Approximation of Controllable Set by Semidefinite Programming for Open-Loop Unstable Systems with Input Saturation

Abraham W. L. Wang, *Member, IAENG*, and Yen-Ming Chen

***Abstract*—In order to test the efficiency of semidefinite programming (SDP), we apply SDP software package to the solution of open-loop unstable systems under input saturation. And we compare the ease of programming and the execution time for solving the problem between the classical approach (which applies a nonlinear equation solver to the Kuhn-Tucker conditions) and the SDP approach (which exploits interior-point algorithms) for three techniques in this paper. It is also shown that, for certain types of optimization problems, SDP is indeed very efficient. However, our examples show that SDP has limitations in solving non-convex optimization problems. It is also shown that the technique we proposed, namely that of approximating the controllable set inside the Lyapunov descent criterion, is better than the controllable set found by SDP, even though the execution time is inferior to the latter.**

***Index Terms*—Controllable Set, Semidefinite Programming (SDP), Lyapunov descent criterion, Kuhn-Tucker Theorem.**

I. Introduction

Semidefinite programming (SDP) is an extension of linear programming (LP) with vector variables replaced by matrix variables and with vector elementwise non-negativity constraints replaced by matrix positive semidefiniteness constraints. Generally speaking, in semidefinite programming, one minimizes a linear function subject to the constraint that a linear combination of symmetric matrices be positive semmidefinite. A typical example of a semidefinite programming problem is

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & F(x) \succeq 0 \end{array} \tag{1}$$

where x is a solution vector in $\Re^n$ and c is a constant vector in $\Re^n$ and F is linear with respect to x. We call $F(x) \succeq 0$ a *linear matrix inequality* because of linearity of F with respect to x and $F(x)$ is a square matrix. Here, $F(x) \succeq 0$ means that $F(x)$ is positive semidefinite.

Manuscript received September 2, 2008. This research was sponsored by the National Science Council, Taiwan, R.O.C. under Grant NSC 94-2213-E-216-026.

Abraham W. L. Wang is currently an Associate Professor with the Department of Information Management, Chung Hua University, Hsin Chu, 30012, Taiwan (Tel: (886) 3-518-6529; fax: (886) 3-518-6546; e-mail: abewang@chu.edu.tw).

Y. M. Chen is currently an Assistant Professor with the Department of Logistics Management, National Kaohsiung First University of Science and Technology (e-mail: yjjchen@ccms.nkfust.edu.tw).

There are similarities between semidefinite programming and linear programming in theory and practice, e.g., duality theory (see Bellman and Fan [5]), the role of complementary slackness, and efficient solution techniques using interior-point methods (see Nesterov and Nemirovski [11], Wright, [20]). Semidefinite programming has been developed both theoretically and practically for the past few yew years, and has become a popular topic due to its efficiency in solving optimization problems with the use of interior-point methods (Stefanatos and Khaneja, [14], Balakrishnan and Vandenberghe [4]).

Semidefinite programming also unifies several standard problems (e.g., linear and quadratic programming) and can be applied to many engineering problems (see Boyd et al. [6], Vandenberghe and Boyd [18]), and combinatorial optimizations (see Alizadeh [1], Goemans [8]). Semidefinite programming is an important numerical tool for analysis and synthesis in control systems theory (see Vandenberghe and Boyd [17], Yao, et al. [21]), and many semidefinite programming problems can be solved very efficiently both in theory and practice (see Alizadeh, et al. [2], Fujisawa, et al. [7], Porta, et al. [13], Toh, et al. [15], and Vandenberghe and Boyd [16]).

In this paper, we applied semidefinite programming to the optimization problem of approximating the controllable set by using the SDPpack, and then compared it with the controllable set proposed by Lee and Hedrick [10], and the Lyapunov controllable set studied in our previous works [19].

Our results show that the commands usage for SDP are only about half of the commands written for the Lagrangian technique. Furthermore, the execution time by SDP is shorter. However, the controllable set found by SDP is slightly larger than the controllable set found by applying the concept proposed by Lee and Hedrick, but is smaller than the Lyapunov controllable set found by our previous work.

II. Linear Matrix Inequality

Many problems in control and systems theory can be formulated as optimization problems in terms of linear matrix inequalities (LMIs), i.e., constraints of the form

$$F(x) \triangleq F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0, \tag{2}$$

where $x \in \Re^m$ is the variable, and the matrices $F_i = F_i^T \in \Re^{n \times n}, i = 0, \cdots, m$, are given symmetric constant matrices, and the inequality $F(x) \succeq 0$ represents the

(Advance online publication: 19 November 2009)

requirement that $F(x)$ be positive semidefinite. There are several equivalent definitions for the function of $F(x)$.

Lemma 2.1 *The following statements are equivalent for a symmetric real matrix* $F \in \Re^{n\times n}$.

1. *F is positive semidefinite.*
2. $z^T Fz \geq 0, \forall z \in \Re^n$.
3. *All the eigenvalues of F are positive or zero.*
4. *There exists a real matrix* $M \in \Re^{n\times n}$ *such that* $F = M^T M$.

The LMI (2) is a convex constraint on x, i.e., the set $\{x \mid F(x) \succeq 0\}$ is convex. The LMI can represent a wide variety of convex constraints on *x*, e.g., linear inequalities, certain forms of quadratic inequalities, matrix norm inequalities, constraints arising in control theory, such as Lyapunov and convex quadratic matrix inequalities. Many conditions can be cast in the form of LMI.

We note that multiple LMIs, $F_1(x)\succeq 0, \cdots, F_m(x)\succeq 0,$ can be expressed as a single LMI as $\mathrm{diag}(F_1(x), \cdots, F_m(x))\succeq 0$, i.e.,

$$F_1(x)\succeq 0,\cdots,F_m(x)\succeq 0 \Leftrightarrow \begin{bmatrix} F_1(x) & 0 & \cdots & 0 \\ 0 & F_1(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_1(x) \end{bmatrix}$$

Nonlinear (convex) inequalities can also be converted into the LMI form by applying Schur decomposition: the LMI; the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} \succ 0, \tag{3}$$

where $Q(x) = Q(x)^T, R(x) = R(x)^T$, and *S(x)* depend affinely on *x*, is equivalently to

$$R(x) \succ 0,\ Q(x) - S(x)R(x)^{-1}S(x)^T \succ 0, \tag{4}$$

i.e., the inequality of (4) can be represented as an LMI (4). Here, $R(x) \succ 0$ represents the requirement that the matrix *R*(*x*) is positive definite.

As another related example, the LMI

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} \succ 0,$$

where $A, B, Q = Q^T, R = R^T$ are given constant matrices of appropriate sizes, and $P = P^T$ is the variable, is equivalent to the algebraic Riccati inequality

$$A^T P + PA + PBR^{-1}B^T P + Q \prec 0,\ R \succ 0.$$

III. SEMIDEFINITE PROGRAMMING

We consider the optimization problem of minimizing a linear function of variable $x \in \Re^m$ subject to an LMI:

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & F(x) \succeq 0 \end{array} \tag{5}$$

where *c* is a constant vector in $\Re^n$ and $F(x)$ is defined as in (1). Then we call the optimization problem (3) a *semidefinite program* (SDP). A semidefinite program is a convex optimization problem since the objective and constraints are convex: if $F(x)\succeq 0$ and $F(y)\succeq 0$, then, for all λ, $0 \leq \lambda \leq 1$,

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) \succeq 0.$$

There are many similarities between semidefinite programs and linear programs both in theory and practice, e.g., in duality theory, the role of complementary slackness, and availability of efficient solution techniques using interior-point methods. Foe instance, consider the following linear program (LP):

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax + b \geq 0, \end{array}$$

in which the inequality denotes a componentwise inequality. A vector v is nonnegative, $v \geq 0$, if and only if the matrix $\mathbf{diag}(v)$ is positive semidefinite $\mathbf{diag}(v) \succeq 0$. Therefore, we can express the LP as a semidefinite program with the linear matrix inequality $F(x) = \mathbf{diag}(Ax + b)$, i.e.,

$$F_0 = \mathbf{diag}(b),\quad F_i = \mathbf{diag}(a_i),\quad i = 1, \cdots, m,$$

where $A = [a_1, \cdots, a_m] \in \Re^{n\times m}$.

A convex quadratic constraint $(Ax+b)^T(Ax+b) - c^T x - d \leq 0$, where $x \in \Re^m$, can be written as

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^T x + d \end{bmatrix} \succeq 0. \tag{6}$$

The left-hand side of equation (4) depends affinely on vector *x*, and hence it can be expressed as a linear matrix inequality,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0,$$

where

$$F_0 = \begin{bmatrix} I & b \\ b^T & d \end{bmatrix},\quad F_i = \begin{bmatrix} 0 & a_i \\ a_i^T & c_i \end{bmatrix},\quad i = 1, \cdots, k.$$

Therefore, a general (convex) quadratically constrained quadratic program (QCQP) problem in $x \in \Re^m$,

$$\begin{array}{ll} \min & f_0(x) \\ \text{subject to} & f_i(x) \leq 0,\quad i = 1, 2, \cdots, k, \end{array} \tag{7}$$

where each $f_i, i = 0, \cdots, k$, is a convex quadratic function of the form

$$f_i(x) = (A_i x + b_i)^T (A_i x + b_i) - c_i^T x - d_i, \tag{8}$$

or equivalently a general quadratically constrained quadratic program problem in $(x, t) \in \Re^{m+1}$,

$$\begin{aligned} &\min && t \\ &\text{subject to} && f_0(x) \le t, \\ &&& f_i(x) \le 0, \quad i = 1,2,\cdots,k, \end{aligned}$$

can be written as

$$\begin{aligned} &\min && t \\ &\text{subject to} && \begin{bmatrix} I & A_0x+b_0 \\ (A_0x+b_0)^T & c_0^Tx+d_0+t \end{bmatrix} \succeq 0, \qquad (9) \\ &&& \begin{bmatrix} I & A_ix+b_i \\ (A_ix+b_i)^T & c_i^Tx+d_i \end{bmatrix} \succeq 0, \quad i=1,2,\cdots,k, \end{aligned}$$

We then can put the above QCQP in the SDP form:

$$\begin{aligned} &\min && t \\ &\text{subject to} && F(t,x) = F_0 + \sum_{i=1}^{m} x_i F_i + tF_{m+1} \succeq 0, \end{aligned} \qquad (10)$$

where the variables are $x \in \Re^m$ and $t \in \Re$.

For a non-convex optimization problem of the form,

$$\begin{aligned} &\min && f_0(x) \\ &\text{subject to} && f_i(x) \le 0, \quad i=1,2,\cdots,k, \end{aligned} \qquad (11)$$

where $f_i(t) = x^T A_i x + 2b_i^T x + c_i, i = 0,1,\cdots,k$, and the matrices A_i may be indefinite, it has been proposed by Shor and others that the lower bounds for the minimum value of $f_0(x)$ for (11) can be obtained by solving the semidefinite programming (with variables t and τ_i),

$$\begin{aligned} &\min && t \\ &\text{subject to} && \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots \\ &&& + \tau_k \begin{bmatrix} A_k & b_k \\ b_k^T & c_k \end{bmatrix} \succeq 0, \qquad (12) \\ &&& \tau_i \ge 0, \quad i=1,2,\cdots,k. \end{aligned}$$

We can easily verify that this semidefinite program yields a lower bound for the minimum value of $f_0(x)$ of (11). Suppose that x satisfies the constraints in the non-convex problem (11), i.e.,

$$f_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0,$$

for $i = 1,\cdots,k$, and that $t, \tau_1, \cdots, \tau_k$ satisfy the constraints in the semidefinite program (10). Then

$$\begin{aligned} 0 &\le \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left\{ \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots \right. \\ &\quad \left. + \tau_k \begin{bmatrix} A_k & b_k \\ b_k^T & c_k \end{bmatrix} \right\} \begin{bmatrix} x \\ 1 \end{bmatrix} \\ &= f_0(t) - t + \tau_1 f_1(x) + \cdots + \tau_k f_k(x) \\ &\le f_0(x) - t. \end{aligned}$$

Therefore, t is indeed a lower bound for the minimum value of $f_0(x)$ in (11).

IV. SOFTWARE PACKAGES FOR SEMIDEFINITE PROGRAMMING

Several software packages have been developed for the past few years for solving the semidefinite program. Here we give a brief introduction for one of the software packages applied in this paper:

SDPPACK

This is a software package for Matlab and is made by Alizadeh et al. [2].

Semidefinite-Quadratic-Linear Program (SQLP)

This package solves the primal mixed semidefinite-quadratic-linear program of the form

$$\begin{aligned} &\min && C_S \bullet X_S + C_Q^T X_Q + C_L^T X_L \\ &\text{subject to} && (A_S) \bullet X_S + (A_Q)_k^T X_Q + (A_L)_k^T X_L = b_k, k = 1,\cdots,m, \\ &&& X_S \succeq 0, \quad X_Q \ge 0, \quad X_L \ge 0, \end{aligned}$$

where X_S is a block diagonal symmetric matrix variable, with block sizes $N_1, N_2, \cdots, N_S$ respectively, each greater than or equal to two; X_Q is a block vector variable, with block sizes $n_1, n_2, \cdots, n_q$ respectively, each greater than or equal to two; and X_L is a vector of length n_0 . The quantities C_Q and $(A_Q)_k, k = 1,\cdots,m$, are also vectors. The quantity $C_S \bullet X_S$ is the trace inner product ($\mathrm{tr} C_S X_S$), i.e., $\sum_{ij} (C_S)_{ij} (X_S)_{ij}$.

Each of the three inequalities in this primal problem has a different meaning.

- The first kind of inequality is a semidefinite constraint. $X_S \succeq 0$ means that the matrix X_S is positive semidefinite.
- The second kind of inequality describes a quadratic cone constraint. Writing $x = X_Q$ for brevity, with the block structure
$$x = [(x^1)^T, (x^2)^T, \cdots, (x^q)^T]^T,$$
where
$$x^i = [x_1^i \; x_2^i \cdots x_{n_i}^i],$$
The constraint $X_Q \ge_Q 0$, i.e., $x \ge_Q 0$ means that, for each block i,
$$x_1^i \ge \sqrt{\sum_{j=2}^{n_i} (x_j^i)^2}.$$
Any convex quadratic constraint can be converted to this form.
- The third kind of inequality is the standard one: $X_L \ge 0$ means each component of vector X_L is nonnegative.

The dual SQLP is

$$\begin{aligned} \max \quad & b^T y \\ \text{subject to} \quad & \sum_{k=1}^{m} y_k (A_S)_k + Z_S = C_S, \\ & \sum_{k=1}^{m} y_k (A_Q)_k + Z_Q = C_Q, \\ & \sum_{k=1}^{m} y_k (A_L)_k + Z_L = C_L, \\ & Z_S \succeq 0,\ Z_Q \succeq 0,\ Z_L \succeq 0, \end{aligned}$$

where Z_S, Z_Q, and Z_L are matrix variables.

Algorithm

This package implements a primal-dual Mehrotra predictor-corrector scheme based on the *XZ*+*ZX* search direction for SDP.

V. FINDING ELLIPSOIDAL CONTROLLABLE SETS BY SEMIDEFINITE PROGRAMMING

Consider a linear time-invariant continuous-time system with input saturation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{13}$$

$$u(t) = -sat(Kx(t)), \tag{14}$$

where $A \in \Re^{n\times n}$ is a given constant matrix, $B \in \Re^{n\times m}$ is a given constant matrix, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control vector, with $u(t) = [u_1(t),..., u_m(t)]$, and $sat(\cdot)$ denotes the saturation function. The one-dimensional version of the saturation function is defined by

$$sat(y) = \begin{cases} 1, & \text{if } y \ge 1 \\ y, & \text{if } y \in (-1,1) \\ -1, & \text{if } y \le -1 \end{cases}, \forall y \in \Re \tag{15}$$

and we componentwise extend its definition to the multi-dimensional version:

$$sat(y) = \begin{bmatrix} sat(y_1) \\ sat(y_2) \\ \vdots \\ sat(y_3) \end{bmatrix}, \forall y \in \Re^m. \tag{16}$$

Here we assume that A is not necessarily asymptotically stable. We also assume that the system (A, B) is linearly stabilizable. In other words, it is assumed that, without saturation, the system would be stabilizable.

Hence there exists at least one matrix K such that

$$\dot{x}(t) = Ax(t) - BKx(t) = (A - BK)x(t)$$

is asymptotically stable. Actually it is possible to select the location of the system eigenvalues (i.e., the eigenvalues of *A-BK*) arbitrarily. Hence we assume that matrix *K* has been selected so as to place the system eigenvalues in the desired location. Since $\tilde{A} = A - BK$ is Hurwitz, for every positive definite matrix $\tilde{Q}$, there exists an unique $P \in \Re^{n\times n}$ satisfying

$$\tilde{A}^T P + P\tilde{A} = -\tilde{Q},$$

and $P > 0$. Our goal is first to find an inner approximation $\Omega(P)$ of the controllable set Ω^* of our system (13) and (14) based on the quadratic Lyapunov function $V(\xi) = \xi^T P \xi$, and then to maximize the approximate controllable set $\Omega(P)$ by varying the approximation parameter *P* in such a way that the resulting matrix $\tilde{Q} = -(\tilde{A}^T P + P\tilde{A})$ remains positive definite.

We denote the *i*-th row of matrix *K* by $k_i, i = 1,..., m$:

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}.$$

We now consider the case of a single input. Define

$$f(\xi) = A\xi - Bsat(K\xi) \tag{17}$$

$$= \begin{cases} (A - BK)\xi, & \text{if } \xi \in H_0 = \{\xi \in \Re^\pi \,\|\, K\xi| < 1\} \\ A\xi - B, & \text{if } \xi \in H_- = \{\xi \in \Re^\pi \mid K\xi \ge 1\} \\ A\xi + B, & \text{if } \xi \in H_+ = \{\xi \in \Re^\pi \mid K\xi \le -1\} \end{cases}. \tag{18}$$

Define $\tilde{V}(t) = V(x(t))$. Taking derivative of $\tilde{V}(t)$ along the trajectory $x(t)$, we obtain the following cases:

Case 1. $x(t) \in H_0$: unsaturated case, i.e., $u(t) = -Kx(t)$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= f(x(t))^T Px(t) + x(t)^T Pf(x(t)) \\ &= ((A - BK)x(t))^T Px(t) + x(t)^T P(A - BK)x(t) \\ &= x(t)^T ((A - BK)^T P + P(A - BK))x(t) \\ &= -x(t)^T \tilde{Q}x(t), \end{aligned} \tag{19}$$

where

$$\tilde{Q} \underline{\underline{\Delta}} -[(A - BK)^T P + P(A - BK)] = -(\tilde{A}^T P + PA). \tag{20}$$

Case 2. $x(t) \in H_+$: positively saturated case, i.e., $u(t) = 1$

$$\begin{aligned} \frac{d}{dt}\tilde{V}(t) &= f(x(t))^T Px(t) + x(t)^T Pf(x(t)) \\ &= (Ax(t) + B_+)^T Px(t) + x(t)^T P(Ax(t) + B_+) \\ &= x(t)^T (A^T P + PA)x(t) + B_+^T Px(t) + x(t)^T PB_+ \\ &= -x(t)^T Qx(t) + B_+^T Px(t) + x(t)^T PB_+, \end{aligned} \tag{21}$$

where

$$Q \underline{\underline{\Delta}} -(A^T P + PA). \tag{22}$$

and

$$B_+ = B.$$

Case 3. $x(t) \in H_-$: negatively saturated case, i.e., $u(t) = -1$

$$\begin{aligned}\frac{d}{dt}\tilde{V}(t) &= f(x(t))^T Px(t) + x(t)^T Pf(x(t)) \\ &= (Ax(t) - B_-)^T Px(t) + x(t)^T P(Ax(t) - B_-) \\ &= x(t)^T (A^T P + PA)x(t) - B_-^T Px(t) - x(t)^T PB_- \\ &= -x(t)^T Qx(t) - B_-^T Px(t) - x(t)^T PB_-,\end{aligned} \tag{23}$$

where $\tilde{Q}$ is defined as in (20) and $B_- = -B$.

Inspired by the right-hand sides (19), (21) and (23) for $\frac{d}{dt}\tilde{V}(t)$, we define

$$\begin{aligned}g_0(\xi) &= -\xi\tilde{Q}\xi, & \xi \in H_0; \\ g_+(\xi) &= -\xi Q\xi + B^T P\xi + \xi^T PB, & \xi \in H_+; \\ g_-(\xi) &= -\xi Q\xi - B^T P\xi - \xi^T PB, & \xi \in H_-;\end{aligned}$$

Combining these three functions into one function, we obtain

$$g(\xi) = \begin{cases} g_0(\xi) & \text{if } \xi \in H_0, \\ g_+(\xi) & \text{if } \xi \in H_+, \\ g_-(\xi) & \text{if } \xi \in H_-, \end{cases} \tag{24}$$

Observe that

$$\frac{d}{dt}\tilde{V}(t) = g(x(t)) \tag{25}$$

We want to find the maximum level set $L(r) = \{\xi \in \Re^n : V(\xi) = \xi^T P\xi \le r\}$ of the Lyapunov function V that is contained in the descent region $R_g \triangleq \{\xi \in \Re^n : g(\xi) \le 0\}$ in which the time derivative of the Lyapunov function is negative, i.e.,

$$r^* = \max\{r \mid L(r) \subset R_g = \{\xi \in \Re^n : g(\xi) \le 0\}\}.$$

We note that, in Case 1 $\tilde{Q} > 0$ because P is selected so that $\tilde{Q} > 0$. In other words, because we use only those P that will make $\tilde{Q} = -(\tilde{A}^T P + P\tilde{A}) > 0$, the right-hand side for $\frac{d}{dt}\tilde{V}(t)$ is negative: $g_0(\xi) \le 0, \forall \xi \ne 0.$ Hence $g(\xi) < 0,$ $\forall \xi \in H_0 - \{0\}.$ Therefore, the equilibrium point is locally asymptotically stable in H_0. However, in Case 2 and Case 3, since the open-loop system may be unstable, matrix A may not be Hurwitz. Given a positive definite matrix P that will make $\tilde{Q} > 0$, the Q defined by (22) may or may not be positive definite.

In order to satisfy the Lyapunov descent condition $g(\xi) < 0$ for a given ξ, we require that for each $\xi \ne 0,$ there exists at least one control value ν satisfying $\|\nu\|_\infty \le 1$ and

$$g(\xi) = -\xi^T Q\xi \pm 2\xi^T PB\nu < 0.$$

Then the state space $\Re^n$ can be divided into the following regions:

(a) $R_0 = \left\{\xi \in \Re^n \,\middle|\, \xi^T \tilde{Q}\xi > 0\right\}$ If $\xi \in R_0$, then $g(\xi) < 0.$

(b) $R_+ = \left\{\xi \in \Re^n \,\middle|\, 2\xi^T PB < \xi^T \tilde{Q}\xi \le 0\right\}$ If $\xi \in R_+$, then set $\nu = 1$ so that $g(\xi) < 0.$

(c) $R_- = \left\{\xi \in \Re^n \,\middle|\, 2\xi^T PB > -\xi^T \tilde{Q}\xi \ge 0\right\}$ If $\xi \in R_-$, then set $\nu = -1$ so that $g(\xi) < 0.$

(d) $\Re^n - \{R_- \cup R_0 \cup R_+\}$. If $\xi \in \Re^n - \{R_- \cup R_0 \cup R_+\}$, then it is not possible to find $\nu \in [-1,1]$, such that $g(\xi) < 0.$

The approach for finding the maximal level set $L_P(c^*) = \left\{\xi \in \Re^n \,\middle|\, V(\xi) = \xi^T P\xi \le c^*\right\}$ which is contained in the union of the regions (a), (b) and (c), i.e.,

$$c^* = \max\left\{c \,\middle|\, L_P(c) \subset R_0 \cup R_+ \cup R_-\right\},$$

can be found by the following maximization problem

$$\begin{aligned} c^* = \quad & \min \quad V(\xi) = \xi^T P\xi \\ & \text{subject to} \quad g_+(\xi) = -\xi^T Q\xi + B^T P\xi + \xi^T PB \ge 0, \\ & \text{and} \quad K\xi + 1 \le 0. \end{aligned} \tag{26}$$

The above optimization problem with Lyapunov descent criterion can be solve by Kuhn-Tucker Theorem, see Wang and Chen [19].

We now apply SDP to the above optimization problem (26).

Since $P > 0$, we can find $\sqrt{P}$ such that $P = \sqrt{P}\sqrt{P}$. Therefore, $V(\xi)$ can be put in the format of (8). But Q may not be positive definite, and thus we may not be able to decompose Q into its square roots.

Note that the optimization problem (25) is exactly in the formulation of QCQP as in (7) and (8), except the inequality constraint. We now attempt to rewrite the above optimization problem in the QCQP format (7) and (8)

$$\begin{aligned} & \min \quad t \\ & \text{subject to} \quad x^T Px \le t, \\ & \qquad x^T Qx - 2B^T Px \le 0, \\ & \qquad K\xi + 1 \le 0. \end{aligned}$$

We note that Q is indefinite, we need to apply the non-convex optimization technique to the optimization problem (7) and (8) to the form of (11) and (12).

Recall that a non-convex optimization problem of the form,

$$\begin{aligned} & \min \quad f_0(x) \\ & \text{subject to} \quad f_i(x) \le 0, \quad i = 1,2,\cdots,k, \end{aligned}$$

where $f_i(t) = x^T A_i x + 2b_i^T x + c_i, i = 0,1,\cdots,k.$ The matrices A_i can be indefinite. The lower bounds for the minimum value of $f_0(x)$ for the above optimization problem can be obtained by solving the semidefinite programming (with variables t and τ_i) in the following SDP formulation:

$$\begin{aligned} &\max \quad t \\ &\text{subject to} \quad \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \\ &\qquad + \tau_2 \begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \succeq 0, \\ &\qquad \tau_i \geq 0, \quad i = 1, 2. \end{aligned} \tag{27}$$

Therefore, we rewrite the optimization problem (26) into the form

$$\begin{aligned} &\max \quad t \\ &\text{subject to} \quad \begin{bmatrix} P & 0 \\ 0 & -t \end{bmatrix} + \tau_1 \begin{bmatrix} Q & -PB \\ -B^T P & 0 \end{bmatrix} \\ &\qquad + \tau_2 \begin{bmatrix} 0 & \frac{1}{2} K^T \\ \frac{1}{2} K & 1 \end{bmatrix} \succeq 0. \end{aligned} \tag{28}$$

VI. EXAMPLE

We first show a famous example of a double integrator (see, Athans, et al. [3], Kirk [8]), a single input plant by applying three techniques for finding the controllable set of the linear time-invariant open-loop unstable system. And then we conclude this paper by applying the techniques to a 3D submarine case.

Example 1 Consider the double integrator, a single input plant of the form

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ -1 \leq u \leq 1. \end{cases}$$

Here, we note that the eigenvalues of the open-loop systems are found to be 0, 0. Since there are two open-loop zeros for the system, the system is open-loop unstable. Suppose that the desired eigenvalues $\lambda_1' = -\sigma + i\omega$ and $\lambda_2' = -\sigma - i\omega$ for the closed-loop system are as follows:

(i) $\sigma_1 = 1, \omega_1 = 1.$

(ii) $\sigma_2 = 2, \omega_2 = 1.$

For a given $\tilde{Q}$, where

$$\tilde{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the feedback gain K_i can be selected by the standard pole placement technique and $P_i, i = 1,2$ can be found, respectively, as follows:

(i) $K_1 = \begin{bmatrix} 4 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 0.375 \end{bmatrix}.$

The eigenvalues of P are as 1.3164 and 0.3086.

(ii) $K_2 = \begin{bmatrix} 7 & 3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.15 & 0.1 \\ 0.1 & 0.15 \end{bmatrix}.$

The eigenvalues of P are as 1.1943 and 0.1937.

Technique 1 Optimize the set inside the linear region.

(Based on the following proposition in [9])

Proposition *Consider the system (13) under feedback (14). Suppose the following are true:*

1. *The open-loop matrix A is unstable;*
2. *The closed-loop matrix $\tilde{A} = A - BK$ is asymptotically stable.*

Let $\tilde{Q} > 0$. Let $P > 0$ be a solution to $\tilde{A}^T P + P\tilde{A} = -\tilde{Q}$. Let $L(r) = \{\xi \in \Re^n \mid \xi^T P \xi \leq r\}$ be a level set, where $r \in \Re_+$, and let $H_0 = \{\xi \in \Re^n \mid |K\xi| \leq 1\}$ be defined as the unsaturated region. Let $\tilde{r}$ be the largest number such that $L(\tilde{r}) \subset H_0$. Then, $L(\tilde{r})$ is an asymptotically stable region.)

According to the above proposition, we have the following optimization problem of the form:

$$\begin{aligned} &\min \quad V(\xi) = \xi^T P \xi \\ &\text{subject to} \quad |K\xi| \geq 1 \end{aligned}$$

Technique 2 Optimize the set inside the Lyapunov descent region. (as stated in (26))

$$\begin{aligned} &\min \quad V(\xi) = \xi^T P \xi \\ &\text{subject to} \quad g_+(\xi) = -\xi^T Q \xi + B^T P \xi + \xi^T PB \geq 0, \\ &\text{and} \quad K\xi + 1 \leq 0. \end{aligned}$$

Technique 3 Approximate the set by semidefinite programming. (as stated in (28))

$$\begin{aligned} &\max \quad t \\ &\text{subject to} \quad \begin{bmatrix} P & 0 \\ 0 & -t \end{bmatrix} + \tau_1 \begin{bmatrix} Q & -PB \\ -B^T P & 0 \end{bmatrix} \\ &\qquad + \tau_2 \begin{bmatrix} 0 & \frac{1}{2} K^T \\ \frac{1}{2} K & 1 \end{bmatrix} \succeq 0. \end{aligned}$$

Figure 1 shows the sets of three approximations for the controllable set of case (i) in example 1. The smallest ellipse is the controllable set by applying technique 1, in which the level is found as $r_1 = 0.0774$. The outer ellipse is the Lyapunov controllable set approximated by Technique 2, in which the level is found as $r_2 = 0.1678$, while the ellipse in between is the controllable set obtained from the non-convex optimization technique of SDP (Technique 3), in which the level is found as $t = 0.1064$.

Figure 2 shows the sets of three approximations for the controllable set of case (ii). The smallest ellipse is the controllable set by applying technique 1, in which the level is found as $r_1 = 0.012$. The outer ellipse is the Lyapunov controllable set approximated by Technique 2, in which the level is found as $r_2 = 0.0412$, while the ellipse in between is the controllable set obtained from the non-convex optimization technique of SDP (Technique 3), in which the level is found as $t = 0.0276$.

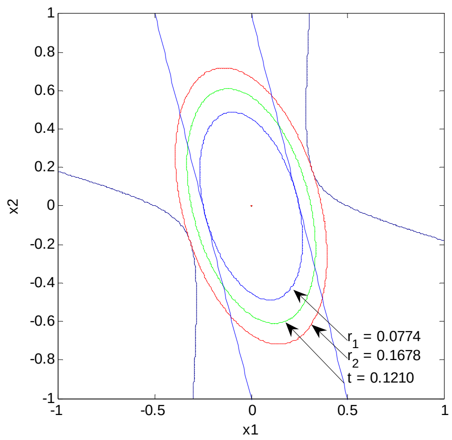

Figure 1 Comparison of the areas of the three approximations for case (i) in Example 1

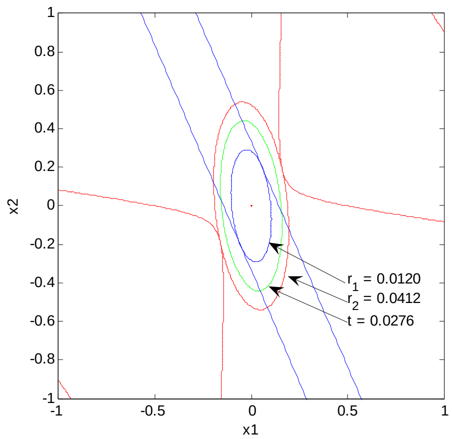

Figure 2 Comparison of the areas of the three approximations for case (ii) in Example 1

For both cases, as for the command usage, it takes 21, 40, and 14 commands for the first, second, and third technique, respectively. And the execution times for SDP is very effective It is obviously that it is very efficient to find the controllable set by using SDP. However, the area of the controllable sets found by the second technique is the largest among all the three techniques.

We now apply the three techniques to the following real case from [12], except that our objective is to find the 3D controllable sets for the model instead of their objective of finding the optimal control.

Example 2 Consider the following linearized model of a submarine obtained from Kockumation AB, Malmö, Sweden [12]:

$$\dot{x} = Ax + bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -0.005 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0.005 \end{bmatrix},$$

with the control constraint

$$|u| \le 0.005.$$

Here we note that the eigenvalues of the open-loop system are found as 0, 0, -0.005. Since there are two open-loop zero eigenvalues for the system, the system is open-loop unstable. Suppose the desired eigenvalues of the closed-loop system are -0.0039, -0.0026 $\pm$0.0021i. Therefore, by the technique of eigenvalues placement, the feedback K is found as

$$K = [8.7126\times10^{-6} \quad 6.29\times10^{-3} \quad 8.2\times10^{-1}]$$

We choose $\tilde{Q}$ as

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the eigenvalues of P are found as

$$\lambda_1 = 4.3051\times10^{11},$$

$$\lambda_2 = 2.0608\times10^{6},$$

$$\lambda_1 = 5.4946\times10^{1}.$$

Hence P is positive definite.

Technique 1 Maximize the stable region inside the linear unsaturated region.

To find the stable region proposed by Lee and Hedrick [10], we formulate the following minimization problem:

$$\min \qquad V(\xi) = \xi^{T} P \xi$$

$$\text{subject to} \quad |K\xi| \ge 0.005.$$

The above optimization problem yields the level $r_1^* = 1.0679\mathrm{e}+007$.

Figure 3 shows the 3D ellipsoidal controllable set approximated by Technique 1 without showing the linear unsaturated region. Figure 4 shows the 3D linear unsaturated region and the controllable set fit inside the linear unsaturated region. It is clear that the ellipsoid is bounded by the linear unsaturated region and tangent to the linear unsaturated region.

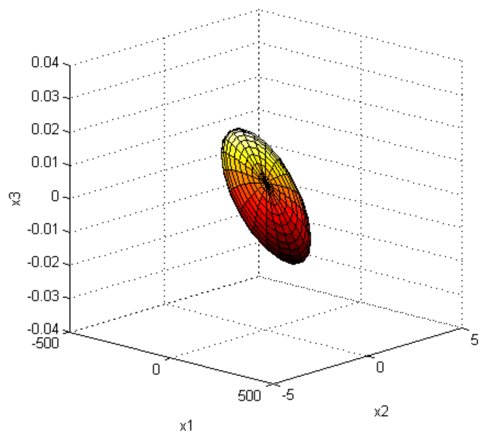

Figure 3 3D ellipsoidal controllable set approximated by Technique 1

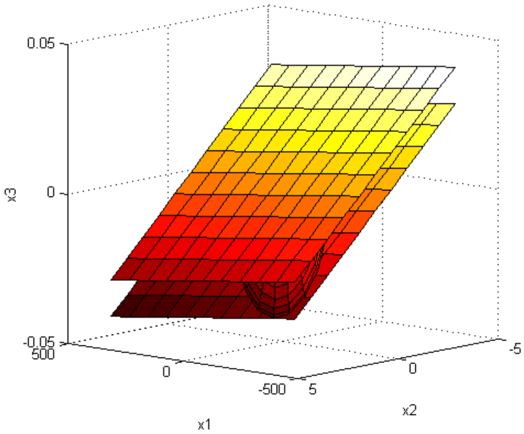

Figure 4 Linear unsaturated region and 3D controllable set approximated by technique 1

Technique 2 Maximize the stable region under Lyapunov descent criterion

To find the stable region under Lyapunov descent criterion, we formulate the following minimization problem:

$$\min \quad V(\xi) = \xi^{\mathrm{T}} P \xi$$

$$\text{subject to} \quad g_{+}(\xi) = \xi^{T}(A^{T}P + PA)\xi - 2\times 0.005 \times B^{T} P \xi \geq 0,$$

$$K\xi \leq -0.005.$$

The above optimization problem yields the level $r_2^* = 1.1115\mathrm{e}+007$, which is about 4% more than that of the level found by Technique 1.

Figure 5 shows the Lyapunov controllable set for approximated by Technique 2 without showing the linear unsaturated region. Figure 6 shows the linear unsaturated region and the Lyapunov controllable set. We note that a portion of the ellipsoid breaking through the linear unsaturated region indicates that indeed the Lyapunov controllable set extends beyond the linear unsaturated region, i.e., the Lyapunov controllable set is larger than the controllable set found inside the linear unsaturated region through Technique 1.

Technique 3 Approximate the set by semidefinite programming.

The optimization problem in the LMI form can be written as:

$$\min \quad t$$

$$\text{subject to} \quad x^{\mathrm{T}} P x \leq t,$$

$$x^{T} Q x - 2\times 0.005 \times B^{T} P x \leq 0,$$

$$K\xi + 0.005 \leq 0.$$

The lower bound of the optimization problem can be obtained by the following SDP formulation:

$$\max \quad t$$

$$\text{subject to} \quad \begin{bmatrix} P & 0 \\ 0 & -t \end{bmatrix} + \tau_1 \begin{bmatrix} Q & -0.005\times PB \\ -0.005\times B^{T}P & 0 \end{bmatrix}$$

$$+\tau_2 \begin{bmatrix} 0 & \frac{1}{2}K^{T} \\ \frac{1}{2}K & 0.005 \end{bmatrix} \succeq 0.$$

The optimal objective value t^* is found as $t^* = 1.1033\times 10^{7}$, which is about 1% less than that of the level found by Technique 2.

Figure 7 shows the controllable set for approximated by SDP without showing the linear unsaturated region. Figure 8 shows the linear unsaturated region and the controllable set approximated by SDP.

We note that even though the commands usage for SDP is only about half of the commands written for the Lagrangian technique of Technique 2, and the execution time for the former is much shorter than that of the latter, but the controllable set approximated by SDP is slightly smaller than the Lyapunov controllable set (about 1%).

VII. Conclusion

In this paper, we applied SDP to the problems of approximating the controllable set for the open-loop unstable system with input saturation. Our examples showed that there is a limitation in applying the SDP to the problem of approximating the Lyapunov controllable set. Therefore, the alternative non-convex technique of the semidefinite programming was applied. Our examples showed that in the two-dimensional case, the level found by the non-convex optimization technique was about 30% smaller than the one for the Lyapunov controllable set and was about 40% more than the one for the controllable set inside the linear unsaturated region; however, the command usage and executing time for the non-convex optimization technique by SDP were far superior to those of the conventional way of finding the Lyapunov controllable set using the Lagrangian technique. In the 3-dimensional submarine example, the level of the inner approximation of the controllable set approximated by SDP was about 1% smaller than the Lyapunov controllable set but 3% more than the level set inside the linear unsaturated region. On the other hand, the command usage for the non-convex optimization technique by SDP was only half of the command usage for the Lyapunov controllable set. Furthermore, the execution time for the former was far faster than the latter.

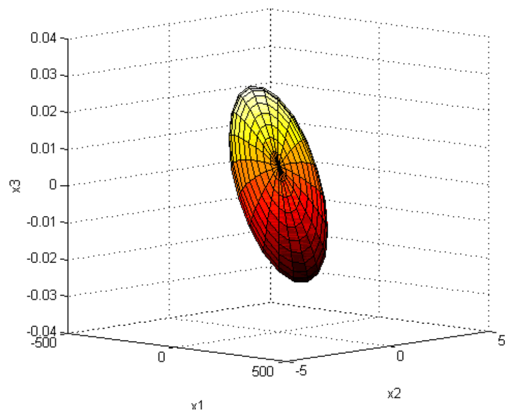

Figure 5 3D ellipsoidal controllable set approximated by Technique 2

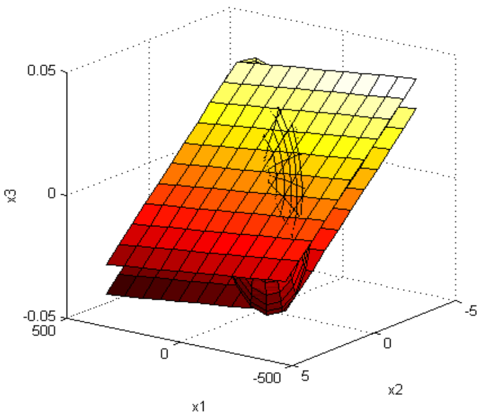

Figure 6 Linear unsaturated region and 3D controllable set approximated by Technique 2

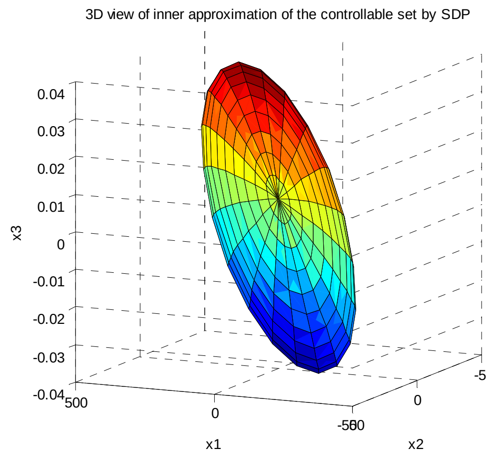

Figure 7 3D controllable set approximated by SDP

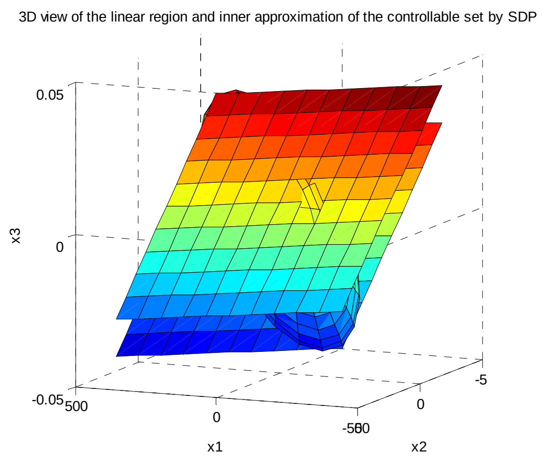

Figure 8 Linear unsaturated region and 3D controllable set approximated by SDP

REFERENCES

[1] Alizadeh, F. "Interior Point Methods in Semidefinite Programming with Applications to Combinatorial Optimization" *SIAM J. Optimization*, Vol. 5, No. 1, pp. 13-51, February 1995.

[2] Alizadeh, F., Haberly, A., Nayakkankuppam, M., Overton, M. and Schmieta, S. *SDPPACK: User's Guide*, June, 1997.

[3] Athans, M. and Falb, P.L. Optimal Control, An Introduction to the Theory and Its Applications, McGraw-Hill, New York, U.S.A. (1966).

[4] Balakrishnan, V., and Vandenberghe, L. "Semidefinite programming duality and linear time-invariant systems," *IEEE Transactions on Automatic Control*, Vol. 48, No. 1, January, 2003, pp. 30-41

[5] Bellman R. and Fan, K. "On systems of linear inequalities in Hermitian matrix variables," *Proceedings of Symposia in Pure mathematics*, Vol. 7, AMS, 1963.

[6] Boyd, S., Ghaoui, L.E., Feron E. and Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory*, Studies in Applied Mathematics 15 (SIAM, Philadelphia, PA, June, 1994).

[7] Fujisawa, K., Kojima, M., and Nakata, K. *SDPA User's Manual*, May, 1998.

[8] Goemans, M. "Semidefinite Programming in Combinatorial Optimization," *Mathematical Programming*, Vol. 79, 1997, pp. 143-161.

[9] Kirk, D.E. *Optimal Control Theory—An Introduction*, Englewood Cliffs, Prentice-Hall, NJ, U.S.A. (1970)

[10] Lee A.W. and Hedrick, J.K. "Some new results on closed-loop stability in the presence of control saturation," *INT. J. CONTROL*, 1995, Vol. 62, No. 3, pp. 619-631.

[11] Nesterov Y.E. and Nemirovski, A.S. *Interior Point Polynomial Algorithm in Convex Programming*, SIAM, Philadelphia, PA, 1994.

[12] Per-Olof Gutman and P. Hagander, "A new design of constrained controllers for linear systems," IEEE Transactions on Automatic Control, AC-30(1), 1985.

[13] Potra, F. Sheng, R. and Brixius, N. *SDPHA User Guide*, August, 1998.

[14] Stefanatos, D., and Khaneja, N., "Semidefinite programming and reachable sets of dissipative bilinear control systems," *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC '05*, 2005, pp. 2811-2816.

[15] Toh, K.C. Todd, M.J. and Tutuncu, R.H. *SDPT3 - A Matlab Software Package for Semidefinite Programming*, 1998.

[16] Vandenberghe, L. and Boyd, S. *Software for Semidefinite Programming: User's Guide*, November 1994.

[17] Vandenberghe, L. and Boyd, S. "Semidefinite Programming," *SIAM Review*, March 1996.

[18] Vandenberghe, L. and Boyd, S. "Applications of semidefinite programming," *Applied Numerical Mathematics*, Vol. 29, 1999, pp. 283-299.

[19] Wang, A., and Chen, Y.M., "3-D Controllable Set of Linear Time-Invariant Open-Loop Unstable Systems with Constrained Input -- A Submarine Case" , *Engineering Letters*, Vol. 16, Issue 2, pp. 185-192.

[20] Wright, S.J. *Primal-Dual Interior-Point Methods*, Argonne National Laboratory, SIAM 1997.

[21] Yao, D.D., Zhang, S., and Zhou, X.Y., "A primal-dual semidefinite programming approach to linear quadratic control," *Transactions on Automatic Control*, Vol. 46, No. 9, September, 2001, pp. 1442-1447.

Dr. Abraham W. L. Wang earned his M. S. and Ph. D. degrees in 1994 and 2000, respectively, both from Washington University in St. Louis, St. Louis, MO, U. S. A. in Systems Science and Mathematics. In 1998, he also received his M. B. A. degree from Fontbonne University with an honorary degree. He earned his B. S. degree in Electrical Engineering from Ohio Northern University, Ada, Ohio, U. S. A. in 1992. He is a member of IAENG. His research interests are in the areas of control theory, optimization theory, multi-criteria decision-making analysis, applications of service quality, and forecasting.

Dr. Yenming J Chen received his B.S. degree in Electrical Engineering and Computer Science from Ming-Chi Institute of Technology and Tamkang University, Taiwan, in 1986 and 1991, respectively. He received the M.Sc. and Ph.D. degrees in 1994 and 1998, respectively, both from Washington University in Systems Science and Mathematics. Dr. Chen's research interests are primarily in the areas of control theory, optimization, and software architecture, with applications to supply chain and marketing behavior study.