

Stochastic Recovery Of Sparse Signals From Random Measurements

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Abstract—Sparse signal recovery from a small number of random measurements is a well known NP-hard to solve combinatorial optimization problem, with important applications in signal and image processing. The standard approach to the sparse signal recovery problem is based on the basis pursuit method. This approach requires the solution of a large convex optimization problem, and therefore suffers from high computational complexity. Here, we discuss a stochastic optimization method, as a low-complexity alternative to the basis pursuit approach.

Keywords: sparse signal processing, random measurements, threshold accepting method

1 Introduction

There has been an increasing interest in the problem of sparse signal recovery from random measurements, which is strongly motivated by the recently introduced framework of Compressive Sensing (CS) (see [1]-[7] and the references within). The main idea of CS is that if the signal is compressible, then a small number of random measurements contain sufficient information for its approximate or exact recovery. CS has promising applications in signal and image processing, and it could potentially lead to interesting models for various interactions performed at the biological level [8].

The problem of sparse signal recovery from random projections requires the minimization of the ℓ_0 norm of the candidate solution. This is generally impossible to solve, since it requires an intractable combinatorial search. A common alternative is to consider the convex problem, known as Basis Pursuit (BP), which requires the minimization of the ℓ_1 norm, as a sparsity-promoting functional. Here, we discuss a Stochastic Optimization (SO) method, which provides a low-complexity alternative to the standard Basis Pursuit (BP) approach used in CS [1]-[7]. The considered SO method has the advantage of a very easy implementation, comparing to the highly complex BP approach. The objective function of the SO method is also different, and it is defined as the product between the ℓ_1 norm and the *spectral entropy* of the candidate solution. This definition of the objective function

improves the signal reconstruction, since it is equivalent with a weighted ℓ_1 norm functional, where the weights correspond to the *self-information* of each component of the signal. Thus, by using such an objective function, the SO minimization method focuses on entries where the weights are small, and which by definition correspond to the non-zero components of the sparse signal.

2 Compressive Sensing Framework

Let us give a short description of the CS framework [1]-[7]. Assume that we acquire a discrete signal:

$$z = [z_1, \dots, z_M]^T \in \mathbb{R}^M, \quad (1)$$

and

$$\Psi = \left\{ \psi^{(m)} \mid \psi^{(m)} \in \mathbb{R}^M, m = 1, \dots, M \right\}, \quad (2)$$

is an orthonormal basis of vectors spanning \mathbb{R}^M (here T stands for transposition of vectors and matrices). We denote by:

$$\hat{\Psi} = [\psi^{(1)} \dots \psi^{(M)}], \quad (3)$$

the matrix with the columns given by the basis vectors. Obviously, the matrix $\hat{\Psi}$ corresponds to a unitary transformation, i.e. it satisfies the orthogonality condition:

$$\hat{\Psi}^T \hat{\Psi} = \hat{\Psi} \hat{\Psi}^T = \hat{I}_M, \quad (4)$$

where \hat{I}_M is the $M \times M$ identity matrix. We say that Ψ provides a sparse representation of z , if z is well approximated by a linear combination of a small set of vectors from Ψ , i.e. there exists a set of indices $\{m_1, \dots, m_K\} \subset \{1, \dots, M\}$, for small $K \ll M$, such that:

$$z = \sum_{k=1}^K x_{m_k} \psi^{(m_k)}, \quad x_{m_k} \neq 0. \quad (5)$$

Therefore:

$$\hat{\Psi}^T z = x = [x_1, \dots, x_M]^T \in \mathbb{R}^M, \quad (6)$$

is a sparse signal, representing the acquired signal $z \in \mathbb{R}^M$ in the basis Ψ . Reciprocally, by knowing x one can easily synthesize z as following:

$$z = \hat{\Psi} x. \quad (7)$$

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Now, let us consider a set of vectors:

$$\Phi = \left\{ \varphi^{(n)} \mid \varphi^{(n)} \in \mathbb{R}^M, n = 1, \dots, N \right\}, \quad (8)$$

and the corresponding matrix:

$$\hat{\Phi} = [\varphi^{(1)} \mid \dots \mid \varphi^{(N)}], \quad (9)$$

such that $N \leq M$. We use these vectors to collect N measurements of the sparse signal x :

$$\hat{\Phi}^T x = y = [y_1, \dots, y_N]^T \in \mathbb{R}^N, \quad (10)$$

such that:

$$y_n = \langle \varphi^{(n)}, x \rangle = \sum_{m=1}^M \varphi_m^{(n)} x_m, \quad n = 1, \dots, N. \quad (11)$$

The CS theory shows that it is possible to construct a set of vectors Φ , such that the measurements y preserve the essential information about the sparse signal x , and therefore the sparse signal x can be recovered from the measurements y . More specifically, the CS theory shows that the matrices $\hat{\Psi}$ and $\hat{\Phi}$ must satisfy the *restricted isometry condition* [1]-[7]:

$$(1 - \delta_z) \|z\|_2 \leq \left\| \hat{\Phi}^T \hat{\Psi}^T z \right\|_2 \leq (1 + \delta_z) \|z\|_2, \quad (12)$$

where $\|\cdot\|_2$ denotes the ℓ_2 (Euclidean) norm. The restricted isometry constant δ_z is defined as the smallest constant for which this property holds for all K -sparse vectors $z \in \mathbb{R}^M$ in the basis Ψ . The matrices $\hat{\Phi}$ and $\hat{\Psi}$ must also satisfy the *incoherence condition*, which requires that the rows of $\hat{\Phi}$ cannot sparsely represent the columns of $\hat{\Psi}$ (and vice versa) [1]-[7]. It has been shown that one can generate such a measurement matrix $\hat{\Phi}$ with high probability, if the elements of the matrix are drawn independently from certain probability distributions, such as the Gaussian distribution or the Bernoulli distribution [1]-[7]. This is a consequence of the fact that in high dimensions the probability mass of certain random variables concentrates strongly around their expectation. Also, recent theoretic considerations have shown that in order to achieve the restricted isometry condition, any $M \times N$ matrix $\hat{\Phi}$ must have at least $N \simeq cK \leq M$ columns, $c = c(M, K) \sim \log(M/K) > 1$, in order for the observation $y \in \mathbb{R}^N$ to allow an accurate reconstruction of x [1]-[7].

Searching for the sparsest x that matches y , subject to the measurements Φ , leads to the ℓ_0 optimization problem:

$$x = \arg \min_{x \in \mathbb{R}^M} \|x\|_0, \quad s.t. \hat{\Phi}^T x = y. \quad (13)$$

Here,

$$\|x\|_0 = \sum_{m=1}^M [1 - \delta(x_m, 0)], \quad (14)$$

is the ℓ_0 norm, measuring the number of nonzero coefficients in the vector x , and

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}, \quad (15)$$

is Dirac's function. Unfortunately, this problem is known to be an NP-hard combinatorial optimization problem, requiring the enumeration of all possible collections of columns in the matrix $\hat{\Phi}^T$ and searching for the smallest collection which best approximates the signal y [1]-[7]. The standard approach to the sparse recovery problem is based on the convexification of the objective function, obtained by replacing the ℓ_0 norm with the ℓ_1 norm:

$$\|x\|_1 = \sum_{m=1}^M |x_m|. \quad (16)$$

The resulting optimization problem:

$$x = \arg \min_{x \in \mathbb{R}^M} \|x\|_1, \quad s.t. \hat{\Phi}^T x = y. \quad (17)$$

is known as BP, and it can be solved using linear programming techniques whose computational complexities are polynomial [1]-[7]. However, in most real applications the BP approach requires the solution of a very large convex, non-quadratic optimization problem, and therefore suffers from high computational complexity [1]-[7].

A summarizing diagram of the CS framework is given in Figure 1. During the encoding process the signal $z \in \mathbb{R}^M$ is first transformed into a sparse signal $x \in \mathbb{R}^M$, using the $M \times M$ orthogonal transform $\hat{\Psi}^T$ (a typical situation corresponds to Fourier and wavelet representations of natural images). In the next step the sparse signal $x \in \mathbb{R}^M$ is compressed into $y \in \mathbb{R}^N$, using the $N \times M$ random projection matrix $\hat{\Phi}^T$, with $N < M$. The decoding process requires only the compressed vector $y \in \mathbb{R}^N$ and the matrices $\hat{\Phi}$ and $\hat{\Psi}$, and consists also in two steps. In the first step, the sparse signal $x \in \mathbb{R}^M$ is recovered by solving the ℓ_1 minimization problem. In the second step the original signal $z \in \mathbb{R}^M$ is synthesized from x , using the orthogonal transform $\hat{\Psi}$.

3 Stochastic Optimization Approach

Our SO approach is based on the observation formulated in the following theorem:

Theorem: The solution x of the BP problem has the following form:

$$x = x' + \xi, \quad (18)$$

where x' is a solution of the underdetermined linear system:

$$\hat{\Phi}^T x' = y, \quad (19)$$

and ξ is an unknown vector from the null subspace of the measurement matrix $\hat{\Phi}^T$.

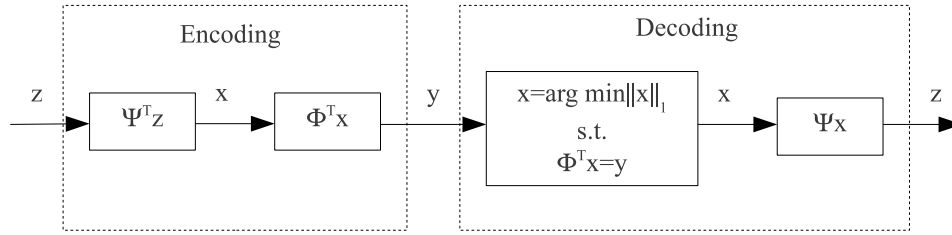


Figure 1: Compressive sensing framework.

Proof: Let us assume that ξ is a vector from the null subspace of the measurement matrix $\hat{\Phi}^T$, i.e. it can be written as a linear combination:

$$\xi = \sum_{m=1}^M a_m q^{(m)}, \quad (20)$$

where $a_m \in \mathbb{R}$, and $q^{(m)}$ are the columns of the null subspace projection operator:

$$\hat{Q} = [q^{(1)} | \dots | q^{(m)}] = \hat{I}_M - \hat{\Phi} \left(\hat{\Phi}^T \hat{\Phi} \right)^{-1} \hat{\Phi}^T. \quad (21)$$

Obviously we have:

$$\hat{\Phi}^T \hat{Q} = 0 \Leftrightarrow \hat{\Phi}^T q^{(m)} = 0, \quad m = 1, \dots, M, \quad (22)$$

and consequently:

$$\hat{\Phi}^T x = \hat{\Phi}^T (x' + \xi) = \hat{\Phi}^T x' = y. \quad (23)$$

Thus, x' must be a solution of the underdetermined linear system: $\hat{\Phi}^T x' = y$.

Reciprocally, let us assume that x' is a solution of the underdetermined linear system: $\hat{\Phi}^T x' = y$. Thus, since $x = x' + \xi$ is the solution of the BP problem, then it also satisfies the constraint $\hat{\Phi}^T x = y$, which implies $\hat{\Phi}^T \xi = 0$, and therefore ξ must be a vector from the null subspace of $\hat{\Phi}^T$ ■.

A good initial solution (with minimum ℓ_2 norm) of the underdetermined linear system $\hat{\Phi}^T x' = y$, is given by:

$$x' = \Phi^\dagger y, \quad (24)$$

where

$$\Phi^\dagger = \Phi \left(\hat{\Phi}^T \hat{\Phi} \right)^{-1}, \quad (25)$$

is the Moore-Penrose pseudo-inverse [9], since obviously we have:

$$\hat{\Phi}^T x' = \hat{\Phi}^T \Phi^\dagger y = \hat{\Phi}^T \Phi \left(\hat{\Phi}^T \hat{\Phi} \right)^{-1} y = \hat{I}_N y = y. \quad (26)$$

Unfortunately, it is hard to find the unique vector $\xi = x - x'$ from the null subspace, which corresponds to the unique BP solution x . This is where we use the stochastic search approach, in order to avoid the difficult to implement deterministic optimization methods.

Our approach is inspired by the Threshold Accepting (TA) method [10]. TA method is a deterministic analog to the well known Simulated Annealing (SA) method [11]. It is a refined search procedure which escapes local minima by accepting solutions which are not worse by more than a given threshold. The algorithm is deterministic in the sense that we fix a number of iterations and explore the neighborhood with a fixed number of random steps during each iteration. Analogously to the SA method, the threshold is decreased successively and approaches zero in the last iteration. The main difference between SA and TA is that for SA the threshold is modeled as a random variable, while for TA the threshold is deterministic. The TA algorithm has the advantage of an easy parametrization, it is robust to changes in problem characteristics and works well for many hard problems.

In our case, the initial solution $x \leftarrow x'$ is randomly perturbed in the null subspace of the matrix $\hat{\Phi}^T$, by adding a small random contribution of the column $q^{(m)}$, $m = 1, \dots, M$, of the matrix \hat{Q} :

$$\tilde{x} = x' + \sigma \alpha q^{(m)}. \quad (27)$$

Here $\alpha > 0$ gives the magnitude of the perturbation, and σ is a uniform binary random variable, which provides the sign of the perturbation, $\sigma \in \{\pm 1\}$. Obviously, at every random transition step, the new candidate solution \tilde{x} still satisfies the linear constraint:

$$\hat{\Phi}^T \tilde{x} = \hat{\Phi}^T \left(x' + \alpha \sigma q^{(m)} \right) = \hat{\Phi}^T x' = y, \quad (28)$$

since $\hat{\Phi}^T q^{(m)} = 0$. Thus, in the framework of ℓ_1 norm minimization, the new random candidate solution \tilde{x} is accepted ($x \leftarrow \tilde{x}$) if:

$$\|\tilde{x}\|_1 - \|x\|_1 \leq \theta, \quad (29)$$

and the process continues with a new perturbation.

We assume that the threshold parameter $\theta \geq 0$ and the magnitude of the perturbations $\alpha \geq 0$, follow an exponential schedule, decreasing by a fixed factor $\lambda \in (0, 1)$:

$$\theta \leftarrow \lambda \theta, \quad \alpha \leftarrow \lambda \alpha, \quad (30)$$

at each iteration. If θ_i and θ_f are the initial and, respectively the final values of the threshold, then we set $\lambda = (\theta_f / \theta_i)^{1/T}$, where T is the total number of iterations.

While this approach provides a relatively simple SO method for ℓ_1 norm minimization, it still can be improved by accelerating its convergence. This can be done by replacing the standard ℓ_1 norm with a weighted ℓ_1 norm, which penalizes the small components of the candidate solution. The weighted ℓ_1 norm can be derived using the concept of *spectral entropy*, as shown below.

We assume that $x = [x_1, \dots, x_M]^T$ is the *spectral decomposition* of some discrete signal $z \in \mathbb{R}^M$ in the basis Ψ :

$$z = \hat{\Psi}x = \sum_{m=1}^M x_m \psi^{(m)} \in \mathbb{R}^M, \quad (31)$$

Thus, $|x_m|$ represents the *spectral amplitude* of the component $\psi^{(m)}$, in the spectral decomposition of $z \in \mathbb{R}^M$.

We define the *spectral signature* of $z \in \mathbb{R}^M$ in the basis Ψ as following:

$$p(x, \Psi) = [p_1, \dots, p_M]^T, \quad (32)$$

where

$$p_m = \frac{|x_m|}{\sum_{i=1}^M |x_i|} = \frac{|x_m|}{\|x\|_1}, \quad m = 1, \dots, M. \quad (33)$$

We also define a probability measure P for x by:

$$P(x_m, \psi^{(m)}) = p_m, \quad (34)$$

which means that p_m is the probability that the spectral decomposition of $z \in \mathbb{R}^M$ in the basis Ψ , has the component $\psi^{(m)}$ with an amplitude $|x_m|$. Obviously, we have:

$$\sum_{m=1}^M p_m = 1, \quad (35)$$

and p is a probability distribution, associated with x .

Therefore, x can be modeled as a random variable in the probability space (\mathbb{R}^M, Ψ, P) . Also, x can be viewed as an *information source*, with its statistics defined by p .

Since $x \in (\mathbb{R}^M, \Psi, P)$ may be viewed as an information source, we can further define its *self-information* provided by the component $\psi^{(m)}$ as following [12, 13]:

$$h(x_m, \psi^{(m)}) = -\log_M P(x_m, \psi^{(m)}) > 0. \quad (36)$$

The average self-information, over all components of the basis Ψ , defines the *spectral entropy* of the source $x \in (\mathbb{R}^M, \Psi, P)$ [12, 13]:

$$H(x, \Psi) = \sum_{m=1}^M P(x_m, \psi^{(m)}) h(x_m, \psi^{(m)}). \quad (37)$$

In the case of $p_m = 0$, we consider $p_m \log_M p_m \equiv 0$, which is consistent with the limit: $\lim_{p \rightarrow 0^+} p \log_M p = 0$.

Obviously, we have $H(x, \Psi) \in [0, 1]$. A high value $H(x, \Psi) \simeq 1$ means that the source $x \in (\mathbb{R}^M, \Psi, P)$ is just noise, i.e. all the components $\psi^{(m)}$ have equiprobable amplitudes, while a smaller value $0 < H(x, \Psi) \ll 1$, means that the source has some intrinsic structure (or order), i.e. some components have a stronger amplitude than others. Equivalently, if $x \in \mathbb{R}^M$ is a sparse signal, then its entropy will be low, while if x is not sparse then its entropy will be high.

Now, let us consider the following functional, defined as the product between the spectral entropy and the ℓ_1 norm:

$$F(x, \Psi) = \|x\|_1 H(x, \Psi). \quad (38)$$

This functional corresponds to the weighted ℓ_1 norm of the vector $x \in \mathbb{R}^M$:

$$\|x\|_{w1} = F(x, \Psi) = \sum_{m=1}^M w_m |x_m|, \quad (39)$$

where the weights are given by the self-information of each component:

$$w_m = h(x_m, \psi^{(m)}). \quad (40)$$

By replacing the standard ℓ_1 norm, $\|x\|_1$, with its weighted version, $\|x\|_{w1}$, one penalizes the small components, x_m , of the candidate solution, since:

$$w_m = \lim_{|x_m| \rightarrow 0} \log_M \frac{\|x\|_1}{|x_m|} = \infty. \quad (41)$$

Thus, in the stochastic iterative process, the solution will concentrate around the small weights, which correspond to the non-zero components of the sparse signal.

In order to avoid the singularity in the weight estimation, we introduce a small parameter $0 < \varepsilon \ll 1/M$, such that:

$$w_m = \log_M \frac{\|x\|_1 + M\varepsilon}{|x_m| + \varepsilon} > 0. \quad (42)$$

Therefore, this analysis shows that from the point of view of solving the sparse signal recovery problem we need to find the source $x \in (\mathbb{R}^M, \Psi, P)$ which minimizes the weighted ℓ_1 norm, subject to a linear constraint system, imposed by the random measurements:

$$x = \arg \min_{x \in \mathbb{R}^M} \|x\|_{w1}, \quad s.t. \hat{\Phi}^T x = y. \quad (43)$$

Using the same threshold accepting approach as before, a new candidate solution \tilde{x} is accepted ($x \leftarrow \tilde{x}$) if:

$$\|\tilde{x}\|_{w1} - \|x\|_{w1} \leq \theta. \quad (44)$$

Thus, the pseudo-code of the proposed SO algorithm can be formulated as following:

```

# SO sparse signal recovery algorithm:
# Initialize the parameters
 $\theta_i, \theta_f, \alpha_i, T;$ 
# Compute  $\lambda$ 
 $\lambda = (\theta_f/\theta_i)^{1/T};$ 
# Initialize the solution
 $x \leftarrow \hat{\Phi}^\dagger y;$ 
 $F \leftarrow \|x\|_{w1};$ 
# Compute the null subspace projection operator
 $\hat{Q}_\perp \leftarrow \hat{I}_M - \hat{\Phi} (\hat{\Phi}^T \hat{\Phi})^{-1} \hat{\Phi}^T;$ 
# Set the initial parameters
 $\theta \leftarrow \theta_i; \alpha \leftarrow \alpha_i;$ 
FOR( $t = 1, \dots, T$ ) {
    FOR( $m = 1, \dots, M$ ) {
        #Compute a candidate solution
         $\sigma \leftarrow \text{sign}(\text{rnd}(1) - 0.5);$ 
         $\tilde{x} \leftarrow x + \sigma \alpha q^{(m)};$ 
         $\tilde{F} \leftarrow \|\tilde{x}\|_{w1};$ 
        # Test the solution
        IF( $\tilde{F} - F \leq \theta$ ) THEN { $x \leftarrow \tilde{x}; F \leftarrow \tilde{F};$ }
    }
    # Compute the new parameters
     $\theta \leftarrow \lambda \theta; \alpha \leftarrow \lambda \alpha;$ 
}
# Return the approximation of  $x$ 
RETURN  $x;$ 
    
```

4 Numerical Results

We have implemented the above SO algorithm in C on a Linux PC, using the GNU Scientific Library [14]. In Figure 2 we give several examples of sparse signal recovery using the SO method. The non-zero coefficients of the sparse signal $x \in \mathbb{R}^M$ are uniformly random generated in the interval $[-1, 1]$. Also, the elements of the measurement matrix $\varphi_m^{(n)}$ are drawn from the Gaussian distribution $\Gamma(0, 1)$. In these particular examples, the length of the sparse signal $x \in \mathbb{R}^M$ and the number of measurements were set to $M = 100$, and respectively $N = 50$. The initial and final thresholds were set to $\theta_i = 0.5$, and respectively $\theta_f = 10^{-5}$. Also we used $\alpha_i = 1$, and $\lambda = 0.95$, such that the number of iterations are $T = \ln(\theta_f/\theta_i)/\ln(\lambda) = 300$. The sparsity of the signal was varied as following: $K = 20, 25, 30$. One can see that by increasing K , and keeping the number of measurements constant, the recovery error

$$E = 100 \times \|x_{\text{recovered}} - x_{\text{original}}\| / \|x_{\text{original}}\| (\%) \quad (45)$$

deteriorates: $E = 8.532 \cdot 10^{-4}\%$ for $K = 20$; $E = 3.145\%$ for $K = 25$; $E = 11.329\%$ for $K = 30$. This result is expected, since the fixed number of measurements cannot hold enough information about the increasing number of non-zero elements. Also, this result suggests that there is a phase transition in the error function, depending on the ratio between the sparsity parameter K and the number of measurements N .

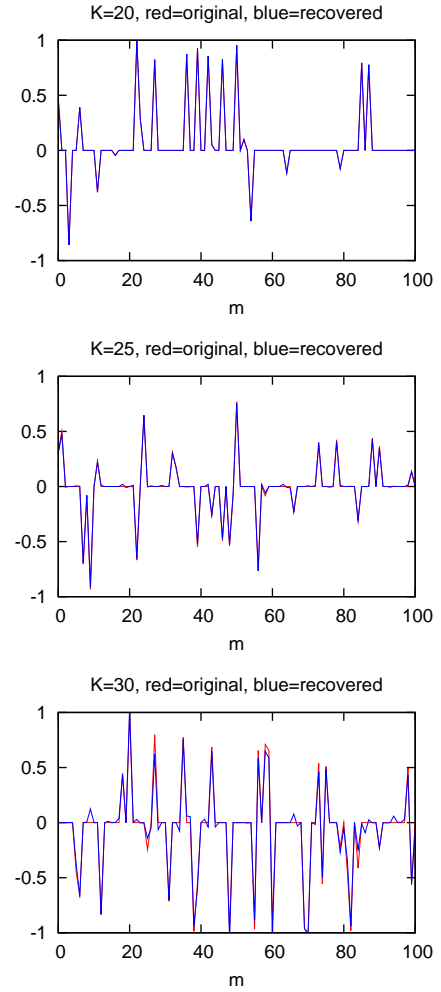


Figure 2: Examples of sparse signal recovery.

The phase transition is illustrated in Figure 3, where we give the relative recovery error $E = E(K, N)$, as a function of the sparsity K and the number of measurements N , obtained by averaging over 100 instances for each value of K and N . One can see that there is a large area (the dark color) where the method performs very well, and recovery is done with an error of less than 10%. Also, the contour lines, corresponding to a fixed error E , have a logarithmic dependence, similar to the ones obtained with the BP approach.

It is interesting to note that the measured vector y is used only once in the whole recovery process, to find the admissible initial solution, which satisfies the linear constraint system. After this initial step, the algorithm simply perturbs the solution such that the new candidate solution always satisfies the linear constraint system. This approach reduces drastically the search space from \mathbb{R}^M to \mathbb{R}^R , where $R = M - N$ is the rank of the null subspace operator \hat{Q} . As expected, by increasing the number of measurements N , the dimensionality R of the search space is reduced and the method will perform better.

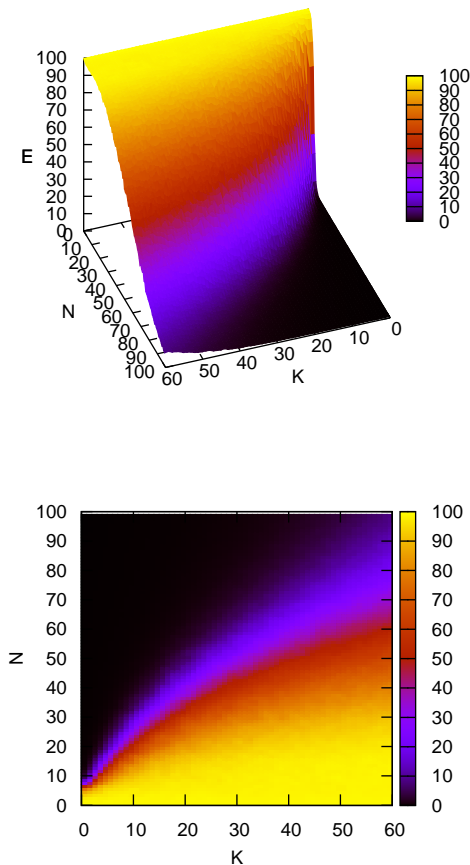


Figure 3: The phase transition in the average recovery error, $E = E(K, N)$.

5 Conclusion

We have presented a SO approach to the problem of sparse signal recovery encountered in the CS framework. The considered SO method has the advantage of a very easy implementation, comparing to the highly complex BP standard approach, used in CS framework. Thus, the proposed method is well suited for applications in which a simple implementation, with an approximate recovery performance of the original sparse signals, is acceptable. The objective function of the SO method is defined as the product between the ℓ_1 norm and the *spectral entropy* of the candidate solution, and it is equivalent with a weighted ℓ_1 norm functional, where the weights correspond to the *self-information* of each component of the signal. As a consequence of using such an objective function, the convergence of the SO minimization method is improved, since it focuses on entries where the weights are small, which by definition correspond to the non-zero components of the sparse signal.

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