}_f^i = \mathbf{q}^i - \mathbf{q}_0^i \quad i = 1..n \tag{6}$$

Using (2), we get the system of equations:

$$\mathbf{q}^i = \mathrm{x}_0 + \mathrm{A}\left(\bar{\mathbf{r}}_0^i + \bar{\mathbf{q}}_f^i\right) \quad i = 1..n \tag{7}$$

where $\bar{\mathbf{q}}_f^i$ is the deformation vector of the *i*-th node in the body frame $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$.. Let $\bar{\mathbf{r}}^i$ denote the position of the *i*-th node in the deformed state in $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$.

$$\bar{\mathbf{r}}^i = \bar{\mathbf{r}}_0^i + \bar{\mathbf{q}}_f^i \quad i = 1..n \tag{8}$$

Substituting $\bar{\mathbf{r}}^i$ in (7), we get

$$\mathbf{q}^i = \mathrm{x}_0 + \mathbf{A}\bar{\mathbf{r}}^i \quad i = 1..n \tag{9}$$

This system of equations can be rewritten as

$$\mathbf{q} = \mathbf{x}_0^\mathbf{n} + \mathbf{A^n}\bar{\mathbf{r}} \tag{10}$$

where $\mathbf{q} = \left(\mathbf{q}^{1\,T} \quad \dots \quad \mathbf{q}^{n\,T}\right)^T$ and $\bar{\mathbf{r}} = \left(\bar{\mathbf{r}}^{1\,T} \quad \dots \quad \bar{\mathbf{r}}^{n\,T}\right)^T$

III. Relation Between Inertia Shape Integral $\bar{\boldsymbol{S}}_{ff}$ and $\boldsymbol{M}_A$

The mass matrix $\mathbf{M_A}$ in the IFF is a symmetric matrix defined as [4]

$$\mathbf{M_A} = \int_V \rho \mathbf{N}^T\mathbf{N}dV \tag{11}$$

where **N** is the matrix of shape functions, setting the relation between the displacements within the body **w** and the vector of nodal displacements **u** in $\{O\ \mathbf{e}^1\mathbf{e}^2\mathbf{e}^3\}$**:**

$$\mathbf{w} = \mathbf{N}\mathbf{u} \tag{12}$$

Let $\bar{\mathbf{N}}$ denote a constant matrix of shape functions in the body frame $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$:

$$\bar{\mathbf{w}} = \bar{\mathbf{N}}\bar{\mathbf{u}} \tag{13}$$

where $\bar{\mathbf{w}}$ is the vector of displacements, expressed in $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$, $\bar{\mathbf{u}}$ is the vector of nodal displacements, expressed in $\{\bar{O}\ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$.

Our goal is to find the relation between $\mathbf{M_A}$ and the time-constant inertia shape integral $\bar{\mathbf{S}}_{ff}$, defined as [3]:

$$\bar{\mathbf{S}}_{ff} = \int_V \rho \bar{\mathbf{N}}^T\bar{\mathbf{N}}dV \tag{14}$$

From the definitions of **N** and $\bar{\mathbf{N}}$ follows that

$$\bar{\mathbf{N}} = \mathbf{A}^T\mathbf{N}\mathbf{A^n} \tag{15}$$

Substituting $\bar{\mathbf{N}}$ in (14), we get:

$$\bar{\mathbf{S}}_{ff} = \int_{\bar{V}} \rho \mathbf{A^n}^T\mathbf{N}^T\mathbf{A}\mathbf{A}^T\mathbf{N}\mathbf{A^n}dV = \mathbf{A^n}^T\mathbf{M_A}\mathbf{A^n} \tag{16}$$

Therefore

$$\mathbf{M_A} = \mathbf{A^n}\bar{\mathbf{S}}_{ff}\mathbf{A^n}^T \tag{17}$$

This equation can be used for the calculation of $\bar{\mathbf{S}}_{ff}$ from of the mass matrix $\mathbf{M_A}|_{t=0}$ and the rotation matix $\mathbf{A}|_{t=0}$, obtained from ANSYS. In future we will assume that the axes $\bar{\mathbf{e}}^1, \bar{\mathbf{e}}^2, \bar{\mathbf{e}}^3$ are initially parallel to the axes $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$, i.e. $\mathbf{A}|_{t=0} = \mathbf{E}$ where **E** is the identity (3,3) matrix. Hence

$$\bar{\mathbf{S}}_{ff} = \mathbf{M_A}|_{t=0} \tag{18}$$

IV. Property of $\bar{\mathbf{S}}_{ff}$

Now we need to prove some important property of $\bar{\mathbf{S}}_{ff}$. Let us consider a situation of the free body fall, i.e. when on the body acts only the gravity force $g\boldsymbol{e}$**,** where $\boldsymbol{e}$ is the direction of the gravitations and g is the value of the force. Assume that the body was initially at rest, i.e.

$$\dot{q}^i\big|_{t=0} = 0 \quad i = 1..n \tag{19}$$

Clear, that in this case deformations should not appear, i.e.

$$\dot{\mathbf{q}}^i = \dot{\mathbf{x}}_0 \quad i = 1..n \tag{20}$$

$$\ddot{\mathbf{q}}^i = \ddot{\mathbf{x}}_\mathbf{0} \quad i = 1..n \tag{21}$$

The equations of motion (1) at the start point will have the following form:

$$\sum_j \mathbf{M}_A^{ij}\big|_{t=0}\ddot{\mathbf{q}}^j\big|_{t=0} = k^i\mathrm{g}\mathbf{e} \quad i = 1..n \tag{22}$$

where k^i is a coefficient, corresponding to the *i*-th node.

Using (18) and (21), we rewrite this group of equations as

$$\left(\sum_j \bar{\mathbf{S}}_{ff}^{ij}\right)\ddot{\mathbf{x}}_\mathbf{0}|_{t=0} = k^i\mathrm{g}\mathbf{e} \quad i = 1..n \tag{23}$$

Clear, that the direction of $\ddot{\mathbf{x}}_\mathbf{0}$ should be equal to the direction of the gravity force $\boldsymbol{e}$ and proportional to the gravity's value g. Therefore

$$\left(\sum_{j=1}^n \bar{\mathbf{S}}_{ff}^{ij}\right) = k^i\mathbf{E} \quad i = 1..n \tag{24}$$

where $\mathbf{E}$ is the identity (3,3) matrix.

The symmetry of $\bar{\mathbf{S}}_{ff}$ implies that

$$\left(\sum_{i=1}^{n}\bar{\mathbf{S}}_{ff}^{ij}\right)=k^{j}\mathbf{E}\quad j=1..n \tag{25}$$

V. Mass and Stiffness Matrices

A. *Mass matrix* M

The kinetic energy T of the body in the IFF can be written as [4]:

$$\mathrm{T}=\tfrac{1}{2}\dot{\mathbf{u}}^{T}\mathbf{M}_{\mathrm{A}}\dot{\mathbf{u}}=\tfrac{1}{2}\dot{\mathbf{q}}^{T}\mathbf{M}_{\mathrm{A}}\dot{\mathbf{q}} \tag{26}$$

Our goal is to find from the initial value of $\mathbf{M_A}|_{t=0}=\bar{\mathbf{S}}_{ff}$ the formula for the calculation of the mass matrix $\mathbf{M}$ in FFRF, used for the calculation of kinetic energy

$$\mathrm{T}=\frac{1}{2}\dot{\mathbf{p}}^{T}\,\mathbf{M}\dot{\mathbf{p}}^{T} \tag{27}$$

where $\mathbf{p}=\left(\mathbf{x}_{0}^{T}\quad \boldsymbol{\theta}_{0}^{T}\quad \bar{\mathbf{q}}_{f}^{T}\right)^{T},\mathbf{q}_{f}\ =\left(\mathbf{q}_{f}^{1\,T}\quad ...\quad \mathbf{q}_{f}^{n\,T}\right)^{T}$

Differentiating (9) , we get:

$$\dot{\mathbf{q}}^{i}\ =\dot{\mathbf{x}}_{0}\ -\mathbf{A}\tilde{\bar{\mathbf{r}}}^{i}\,\bar{\boldsymbol{\omega}}+\mathbf{A}\dot{\bar{\mathbf{q}}}_{f}^{i}\qquad i=1..n \tag{28}$$

where $\bar{\boldsymbol{\omega}}$ is the angular velocity vector defined in $\{\bar{O}\ \bar{\mathbf{e}}^{i}\bar{\mathbf{e}}^{2}\bar{\mathbf{e}}^{3}\}$. The vector $\bar{\boldsymbol{\omega}}$ can be expressed using the time derivation $\dot{\boldsymbol{\theta}}_{0}$ of the body rotational coordinates**:**

$$\bar{\boldsymbol{\omega}}=\bar{\mathbf{T}}(\boldsymbol{\theta}_{0}\,)\dot{\boldsymbol{\theta}}_{0} \tag{29}$$

Substituting (29) in (28), we get

$$\dot{\mathbf{q}}^{i}\ =\dot{\mathbf{x}}_{0}\ -\mathbf{A}\tilde{\bar{\mathbf{r}}}^{i}\,\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}\ +\mathbf{A}\dot{\bar{\mathbf{q}}}_{f}^{i}\qquad i=1..n \tag{30}$$

It will be more convenient to rewrite this system of equations as a matrix equation:

$$\dot{\mathbf{q}}=\dot{\mathbf{x}}_{0}^{\mathbf{n}}-\mathbf{A}^{\mathbf{n}}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}\ +\mathbf{A}^{\mathbf{n}}\dot{\bar{\mathbf{q}}}_{f} \tag{31}$$

where $\tilde{\bar{\mathbf{r}}}=(\tilde{\bar{\mathbf{r}}}^{1\,T}\quad ...\quad \tilde{\bar{\mathbf{r}}}^{n\,T})^{T}$.

Substituting (31), (17) in (26), we get:

$$\mathrm{T}=\left(\dot{\mathbf{x}}_{0}^{\mathbf{n}}-\mathbf{A}^{\mathbf{n}}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}\ +\mathbf{A}^{\mathbf{n}}\dot{\bar{\mathbf{q}}}_{f}\,\right)^{T}\cdot\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{f,f}\mathrm{A}^{\mathbf{n}^{T}}\cdot \cdot\left(\dot{\mathbf{x}}_{0}^{\mathbf{n}}-\mathbf{A}^{\mathbf{n}}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}\ +\mathbf{A}^{\mathbf{n}}\dot{\bar{\mathbf{q}}}_{f}\,\right) \tag{32}$$

Or, in another form:

$$\mathrm{T}=\mathrm{T}_{xx}+2\mathrm{T}_{x\theta}+2\mathrm{T}_{xf}+\mathrm{T}_{\theta\theta}+2\mathrm{T}_{\theta f}+\mathrm{T}_{ff} \tag{33}$$

where

$$\mathrm{T}_{xx}=\tfrac{1}{2}\dot{\mathbf{x}}_{0}^{\mathbf{n}^{T}}\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\boldsymbol{A}^{\mathbf{n}^{T}}\dot{\mathbf{x}}_{0}^{\mathbf{n}}$$
$$\mathrm{T}_{x\theta}=-\tfrac{1}{2}\dot{\mathbf{x}}_{0}^{\mathbf{n}^{T}}\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}$$
$$\mathrm{T}_{xf}=\tfrac{1}{2}\dot{\mathbf{x}}_{0}^{\mathbf{n}^{T}}\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\dot{\bar{\mathbf{q}}}_{f}$$
$$\mathrm{T}_{\theta\theta}=\tfrac{1}{2}(\mathbf{A}^{\mathbf{n}}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}\,)^{T}\mathbf{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}$$
$$\mathrm{T}_{\theta f}=-\tfrac{1}{2}(\mathbf{A}^{\mathbf{n}}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0}\,)^{T}\mathbf{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\dot{\bar{\mathbf{q}}}_{f}$$
$$\mathrm{T}_{ff}=\tfrac{1}{2}\dot{\bar{\mathbf{q}}}_{f}^{T}\bar{\mathbf{S}}_{ff}\dot{\bar{\mathbf{q}}}_{f}$$

Let us consider now the component of the sum in (32) more precisely.

The part T_{xx} is equal to the kinetic energy of the translation of the non-deformed body:

$$\mathrm{T}_{xx}=\frac{1}{2}\dot{\mathbf{x}}_{0}^{T}\mathrm{A}\left(\sum_{i,j}\bar{\mathbf{S}}_{ff}^{ij}\right)\boldsymbol{A}^{T}\dot{\mathbf{x}}_{0}\ =\frac{1}{2}\dot{\mathbf{x}}_{0}^{T}\bar{\mathbf{S}}_{xx}\dot{\mathbf{x}}_{0} \tag{34}$$

where $\bar{\mathbf{S}}_{xx}$ is a (3,3) matrix:

$$\bar{\mathbf{S}}_{xx}=\sum_{i,j}\bar{\mathbf{S}}_{ff}^{ij} \tag{35}$$

On the other hand, the kinetic energy of the translation of the non-deformed body should be equal to $\frac{1}{2}m\dot{\mathbf{x}}_{0}^{T}\dot{\mathbf{x}}_{0}$ **,** where m is the body mass. Therefore, $\bar{\mathbf{S}}_{xx}$ is a constant matrix:

$$\bar{\mathbf{S}}_{xx}=\begin{pmatrix}m&0&0\\0&m&0\\0&0&m\end{pmatrix} \tag{36}$$

Summarizing k^{i} from (24) and using the definition of $\bar{\mathbf{S}}_{xx}$**,** we get the following important property of k^{i}:

$$\sum_{i=1}^{n}k^{i}\mathbf{E}=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\bar{\mathbf{S}}_{ff}^{ij}\right)=\bar{\mathbf{S}}_{xx}=\begin{pmatrix}m&0&0\\0&m&0\\0&0&m\end{pmatrix} \tag{37}$$

Consider now the component $\mathrm{T}_{x\theta}$:

$$\mathrm{T}_{x\theta}=-\frac{1}{2}\dot{\mathbf{x}}_{0}^{\mathbf{n}^{T}}\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0} =-\frac{1}{2}\dot{\mathbf{x}}_{0}^{T}\mathbf{A}\left(\sum_{i,j=1}^{n}\bar{\mathbf{S}}_{ff}^{ij}\tilde{\bar{\mathbf{r}}}^{j}\right)\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0} \tag{38}$$

Using (25), we obtain:

$$\sum_{i,j=1}^{n}\bar{\mathbf{S}}_{ff}^{ij}\tilde{\bar{\mathbf{r}}}^{j}\ =\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\bar{\mathbf{S}}_{ff}^{ij}\right)\tilde{\bar{\mathbf{r}}}^{j}\ =\tilde{\mathbf{s}}_{x\theta} \tag{39}$$

where

$$\mathbf{s}_{x\theta}=\sum_{j=1}^{n}k^{j}\bar{\mathbf{r}}^{j} \tag{40}$$

Hence, we can rewrite (38) as

$$\mathrm{T}_{x\theta}=-\frac{1}{2}\dot{\mathbf{x}}_{0}^{T}\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_{0} \tag{41}$$

The formula for T_{xf} can be rewritten as

$$\mathrm{T}_{xf} = \frac{1}{2}\dot{\mathbf{x}}_0^{\mathrm{n}T}\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{f,f}\dot{\bar{\mathbf{q}}}_f = \frac{1}{2}\dot{\mathbf{x}}_0^{\mathrm{n}T}\mathbf{A}\bar{\mathbf{S}}_{xf}\dot{\bar{\mathbf{q}}}_f \tag{42}$$

where $\bar{\mathbf{S}}_{xf}$ is a constant (3,3n) matrix

$$\bar{\mathbf{S}}_{xf} = \left(\sum_{i=1}^{n}\bar{\mathbf{S}}_{ff}^{i1} \quad \dots \quad \sum_{i=1}^{n}\bar{\mathbf{S}}_{ff}^{in}\right) \tag{43}$$

Substituting (25), we obtain

$$\bar{\mathbf{S}}_{xf} = (k^1\mathbf{E} \quad \dots \quad k^n\mathbf{E}) \tag{44}$$

From the definition of $\mathrm{T}_{\theta f}$ follows that:

$$\mathrm{T}_{\theta f} = -\frac{1}{2}\dot{\boldsymbol{\theta}}_0^T\bar{\mathbf{T}}^T\tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{ff}\dot{\bar{\mathbf{q}}}_f = -\frac{1}{2}\dot{\boldsymbol{\theta}}_0^T\bar{\mathbf{T}}^T\mathbf{S}_{\theta f}\dot{\bar{\mathbf{q}}}_f \tag{45}$$

where

$$\mathbf{S}_{\theta f} = \tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{ff} = \left(\sum_{i=1}^{n}\tilde{\bar{\mathbf{r}}}^{i\ T}\bar{\mathbf{S}}_{ff}^{i1} \quad \dots \quad \sum_{i=1}^{n}\tilde{\bar{\mathbf{r}}}^{i\ T}\bar{\mathbf{S}}_{ff}^{in}\right) \tag{46}$$

We also simplify equations for $\mathrm{T}_{\theta\theta}$:

$$\begin{aligned}\mathrm{T}_{\theta\theta} &= \frac{1}{2}(\mathbf{A}^{\mathbf{n}}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0)^T\mathrm{A}^{\mathbf{n}}\bar{\mathbf{S}}_{ff}\tilde{\bar{\mathbf{r}}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0 = \\ &= \frac{1}{2}\dot{\boldsymbol{\theta}}_0^T\bar{\mathbf{T}}^T\mathbf{S}_{\theta\theta}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0\end{aligned} \tag{47}$$

where

$$\mathbf{S}_{\theta\theta} = \tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{ff}\tilde{\bar{\mathbf{r}}} = \mathbf{S}_{\theta f}\tilde{\bar{\mathbf{r}}} \tag{48}$$

Substituting $\mathrm{T}_{xx}, \mathrm{T}_{x\theta}, \mathrm{T}_{xf}, \mathrm{T}_{\theta\theta}, \mathrm{T}_{\theta f}, \mathrm{T}_{ff}$ in (33), we obtain the mass matrix $\mathbf{M}$ in the FFRF:

$$\mathbf{M} = \begin{pmatrix} \bar{\mathbf{S}}_{xx} & -\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\mathbf{T}} & \mathbf{A}\bar{\mathbf{S}}_{xf} \\ -\left(\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\mathbf{T}}\right)^T & \bar{\mathbf{T}}^T\mathbf{S}_{\theta\theta}\bar{\mathbf{T}} & -\bar{\mathbf{T}}^T\mathbf{S}_{\theta f} \\ \left(\mathbf{A}\bar{\mathbf{S}}_{xf}\right)^T & -\mathbf{S}_{\theta f}^T\bar{\mathbf{T}} & \bar{\mathbf{S}}_{ff} \end{pmatrix} \tag{49}$$

B. Stiffness matrix $\bar{\boldsymbol{K}}$.

The potential energy of elastic forces can be written as

$$\Pi_f = \frac{1}{2}\bar{\mathbf{q}}_f^T\bar{\mathbf{K}}_{ff}\bar{\mathbf{q}}_f \tag{50}$$

where $\bar{\mathbf{K}}_{ff}$ is a constant stiffness matrix, equal in the beginning to the stiffness matrix $\mathbf{K}_A$ generated by ANSYS $\bar{\mathbf{K}}_{ff} = \mathbf{K}_{\mathrm{A}}|_{t=0}$.

Then the elastic forces $\mathbf{f}_k$ can be written as:

$$\mathbf{f}_k = -\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{K}}_{ff} \end{pmatrix}\begin{pmatrix} \mathbf{x}_0 \\ \boldsymbol{\theta}_0 \\ \bar{\mathbf{q}}_f \end{pmatrix} = -\bar{\mathbf{K}}\mathbf{p} \tag{51}$$

VI. EQUATIONS OF MOTION IN THE FFRF

Lagrange's equation for a flexible body can be written in the following form [3]

$$\begin{aligned}\begin{pmatrix} \mathbf{M}_{xx} & \mathbf{M}_{x\theta} & \mathbf{M}_{xf} \\ & \mathbf{M}_{\theta\theta} & \mathbf{M}_{\theta f} \\ \text{symmetric} & & \mathbf{M}_{ff} \end{pmatrix}\ddot{\mathbf{p}} + \mathbf{G}^T\boldsymbol{\lambda} \\ = \mathbf{f}_e + \mathbf{f}_v - \bar{\mathbf{K}}\mathbf{p}\end{aligned} \tag{52}$$

where $\mathbf{G}$ is the constraint Jacobian matrix, $\mathbf{f}_e$ is the vector of externally applied forces and $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, $\mathbf{M}_{xx}, \mathbf{M}_{x\theta}$, $\mathbf{M}_{xf}$, $\mathbf{M}_{\theta\theta}$, $\mathbf{M}_{\theta f}$, $\mathbf{M}_{ff}$ are the correspondent parts of $\mathbf{M}$, and $\mathbf{f}_v$ is the quadratic velocity vector

$$\mathbf{f}_v = \begin{pmatrix} \mathbf{f}_{vr} \\ \mathbf{f}_{v\theta} \\ \mathbf{f}_{vf} \end{pmatrix} = -\dot{\mathbf{M}}\dot{\mathbf{p}} + \left[\frac{\partial}{\partial\mathbf{p}}\left(\frac{1}{2}\dot{\mathbf{p}}^T\mathbf{M}\dot{\mathbf{p}}\right)\right]^T \tag{53}$$

Using the formula [3]

$$\boldsymbol{\alpha}_0 = \mathbf{A}\bar{\mathbf{T}}\ddot{\boldsymbol{\theta}}_0 \tag{54}$$

we can transform (52) to the equations of motion in the following form

$$\begin{aligned}\begin{pmatrix} \mathbf{M}_{xx} & \mathbf{M}_{x\alpha} & \mathbf{M}_{xf} \\ & \mathbf{M}_{\alpha\alpha} & \mathbf{M}_{\alpha f} \\ \text{symmetric} & & \mathbf{M}_{ff} \end{pmatrix}\begin{pmatrix} \ddot{\mathbf{x}}_0 \\ \boldsymbol{\alpha}_0 \\ \ddot{\bar{\mathbf{q}}}_f \end{pmatrix} + \begin{pmatrix} \mathbf{G}_x^T \\ \mathbf{G}_\alpha^T \\ \mathbf{G}_f^T \end{pmatrix}\boldsymbol{\lambda} \\ = \begin{pmatrix} \mathbf{f}_{ex} \\ \mathbf{f}_{e\alpha} \\ \mathbf{f}_{ef} - \bar{\mathbf{K}}_{ff}\bar{\mathbf{q}}_f \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{vx} \\ \mathbf{f}_{v\alpha} \\ \mathbf{f}_{vf} \end{pmatrix}\end{aligned} \tag{55}$$

where $\mathbf{M}_{x\alpha} = -\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\mathbf{A}^T$, $\mathbf{M}_{\alpha\alpha} = \mathbf{A}\mathbf{S}_{\theta\theta}\mathbf{A}^T$, $\mathbf{M}_{\alpha f} = -\mathbf{A}\mathbf{S}_{\theta f}$ and

$$\begin{aligned}\bar{\mathbf{T}}^T\mathbf{A}^T\mathbf{f}_{e\alpha} &= \mathbf{f}_{e\theta} \\ \bar{\mathbf{T}}^T\mathbf{A}^T\mathbf{f}_{v\alpha} &= \mathbf{f}_{v\theta}\end{aligned} \tag{56}$$

VII. FORCES AND QUADRATIC VELOCITY VECTOR IN THE FFRF

A. Gravitational forces

Using (44) and (10), we can formulate the potential energy of the gravitational forces in the IFF in the following form:

$$\Pi = -\mathbf{g}^T \sum_i k^i \mathbf{q}^i = -\mathbf{g}^T \bar{\mathbf{S}}_{xf}\mathbf{q} \\ = \mathbf{g}^T(\bar{\mathbf{S}}_{xf}\mathbf{x}_0^{\mathbf{n}} + \bar{\mathbf{S}}_{xf}\mathbf{A}^{\mathbf{n}}\bar{\mathbf{r}} \quad) \tag{57}$$

where $\mathbf{g} = g\mathbf{e}$, g is the value of the free fall acceleration and $\mathbf{e}$ is the gravitational direction. The vector of gravitational forces $\mathbf{f}_g = \left(\mathbf{f}_{gx}^T \quad \mathbf{f}_{g\alpha}^T \quad \mathbf{f}_{gf}^T\right)^T$ in the FFRF can be calculated from

$$\begin{pmatrix}\mathbf{f}_{gx}\\ \mathbf{f}_{g\theta}\\ \mathbf{f}_{gf}\end{pmatrix} = -\left(\frac{\partial \Pi}{\partial \mathbf{x}_0} \quad \frac{\partial \Pi}{\partial \boldsymbol{\theta}_0} \quad \frac{\partial \Pi}{\partial \bar{\mathbf{q}}_f}\right)^T \tag{58}$$

where

$$\bar{\mathbf{T}}^T \mathbf{A}^T \mathbf{f}_{e\alpha} = \mathbf{f}_{e\theta} \tag{59}$$

Substituting (44) in (57) and using (37), we get

$$\frac{\partial \Pi}{\partial \mathbf{x}_0} = -\mathbf{g}^T\left(\frac{\partial \bar{\mathbf{S}}_{xf}\mathbf{x}_0^{\mathbf{n}}}{\partial \mathbf{x}_0}\right) = -\mathbf{g}^T \frac{\partial}{\partial \mathbf{x}_0}\left(\sum_i k^i \, \mathbf{x}_0\right) \\ = -\mathbf{g}^T \bar{\mathbf{S}}_{xx} = -m\mathbf{g}^T \tag{60}$$

From (58) follows that

$$\mathbf{f}_{gx} = -\left(\frac{\partial \Pi}{\partial \mathbf{x}_0}\right)^T = m\mathbf{g} \tag{61}$$

Now we want to find the formula for $\mathbf{f}_{g\theta}$. Substituting (57) in (58) and using (44), we get:

$$\mathbf{f}_{g\theta}^T = -\frac{\partial \Pi}{\partial \boldsymbol{\theta}_0} = \mathbf{g}^T \frac{\partial}{\partial \boldsymbol{\theta}_0}\left(\bar{\mathbf{S}}_{xf}\mathbf{A}^{\mathbf{n}}\bar{\mathbf{r}}\right) \\ = \mathbf{g}^T \frac{\partial}{\partial \boldsymbol{\theta}_0}\left(\sum_i k^i \mathbf{A}\mathbf{r}^i\right) = \mathbf{g}^T \sum_i k^i \frac{\partial(\mathbf{A}\mathbf{r}^i)}{\partial \boldsymbol{\theta}_0} \tag{62}$$

Using (29), we obtain:

$$\frac{\partial(\mathbf{A}\mathbf{r}^i)}{\partial \boldsymbol{\theta}_0}\dot{\boldsymbol{\theta}}_0 = \dot{\mathbf{A}}\mathbf{r}^i = \mathbf{A}\tilde{\bar{\boldsymbol{\omega}}}\bar{\mathbf{r}}^i = -\mathbf{A}\tilde{\bar{\mathbf{r}}}^i \, \bar{\boldsymbol{\omega}} \\ = -\mathbf{A}\tilde{\bar{\mathbf{r}}}^i \, \bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0 \tag{63}$$

Therefore

$$\frac{\partial(\mathbf{A}\mathbf{r}^i)}{\partial \boldsymbol{\theta}_0} = -\mathbf{A}\tilde{\bar{\mathbf{r}}}^i \, \bar{\mathbf{T}} \tag{64}$$

Substituting (64) in (62), and using (40), we get

$$\mathbf{f}_{g\theta}^T = -\mathbf{g}^T \sum_i k^i \, \mathbf{A}\tilde{\bar{\mathbf{r}}}^i \, \bar{\mathbf{T}} = -\mathbf{g}^T\mathbf{A}\left(\sum_i k^i \, \tilde{\bar{\mathbf{r}}}^i\right)\bar{\mathbf{T}} \\ = -\mathbf{g}^T\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\mathbf{T}} \tag{65}$$

Taking into account that $\tilde{\mathbf{s}}_{x\theta}^T = -\tilde{\mathbf{s}}_{x\theta}$ and using (59), we obtain the required formula for $\mathbf{f}_{e\alpha}$:

$$\mathbf{f}_{e\alpha} = \mathbf{A}\tilde{\mathbf{s}}_{x\theta}\mathbf{A}^T\mathbf{g} \tag{66}$$

Calculating $\mathbf{f}_{gf}$ **,** we substitute (57) in (58) and get:

$$\mathbf{f}_{gf}^T = -\frac{\partial \Pi}{\partial \bar{\mathbf{q}}_f} = \mathbf{g}^T \frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\bar{\mathbf{S}}_{xf}\mathbf{A}^{\mathbf{n}}\bar{\mathbf{r}} \quad\right) = \mathbf{g}^T\bar{\mathbf{S}}_{xf}\mathbf{A}^{\mathbf{n}} \\ = \mathbf{g}^T\mathbf{A}\bar{\mathbf{S}}_{xf} \tag{67}$$

Therefore

$$\mathbf{f}_{gf} = \bar{\mathbf{S}}_{xf}^T\mathbf{A}^T\mathbf{g} \tag{68}$$

Finally, we get the formula for $\mathbf{f}_g$:

$$\mathbf{f}_g = \begin{pmatrix} m\mathbf{g} \\ \mathbf{A}\tilde{\mathbf{s}}_{x\theta}\mathbf{A}^T\mathbf{g} \\ \bar{\mathbf{S}}_{xf}^T\mathbf{A}^T\mathbf{g}\end{pmatrix} = \begin{pmatrix} m\mathbf{g} \\ -\mathbf{M}_{x\alpha}\mathbf{g} \\ \mathbf{M}_{xf}^T\mathbf{g}\end{pmatrix} \tag{69}$$

B. External force $\boldsymbol{f_p}$

Let us consider a vector of external force $\mathbf{F}_p$, acting on the i-th node of the deformable body. The virtual work of the force is defined as

$$\delta \mathrm{W}_i = \mathbf{F}_\mathbf{p}^T \delta \mathbf{q}^i \tag{70}$$

Substituting (9), we obtain that

$$\delta \mathbf{q}^i = \delta \mathrm{x}_0 + \frac{\partial(\mathbf{A}\mathbf{r}^i)}{\partial \boldsymbol{\theta}_0}\,\delta\boldsymbol{\theta}_0 + \mathbf{A}\delta\bar{\mathbf{q}}_f \tag{71}$$

Using (71), (64), we get that the vector $\mathbf{f}_p$ of generalized forces, correspondent to the force $\mathbf{F}_p(\mathbf{q}, \mathrm{t})$, can be written as

$$\mathbf{f}_p(\mathbf{F}_p) = \left(\mathbf{f}_{px}^T \quad \mathbf{f}_{p\alpha}^T \quad \mathbf{f}_{pf}^T\right)^T \tag{72}$$

where

$$\begin{aligned}
\mathbf{f}_{px} &= \mathbf{F}_p \\
\mathbf{f}_{p\alpha} &= \mathbf{A}\tilde{\bar{\mathbf{r}}}^i \, \mathbf{A}^T\mathbf{F}_p \\
\mathbf{f}_{pf} &= \left(\mathbf{0} \quad \ldots \quad \mathbf{0} \quad \left(\mathbf{f}_{pf}^i\right)^T \quad \mathbf{0} \quad \ldots \quad \mathbf{0}\right)^T \\
\mathbf{f}_{pf}^i &= \mathbf{A}^T \, \mathbf{F}_p
\end{aligned} \tag{73}$$

C. Quadratic velocity vector $\boldsymbol{f_v}$

From (53) follows that

$$\mathbf{f}_v = \begin{pmatrix}\mathbf{f}_{vx}\\ \mathbf{f}_{v\theta}\\ \mathbf{f}_{vf}\end{pmatrix} = -\begin{pmatrix}\mathbf{0} & \dot{\mathbf{M}}_{x\theta} & \dot{\mathbf{A}}\bar{\mathbf{S}}_{xf}\\ \dot{\mathbf{M}}_{\theta x}^T & \dot{\mathbf{M}}_{\theta\theta} & \dot{\mathbf{M}}_{\theta f}\\ \left(\dot{\mathbf{A}}\bar{\mathbf{S}}_{xf}\right)^T & \dot{\mathbf{M}}_{\theta f}^T & 0\end{pmatrix}\begin{pmatrix}\dot{\mathbf{x}}_0\\ \dot{\boldsymbol{\theta}}_0\\ \dot{\bar{\mathbf{q}}}_f\end{pmatrix} + \left[\frac{\partial}{\partial \mathbf{p}}\left(\frac{1}{2}\dot{\mathbf{p}}^T\mathbf{M}\dot{\mathbf{p}}\right)\right]^T \tag{74}$$

If we calculate $\mathbf{f}_v$ then, using (56), we can easy calculate the desired vector $\left(\mathbf{f}_{vx}^T \quad \mathbf{f}_{v\alpha}^T \quad \mathbf{f}_{vf}^T\right)^T$.

Formulas for calculation of $\mathbf{f}_{vx}$ and $\mathbf{f}_{v\theta}$ are written in [3]:

$$\mathbf{f}_{vx} = -\mathbf{A}\left[\,(\tilde{\bar{\boldsymbol{\omega}}})^2\mathbf{s}_{x\theta} + 2\tilde{\bar{\boldsymbol{\omega}}}\bar{\mathbf{S}}_{xf}\dot{\bar{\mathbf{q}}}_f\,\right] \tag{75}$$

$$\mathbf{f}_{v\theta} = -\dot{\bar{\mathbf{T}}}^T\left(2\mathbf{S}_{\theta\theta}\bar{\boldsymbol{\omega}} + 2\mathbf{S}_{\theta f}\dot{\bar{\mathbf{q}}}_f\right) - \bar{\mathbf{T}}^T\dot{\mathbf{S}}_{\theta\theta}\bar{\boldsymbol{\omega}} \tag{76}$$

Using (56) and simplifying, we get

$$\mathbf{f}_{v\alpha} = -\mathbf{A}\left(\tilde{\bar{\boldsymbol{\omega}}}\left[\mathbf{S}_{\theta\theta}\bar{\boldsymbol{\omega}} + \mathbf{S}_{\theta f}\dot{\bar{\mathbf{q}}}_f\right] + \dot{\mathbf{S}}_{\theta\theta}\bar{\boldsymbol{\omega}}\right) \tag{77}$$

where $\dot{\mathbf{S}}_{\theta\theta}$ is calculated using the formula:

$$\begin{aligned}\dot{\mathbf{S}}_{\theta\theta} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(\tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{f,f}\tilde{\bar{\mathbf{r}}}\right) = \tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{f,f}\frac{\mathrm{d}\tilde{\bar{\mathbf{r}}}}{\mathrm{dt}} + \left(\tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{f,f}\frac{\mathrm{d}\tilde{\bar{\mathbf{r}}}}{\mathrm{dt}}\right)^T\\ &= \tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{f,f}\widetilde{\left(\dot{\bar{\mathbf{q}}}_f\right)} + \widetilde{\left(\dot{\bar{\mathbf{q}}}_f\right)}^T\bar{\mathbf{S}}_{f,f}\tilde{\bar{\mathbf{r}}}\end{aligned} \tag{78}$$

The equations for $\mathbf{f}_{vx}$ and $\mathbf{f}_{v\alpha}$ can be proved by substituting in (74) correspondent equations for parts of $\mathbf{M}$.

Now we need to find the formula for the calculation of $\mathbf{f}_{vf}$. From (74) follows that:

$$\mathbf{f}_{vf} = -\left(\dot{\mathbf{A}}\bar{\mathbf{S}}_{xf}\right)^T\dot{\mathbf{x}}_0 - \dot{\mathbf{M}}_{\theta f}^T\dot{\boldsymbol{\theta}}_0 + \left[\frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\frac{1}{2}\dot{\mathbf{p}}^T\mathbf{M}\dot{\mathbf{p}}\right)\right]^T \tag{79}$$

Using the definition of the angular velocity $\bar{\boldsymbol{\omega}}$, we can rewrite the first term of (79) as:

$$\begin{aligned}\left(\dot{\mathbf{A}}\bar{\mathbf{S}}_{xf}\right)^T\dot{\mathbf{x}}_0 &= \bar{\mathbf{S}}_{xf}^T\dot{\mathbf{A}}^T\dot{\mathbf{x}}_0 = \bar{\mathbf{S}}_{xf}^T\left(\dot{\mathbf{x}}_0^T\dot{\mathbf{A}}\right)^T\\ &= \bar{\mathbf{S}}_{xf}^T(\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\bar{\boldsymbol{\omega}}})^T = -\bar{\mathbf{S}}_{xf}^T\tilde{\bar{\boldsymbol{\omega}}}\mathbf{A}^T\dot{\mathbf{x}}_0\end{aligned} \tag{80}$$

From the definition of $\mathbf{M}_{\theta f}$ we obtain formula for the second term of (79)

$$\dot{\mathbf{M}}_{\theta f}^T\dot{\boldsymbol{\theta}}_0 = -\frac{\mathrm{d}\mathbf{S}_{\theta f}^T}{\mathrm{dt}}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0 - \bar{\mathbf{S}}_{\theta f}^T\frac{\mathrm{d}\bar{\mathbf{T}}}{\mathrm{dt}}\dot{\boldsymbol{\theta}}_0 \tag{81}$$

It can be easily proved that

$$\frac{\mathrm{d}\bar{\mathbf{T}}}{\mathrm{dt}}\dot{\boldsymbol{\theta}}_0 = \mathbf{0} \tag{82}$$

Using this formula and substituting (46) in (81), we get

$$\begin{aligned}\dot{\mathbf{M}}_{\theta f}^T\dot{\boldsymbol{\theta}}_0 &= -\frac{\mathrm{d}\mathbf{S}_{\theta f}^T}{\mathrm{dt}}\bar{\boldsymbol{\omega}} = -\frac{\mathrm{d}\left(\tilde{\bar{\mathbf{r}}}^T\bar{\mathbf{S}}_{ff}\right)^T}{\mathrm{dt}}\bar{\boldsymbol{\omega}}\\ &= -\bar{\mathbf{S}}_{ff}^T\frac{\mathrm{d}\tilde{\bar{\mathbf{r}}}}{\mathrm{dt}}\bar{\boldsymbol{\omega}} = -\bar{\mathbf{S}}_{ff}\widetilde{\left(\dot{\bar{\mathbf{q}}}_f\right)}\bar{\boldsymbol{\omega}}\end{aligned} \tag{83}$$

Let us now consider the third term of (79):

$$\left[\frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\frac{1}{2}\dot{\mathbf{p}}^T\mathbf{M}\dot{\mathbf{p}}\right)\right]^T = \left[\frac{\partial \mathrm{T}}{\partial\bar{\mathbf{q}}_f}\right]^T \tag{84}$$

where T is the kinetic energy of the body. Substituting (33) and using the fact that T_{xx}, T_{xf}, T_{ff} do not depend on $\bar{\mathbf{q}}_f$, we get:

$$\left[\frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\frac{1}{2}\dot{\mathbf{p}}^T\mathbf{M}\dot{\mathbf{p}}\right)\right]^T = \left[2\frac{\partial\mathrm{T}_{x\theta}}{\partial\bar{\mathbf{q}}_f} + \frac{\partial\mathrm{T}_{\theta\theta}}{\partial\bar{\mathbf{q}}_f} + 2\frac{\partial\mathrm{T}_{\theta f}}{\partial\bar{\mathbf{q}}_f}\right]^T \tag{85}$$

Let us consider the first term in the right part. From the definition of $\mathrm{T}_{x\theta}$ follows that

$$\frac{\partial\mathrm{T}_{x\theta}}{\partial\bar{\mathbf{q}}_f} = -\frac{1}{2}\frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0\right) \tag{86}$$

The partial derivation in this equation can be simplified:

$$\begin{aligned}\frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\mathbf{T}}\dot{\boldsymbol{\theta}}_0\right) &= \frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\mathbf{s}}_{x\theta}\bar{\boldsymbol{\omega}}\right)\\ &= -\frac{\partial}{\partial\bar{\mathbf{q}}_f}\left(\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\bar{\boldsymbol{\omega}}}\mathbf{s}_{x\theta}\right) = -\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\bar{\boldsymbol{\omega}}}\frac{\partial\mathbf{s}_{x\theta}}{\partial\bar{\mathbf{q}}_f}\end{aligned} \tag{87}$$

Substituting this equation in (86), we get

$$\frac{\partial\mathrm{T}_{x\theta}}{\partial\bar{\mathbf{q}}_f} = \frac{1}{2}\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\bar{\boldsymbol{\omega}}}\frac{\partial\mathbf{s}_{x\theta}}{\partial\bar{\mathbf{q}}_f} \tag{88}$$

From the definition of $\mathbf{s}_{x\theta}$ follows that

$$\frac{\partial\mathbf{s}_{x\theta}}{\partial\bar{\mathbf{q}}_f} = \frac{\partial\sum_j \mathrm{k}_j\bar{\mathbf{r}}^j}{\partial\bar{\mathbf{q}}_f} = (\mathrm{k}_1\mathbf{E} \quad \ldots \quad \mathrm{k}_n\mathbf{E}) = \bar{\mathbf{S}}_{xf} \tag{89}$$

Therefore, we obtain that

$$\frac{\partial\mathrm{T}_{x\theta}}{\partial\bar{\mathbf{q}}_f} = \frac{1}{2}\dot{\mathbf{x}}_0^T\mathbf{A}\tilde{\bar{\boldsymbol{\omega}}}\bar{\mathbf{S}}_{xf} \tag{90}$$

Consider now the second term of (85). Using formulas (47), (29) and (48) for $\mathrm{T}_{\theta\theta}$, $\bar{\boldsymbol{\omega}}$, and $\mathbf{S}_{\theta\theta}$, we get:

$$\frac{\partial \mathrm{T}_{\theta\theta}}{\partial \bar{\mathbf{q}}_f} = \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\bar{\boldsymbol{\omega}}^T \mathbf{S}_{\theta\theta}\bar{\boldsymbol{\omega}}\right) = \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\bar{\boldsymbol{\omega}}^T \tilde{\bar{\mathbf{r}}}^T \bar{\mathbf{S}}_{ff}\tilde{\bar{\mathbf{r}}}\bar{\boldsymbol{\omega}}\right) \quad (91)$$

Let us define is a symmetric matrix $\mathbf{J}_{ww}$, consisting of *n* blocks $\mathbf{J}_{ww}^{i,j}$ size (3,3), calculated from

$$\mathbf{J}_{ww}^{i,j} = \tilde{\bar{\boldsymbol{\omega}}}^T \bar{\mathbf{S}}_{ff}^{ij}\tilde{\bar{\boldsymbol{\omega}}} \quad i,j = 1..n \quad (92)$$

Then (91) can be rewritten as

$$\frac{\partial \mathrm{T}_{\theta\theta}}{\partial \bar{\mathbf{q}}_f} = \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\bar{\mathbf{r}}^T \mathbf{J}_{ww}\bar{\mathbf{r}}\right) = \bar{\mathbf{r}}^T \mathbf{J}_{ww} \quad (93)$$

Consider now the third term of (85)

$$\frac{\partial \mathrm{T}_{\theta f}}{\partial \bar{\mathbf{q}}_f} = -\frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\bar{\boldsymbol{\omega}}^T \tilde{\bar{\mathbf{r}}}^T \bar{\mathbf{S}}_{ff}\dot{\bar{\mathbf{q}}}_f\right) \quad (94)$$

Let us define the matrix $\mathbf{J}_w$ matrix, consisting of *n* blocks $\mathbf{J}_w^{i,j}$ size (3,3), calculated from

$$\mathbf{J}_w^{i,j} = \tilde{\bar{\boldsymbol{\omega}}}^T \bar{\mathbf{S}}_{ff}^{ij} \quad i,j = 1..n \quad (95)$$

Then (94) can be rewritten as

$$\frac{\partial \mathrm{T}_{\theta f}}{\partial \bar{\mathbf{q}}_f} = \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\bar{\mathbf{r}}^T \mathbf{J}_w \dot{\bar{\mathbf{q}}}_f\right) = \frac{1}{2}\dot{\bar{\mathbf{q}}}_f^T \mathbf{J}_w^T = \frac{1}{2}\left[\mathbf{J}_w \dot{\bar{\mathbf{q}}}_f\right]^T \quad (96)$$

No we substitute (90), (93), (96) in (85):

$$\begin{aligned}&\left[\frac{\partial}{\partial \bar{\mathbf{q}}_f}\left(\frac{1}{2}\dot{\mathbf{p}}^T \mathbf{M}\dot{\mathbf{p}}\right)\right]^T = \\ &= \left[\dot{\mathbf{x}}_0^T \mathbf{A}\tilde{\bar{\boldsymbol{\omega}}}\bar{\mathbf{S}}_{xf}\right]^T + \left[\bar{\mathbf{r}}^T \mathbf{J}_{ww}\right]^T + \mathbf{J}_w \dot{\bar{\mathbf{q}}}_f = \\ &= -\bar{\mathbf{S}}_{xf}^T \tilde{\bar{\boldsymbol{\omega}}}\mathbf{A}^T \dot{\mathbf{x}}_0 + \mathbf{J}_{ww}\bar{\mathbf{r}} + \bar{\mathbf{S}}_{ff}\widetilde{\left(\dot{\bar{\mathbf{q}}}_f\right)}\bar{\boldsymbol{\omega}}\end{aligned} \quad (97)$$

Finally, substituting (80), (83), (97) in (79), we obtain the desired formula for $\mathbf{f}_{vf}$:

$$\mathbf{f}_{vf} = 2\bar{\mathbf{S}}_{ff}\widetilde{\left(\dot{\bar{\mathbf{q}}}_f\right)}\bar{\boldsymbol{\omega}} + \mathbf{J}_{ww}\bar{\mathbf{r}} \quad (98)$$

VIII. Definition of Body Reference Frame

The equations of motion (55) are valid for each body frame $\{\bar{O}\ \bar{\mathbf{e}}^1\bar{\mathbf{e}}^2\bar{\mathbf{e}}^3\}$, initially coinciding with the global inertia frame $\{O\ \mathbf{e}^1\mathbf{e}^2\mathbf{e}^3\}$. A unique representation of the body motion requires six additional equations, defining the body reference frame. We propose to use the definition of the body frame based on three points of the body, proposed by [5].

Let in the beginning the absolute coordinates of the first node are equal to null, the second node lies on the *x*-axis and the third node lies on the x-y plane:

$$\begin{aligned}&\mathbf{q}^1\big|_{t=0} = \mathbf{0} \\ &\mathbf{q}^2\big|_{t=0} = \left(q_x^2\big|_{t=0} \quad 0 \quad 0\right)^T \\ &\mathbf{q}^1\big|_{t=0} = \left(q_x^3\big|_{t=0} \quad q_y^3\big|_{t=0} \quad 0\right)^T\end{aligned} \quad (99)$$

Then we can set the local frame, using the following conditions:

1. the origin $\bar{O}$ is attached to the first node:

$$\bar{\mathbf{r}}^1 = \mathbf{0} \quad (100)$$

2. $\bar{\mathbf{e}}^1$ is on the line between $\bar{O}$ and the second node:

$$\bar{\mathrm{r}}^{2,y} = 0 \quad \bar{\mathrm{r}}^{2,z} = 0 \quad (101)$$

3. $\bar{\mathbf{e}}^3$ is perpendicular to the line between $\bar{O}$ and the third node:

$$\bar{\mathrm{r}}^{3,z} = 0 \quad (102)$$

This implies six limitations on the vector of nodal deformations $\bar{\mathbf{q}}_f$:

$$\bar{\mathbf{q}}_f^1 = \mathbf{0}, \bar{\mathrm{q}}_f^{2,y} = 0, \bar{\mathrm{q}}_f^{2,z} = 0, \bar{\mathrm{q}}_f^{3,z} = 0 \quad (103)$$

Therefore, we can easily exclude these six positions from $\bar{\mathbf{q}}_f$ and six correspondent rows and columns from matrices and vectors in (55). The matrix **M** after this procedure becomes invertible. Hence, the values of accelerations can be uniquely calculated from the modified (55).

IX. Modal Transformation

A major advantage of using the FFRF is that the finite element nodal coordinates can be easily reduced using modal analysis techniques, based on using a reduced set of eigenvectors **a** (also called *mode shapes*) of the free vibration equations of motion [3]:

$$\bar{\mathbf{K}}_{ff}\mathbf{a} = \omega^2 \mathbf{M}_{ff}\mathbf{a} \quad (104)$$

The effect of elimination of high-frequency mode shape on the computational speed is twofold. Firstly, the number of numerical operations on each time step decreases because the size of matrices in the equations of motions is much less than in non-reduced case. Secondly, a larger integration time step can be used.

X. Implementation

We have implemented the method and developed in Maple a module VSDFlex for the simulation of dynamics of flexible multibodies. The module is integrated with ANSYS, where we get matrices of the initial values of $\mathbf{q}|_{t=0}$, $\mathbf{M}_{\mathbf{A}}|_{t=0}$, $\mathbf{K}_{\mathbf{A}}|_{t=0}$, describing the motion of bodies in the absence of constraints. The types and parameters of constrains should be defined in VSDFlex.

The simulation in VSDFlex consists on three standard steps:

1. **Preprocessing Step**: Starting from the values of $\mathbf{q}|_{t=0}$, $\mathbf{M}_{\mathbf{A}}|_{t=0}$, $\mathbf{K}_{\mathbf{A}}|_{t=0}$, VSDFlex calculates integrals $\bar{\mathbf{S}}_{xf}$, $\bar{\mathbf{S}}_{ff}$, $\bar{\mathbf{S}}_{xx}$ and a modal transformation matrix **B**.
2. **Dynamic analysis:** VSDFlex calculates the dynamics of the simulated system, using the modal

transformation.

3. **Postprocessing Step**: Starting from trajectory of modal coordinates, obtained during the dynamic analysis, the trajectories positions, velocities and accelerations of nodes, applied forces as well as stresses and strains can be evaluated.

XI. Flexible Pendulum Example

The example we considered is a flexible pendulum, articulated to the foundation through a revolute joint. The model of the pendulum, shown in Fig. 2, was developed in ANSYS as a rod, consisting of 16 SOLID45 elements.

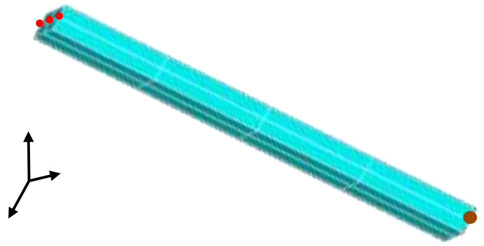

Fig. 2 Model of flexible pendulum

The revolute joint is modeled in VSDFlex as a fixation of three nodes, colored in Fig. 2 by red, in the inertial system of coordinates. The physical properties, expressed in SI are: length 0.1, wide 0.01, thickness 0.015, elasticity modulus 10^7, Poisson ratio 0.49, mass 0.26, moment of inertia $3.3\cdot10^{-3}$. The simulation was performed using the modal transformation, including the nine mode shapes, correspondent to the first nine eigenfrequencies (from 5 to 124 Hz).

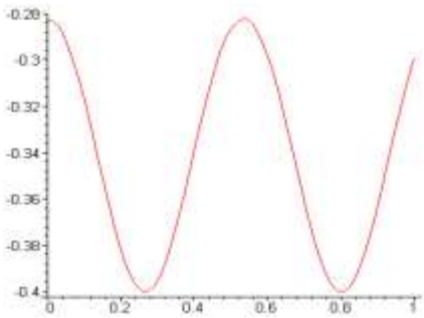

Fig. 3 Z-coordinate of the end node

Fig. 3 shows the trajectory of the *z*-coordinate of the node at the end of the rod (marked in Fig. 2 by brown). Due to model order reduction a simulation step size of $5\cdot10^{-5}$ ms guarantees stable results. Obviously, the high elasticity modulus implies little displacements on the coordinate level.

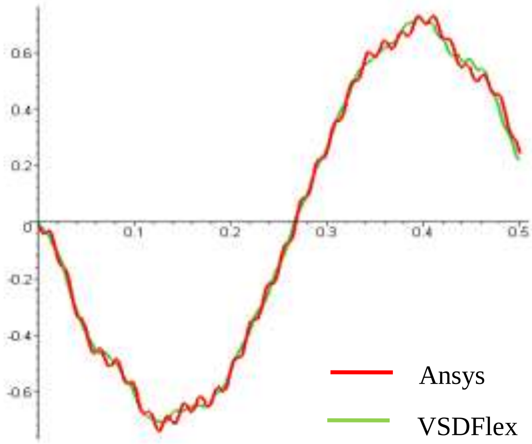

Fig. 4 z-velocities of the end node

In order to prove the VSDFlex simulation we simulated the dynamics of the model in ANSYS. In Fig. 4 are shown the velocity in *z*-direction of the same node, obtained in VSDFlex and in ANSYS. The difference between ANSYS and VSDFlex solutions appears because of the accounting of non-linear large-deflection effects in ANSYS. The divergences on the coordinate and on the velocity levels are limited by $4\cdot10^{-4}$ and $4\cdot10^{-2}$ correspondently.

XII. Conclusion

The article shows a method of transformation of equations of motion in the inertial frame formulation (IFF), used for FEM models of flexible multibody systems, to the equations in the floating frame of reference formulation (FFRF). The method was implemented in VSDFlex module, performing the simulation of flexible multibody systems. Starting from the initial values of mass matrix $\mathbf{M_A}|_{t=0}$and stiffness matrix $\mathbf{K_A}|_{t=0}$, generated by ANSYS in the IFF, the module generates the FFRF equations of motion. The use of the modal transformation during the simulation greatly reduces the computation time. Results of the simulation of flexible pendulum example show the method's stability.

It seems needful to integrate VSDFlex with the object-oriented simulation software VSD in order to improve the software usability and areas of implementation.

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