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Monte Carlo Analysis of Risk Measures for Blackjack Type Optimal Stopping Problems

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***Abstract*—In the paper a blackjack type optimal stopping problem is considered. A decision maker (DM) observes sequentially the values of a finite sequence of nonnegative random variables. After each observation the DM decides whether to stop or to continue. If the DM decides to stop at a given moment they obtain a payoff dependent on the sum of already observed values. The greater the sum, the more the DM gains, unless the sum exceeds a given positive number. If so, the decision maker loses all or part of the payoff. It can be shown that under some elementary assumptions the optimal stopping rule (OSR) for such a problem has so-called threshold structure. However, even in such a case the relationship between the problem design parameters and the risk connected with the OSR cannot be examined via any formal method due to its very complex probabilistic nature. Thus in the paper it is proposed to make use of the Monte Carlo simulations combined with regression analysis to study this aspect of the OSR performance.**

***Index Terms*— Monte Carlo simulations, optimal stopping rules, risk analysis, sequential decision making.**

I. INTRODUCTION

OPTIMAL stopping problems form a class of optimization problems with a wide range of applications in mathematical statistics, engineering, industry, economics, and mathematical finance. The most interesting include e.g. the system-maintenance management [7], job-search and house-hunting problems [1],[6],[21], the pricing of perpetual American options as well as the optimal timing to invest in a project or capitalizing an asset, [2], [10],[12],[20],[23].

In this paper we consider a "blackjack type problem" (BTP). The BTP models a class of optimal stopping decision tasks in which a decision maker observes sequentially the values of a finite sequence $X_1, X_2,\ldots,X_N$ of nonnegative random variables. After each observation a decision maker (DM) decides whether to stop or to continue. If the DM decides to stop at the moment k he/she obtains a payoff dependent on the sum $X_1+\ldots+X_k$. The greater the sum, the more the DM gains, unless the sum exceeds a given number T – a limit given in the problem. If so, the DM loses all or part of the payoff. Such problems can represent various real world situations which can be observed in engineering, economics, finance or social life. To illustrate this class of optimal stopping problems consider three examples from totally different domains. The first example is *service with work time limit*. A DM controls a mechanism which should not work longer than a given time period T. He/she has several jobs to process in sequential order with each job requiring a random time for its execution. After each job the DM must decide whether to start next one or to stop. Every initiated job must be completed. The longer the mechanism works, the more the DM gains but if the work time exceeds the limit T the DM will be punished in some way.

The second example is a problem of *loading a device with a limit of load bearing capacity.* Many types of machines (trucks, cranes etc.) or other engineering structures (such as dams, roofs, bridges, computer servers) may be subjected to excessive overloads resulting in possibly breakage of the mechanism or structure. Assume a DM observes a process of loading of such a device. During the loading process the load is increased in random steps, as for example during a flood (a dam) or heavy snowfall (a roof). Assume that the limit of load bearing capacity of the device is given. After each observation the DM decides whether to stop (in order to prevent a dangerous overloading) or to continue the process of loading. The DM wants the device to bear as much load as possible. However on the other hand if the limit of the load bearing capacity is crossed than the gain for the DM is dramatically decreased.

The third problem is *blackjack type game*. One such game is played on a points system that gives numeric values to every card in a single deck of playing cards. The cards are given to a player sequentially until he/she decides to stop. The score is the sum of the values in hand. The player with the highest total score wins as long as it doesn't exceed a given limit number. If a player's cards exceed the limit then the player loses and his/her bet is taken by the dealer.

Typically in the theory of optimal stopping, see e.g. [4],[15], [22], the solution of any optimal stopping problem consists of the optimal stopping rule (OSR) and the value of the problem, i.e. the greatest expected payoff possible to achieve. A solution for BTP satisfying some general assumptions is given in [9]. It appears that under some elementary conditions, the OSR is of so-called threshold type. Such OSRs are especially interesting because of their very simple structure, [5],[10],[16],[24]. However, the dependence between the expected gain and the design parameters of the problem is rather complex. Even more complex is the relation between these parameters and various characteristics of the risk connected with a given OSR. Usually there is no analytical expression relating the design parameters of the decision problem to the corresponding risk characteristics of the decision rule. It is a common situation - in most optimal stopping problems, or more generally, sequential decision

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problems, the relation cannot be studied via formal-theoretical methods. In many problems where the relationship between some dependent and the independent variables is extremely complex or unknown the Monte Carlo simulations approach can be adopted, see e.g. [3], [11], [13], [14], [17], [25], [26]. However, the Monte Carlo methods allow us to solve a specific given problem rather than to obtain some general expressions describing the relation in which we are interested. Thus in order to obtain some more general results we propose combining the Monte Carlo method with the regression analysis. The latter helps us to estimate and express analytically the relationship between some risk characteristics of the OSR and the problem design parameters. Similar approach to study some optimal stopping problems may be found in [3],[11].

The paper is organized as follows. In Section II we formally state a general BTP and summarize some important results from the theory of optimal stopping. In Subsection II.C we present an important class of BTPs which can be effectively solved and which will be studied in details. Next, in Subsections II.D and II.E we describe the design parameters of the BTP and define considered risk characteristics. In Subsection III.A we present a Monte Carlo experiment which we use in order to generate data containing the necessary information about the relations which we want to study. In Subsection III.B, as a kind of benchmark, we make use of the proposed methodology to build a model for the expected payoff as a function of the problem design parameters and compare the result with the known theoretical formula. Finally, in Subsections III.D, E and F, we present and discuss regression models obtained for the considered risk characteristics.

II. Formal Statement of the Problem

The formal model for the class of problems we consider in this paper is the following. Let X_1 , X_2 , …, X_N be a finite sequence of random variables. A DM observes sequentially the values of the variables and decides whether to stop or to continue. If the process is stopped at the moment k the DM gains a value $W(y+\sum_{i=1}^{k} X_i)$, where W is a given real function and $y \geq 0$ is an initial state of the process. The function W is positive and nondecreasing on the interval $(0, T]$ and is nonincreasing for arguments greater than T. Such a problem will be called *blackjack type problem* (BTP) if the random variables are *nonnegative* and the payoff function W achieves its *only* maximum for $y+\sum_{i=1}^{k} X_i = T$.

Our task is to find a stopping rule which maximizes the expected payoff for a decision maker.

A. Optimal Stopping Theory – Necessary Definitions and Results

In order to present solutions of certain BTPs, we need some formal definitions and results from the theory of optimal stopping. They can be found e.g. in [4], [22].

Let X_1 , X_2 , … be a sequence of independent random variables. Let $\mathcal{F}_n$ denote the σ-algebra generated by the random variables X_1 , X_2 , …, X_n in an underlying probability space (Ω, $\mathcal{F}$,P) *A stopping rule* is a random variable τ with values in a set of natural numbers such that $\{\tau=n\}\in \mathcal{F}_n$ for n=1,2,… and P(τ<¶)=1. Let $\mathbf{M}(n)$ be a class of all stopping rules τ such that P($\tau\leq n$)=1.

Let $(Y_n,\mathcal{F}_n)$, n=1,2,… , be a homogenous Markov chain with values in a state space $(\Upsilon,\mathcal{B})$. Let W:R_+ØR be a Borel measurable function which values $W(y)$ will be interpreted as the gain for a DM when the chain $(Y_n,\mathcal{F}_n)$ is stopped at the state y. Assume that for a given state y and for a given stopping rule τ the expectation $E(W(Y_\tau)|Y_0=y)$ exists. Then it is natural to interpret the value - denoted by $E_yW(Y_\tau)$ - as the mean gain corresponding to a chosen stopping rule τ.

Let us define a function V_N by the equation:

$$V_N(y) = \sup_{\tau\in \mathbf{M}_W(N)} E_yW(Y_\tau) \tag{3}$$

where $\mathbf{M}_W(N)$ is a set of all stopping rules belonging to $\mathbf{M}(N)$ for which the expectations $E_yW(Y_\tau)$ exist for all $y\in \Upsilon$ and are larger than -¶. The value V_N (y) is called a *value* of the problem of optimal stopping when the *initial state* of the process is y and the *boundary* for the possible number of steps is N.

A stopping rule $\tau^*\in \mathbf{M}_W(N)$ which for all $y\in \Upsilon$ satisfies the condition

$$E_yW(Y_{\tau^*})= V_N(y) \tag{4}$$

is called an *optimal stopping rule*.

Let B denote a class of *all* Borel measurable functions W for which the expectations $E_yW(Y_1)$ exist for all $y\in \Upsilon$. Let us define an operator Q operating on functions $W \in B$ by $QW(y) = \max\{W(y), E_yW(Y_1)\}$. Consequantly, by definition, $Q^nW(y) = \max\{Q^{n-1}W(y) , E_yQ^{n-1}W(y)\}$.

The following theorem, which can be found in [4], provides us with the solution to the optimal stopping problem in the considered case.

Theorem Assume that $W \in B$. Then:

i. $V_n(y) = Q^nW(y)$, n=1,2,…

ii. $V_n(y) = \max\{W(y), E_yV_{n-1}(Y_1)\}$, where $V_0(y) = W(y)$

iii. A stopping rule τ_n^* defined by

$$\tau_n^* = \min\{ 1 \leq k \leq n : V_{n-k}(Y_k) = W(Y_k)\}$$

is an optimal stopping rule in a class $\mathbf{M}_W(n)$

If $E_y|W(Y_k)| <$ ¶ , for k=1,…,n , then the stopping rule τ_n^* is optimal in the class $\mathbf{M}(n)$.

It results from the Theorem that the DM should compare the gain resulting from "stopping" with the optimal expected gain resulting from "continuing" and should stop at the first moment when the both values are equal. However this general result, as well as other results from the optimal stopping theory, see e.g [4], [22], describe only the qualitative features of the solution but they do not provide us with any effective method for finding the OSR and the value V_N of the problem. As it was emphasized in the literature, see e.g. [15], from practical point of view it is always of interest to find both, OSR and V_N in an explicit form. One such result will be presented in the sequel.

B. Effective Solutions of some Classes of Blackjack Type Problems

Let us consider the BTP and let $Y_n = y + \sum_{i=1}^{n} X_i$, where $y \geq 0$. It is easy to see that the sequence $(Y_n, \mathcal{F}_n)$, $n=1,2,\ldots$, forms a Markov chain with the initial state of the process $Y_0=y$. Thus our BTP is a special case of the more general problem described in Subsection IIA. It follows from the Theorem that an optimal stopping rule $\tau_n^* \in \mathbf{M}_W(n)$ may be expressed in the following way:

$$\tau_n^* = \min\{1 \leq k \leq n : Y_k \in \mathbf{S}_k^*\} \tag{5}$$

with the stopping sets $\mathbf{S}_k^*$, $k=1,\ldots,n$ given by:

$$\mathbf{S}_k^* = \{y \in \mathrm{Y} : V_{n-k}(y) = W(y)\} \tag{6}$$

It appears that particularly interesting situation occurs when there exists a positive number t^* such that the following conditions hold:

$$W(y) < \mathrm{E}_y W(Y_1) \text{ for } 0 \leq y < t^* \tag{7}$$

and

$$W(y) \geq \mathrm{E}_y W(Y_1) \text{ for } y \geq t^* \tag{8}$$

It was proved in [9], that in any BTP the conditions (7) and (8) yilde the following inequalities:

$W(y) < E_y V_{n-1}(Y_1)$ for $0 \leq y < t^*$

and

$W(y) \geq E_y V_{n-1}(Y_1)$ for $y \geq t^*$, $n=2,\ldots,N$,

Consequently, in such a case the following Proposition results immediately from the Theorem.

Proposition. If there exists a real number t^*, $0< t^* <T$, such that the conditions (7) and (8) hold then the OSR for the BTP is given by formulae (5) and (6) with the *threshold* type stopping sets

$$\mathbf{S}_k^* = [t^*, \infty) \text{ , } k=1,\ldots,N\text{-}1 \tag{9}$$

The value $V_N(y)$ of the problem can be calculated for $y< t^*$ with the help of the following recursive equation:

$$V_n(y) = \int_0^{t^*-y} V_{n-1}(y+x) f(x)dx + \int_{t^*-y}^{\infty} W(y+x) f(x)dx$$

$$n=2,\ldots,N, \tag{10}$$

with the initial condition $V_1(y) = \int_0^{\infty} W(y+x) f(x)dx$. $\blacksquare$

The condition stated in the Proposition is fulfilled in many practically interesting problems, see [9]. One such problem will be considered in the next Subsection.

C. Blackjack Type Problem with Linear Payoff and Exponential Step.

The following important class of optimal stopping problems will be considered in details. The class can model various BTPs, especially those connected with the queuing theory, such as the *service with work time limit problem* described in the introduction.

Let us assume that the DM observe a sequence of i.i.d. random variables having an exponential distribution with the density function:

$$f(t) = \frac{1}{\lambda}\exp(-\frac{t}{\lambda})\mathbf{1}_{[0,\infty)}(t), \quad \lambda > 0 \tag{11}$$

So, intuitively we can think, that in this problem the DM approaches the limit T with exponential steps of an average length λ.

Let the payoff function W be given by the following equation:

$$W(y) = \begin{cases} B \cdot y & , \quad y \leq T \\ 0 & , \quad y > T \end{cases} \tag{12}$$

with $B > 0$.

According the formula (12) the DM obtains positive payoff which is proportional to the state y of the process, unless the state is greater than the limit T. If so, then the player gains 0.

It was shown in [8], that such a problem satisfies the conditions (7) and (8) with the threshold

$$t^* = T - \lambda \ln(1 + \frac{T}{\lambda}) \tag{13}$$

So, the OSR τ^* given by (5) and (9) tells us to continue the observation as long as the sum of the initial state and already observed values do not exceed the above given value t^*. Note, that the threshold t^* is independent of N and B. In order to compute its value we need to know only two parameters of the problem, T and λ.

D. The Design Parameters of the Problem

The design parameters of the problem are the following: the limit T, the step distribution parameter λ, the upper bound for the number of possible steps N, and the payoff function parameter B. Let us assume that the initial state y of the process equals 0 and let us confine ourselves to these situations where the value of the problem $V_N(0)$ is positive. It reflects the case where the optimal stopping rule tells the decision maker to make at least one observation. For simplicity, from now on the symbol V_N denotes $V_N(0)$

For a given problem design the value V_N can be computed with the help of the recursive equation (10). Usually the calculations are rather arduous but fortunately in the considered case they can be done by any symbolic manipulation software. All calculations and simulations presented in this paper were coded and performed in *Wolfram Mathematica.*

As an example we present here an analytical expression for the function relating the design parameters and the value V_{10}. We will make use of this expression in Subsection III.B where the theoretical values will be compared with their Monte Carlo estimates. Below presented formula was calculated by *Mathematica* :

$$\begin{aligned}
V_{10} = {} & Be^{-K}T(362880K)^{-1}[3628800+3628800e^K \\
& - 6894720K - 4717440K^2 - 1874880K^3 - 514080K^4 \\
& - 105840K^5 - 17136K^6 - 2232K^7 - 234K^8 - 19K^9 - K^{10} \\
& + 9(362880 + 685440K + 463680K^2 + 181440K^3 + 48720K^4 \\
& + 9744K^5 + 1512K^6 + 184K^7 + 17K^8 + K^9)\ln(1+K) \\
& - 36(40320 + 75600K + 50400K^2 + 19320K^3 + 5040K^4 \\
& + 966K^5 + 140K^6 + 15K^7 + K^8)\ln(1+K)^2 + 84(5040 + 9360K + \\
& 6120K^2 + 2280K^3 + 570K^4 + 102K^5 + 13K^6 + K^7)\ln(1+K)^3 - \\
& 126(720 + 1320K + 840K^2 + 300K^3 + 70K^4 + 11K^5 \\
& + K^6)\ln(1+K)^4 + 126(120 + 216K + 132K^2 + 44K^3 + 9K^4 \\
& + K^5)\ln(1+K)^5 - 84(24 + 42K + 24K^2 + 7K^3 + K^4)\ln(1+K)^6 \\
& + 36(6+10K + 5K^2 + K^3)\ln(1+K)^7 - 9(2+3K + K^2)\ln(1+K)^8 \\
& + (1+ K)\ln(1+K)^9]
\end{aligned} \tag{14}$$

The parameter K appearing in the above expression is defined as the ratio T/λ. As we see the value of the problem depends on T and λ only through this ratio. Intuitively, the parameter K can be interpreted as an average number of steps needed to cover the distance T.

Formulae similar to (14) can be obtained for any N but for larger values of the boundary the calculations are very time-consuming and the results are too big to place them in the text. Note that there is no explicit formula relating the value of the problem and the boundary for the number of steps N.

Because the optimal stopping rule is independent of B and the expected value of the payoff as well as the value of the problem are linear functions of B we assume without loss of generality that B=1.

E. Important Characteristics of the OSR

In the situation where we deal with the decision making under uncertainty the most important features of the decision rules are the *expected payoff* and the *risk* connected with the rule. In the classical theory of optimal stopping the expected payoff is defined naturally and it is the criterion index for the optimality of a stopping rule, see (3) and (4). As we have seen, compare (14), even in the cases where the expected payoff connected with the OSR can be expressed in terms of elementary functions the formulae can be very long and really sophisticated. In [9] one can also find some examples of BTPs for which the formulae even in their simplest form need a few standard pages to be written down.

An even more difficult task is to investigate the relation between a given risk characteristic and the design parameters of the decision problem. The theory of optimal stopping hardly provides us with any formal results devoted to any risk measures connected with the OSR. However, the general decision theory propose taking into account many risk concepts and related risk characteristics. From theoretical and practical point of view two basic types of risk can be distinguished, see e.g. [13],[18],[19]:

- the risk connected with the variability of the results around a specific value of the payoff,
- the risk connected with the possibility of occurrence of undesired results.

In the first case the risk is treated as both, a possibility of getting less as well as a chance to get more than the specific value of the payoff we are interested in. Usually in this setup we use the variance or standard deviation to measure the variability of the possible payoffs around the expected payoff. In the second case we consider the risk as the probability of a failure. In the sequel of the paper we adopt both concepts of risk. Namely, we consider here risk characteristics formally defined as follows.

Let Z be a random payoff connected with the optimal stopping rule τ^*, i.e. $Z=W(Y_{\tau^*})$. Let σ_Z denote the standard deviation of the optimal payoff Z. In our studies the following risk measures connected with rule τ^* will be taken into account:

- the ratio SV of the standard deviation of the random payoff to the expected payoff, i.e. $SV=\sigma_Z/V_N(0)$,
- the probability PrF that the decision maker will cross the limit T, or equivalently in our problem, that the gain equals 0, i.e. $PrF=\Pr(Z=0)$.

Our problem now is to obtain the model for the relation between the risk characteristics SV and PrF of the OSR and the design parameters of the BTP.

It is easy to verify, that in the considered case both risk characteristics depend on T and λ only through their ratio $K=T/\lambda$. So, we can conclude, that the only design parameters which contain all significant information for the model building are K and N. For any given pair of the parameters (K,N) the values of SV and PrF can be computed analytically, but the calculations are much more arduous then those performed to obtain the value of the problem (in each case they require to compute N multiple integrals with dimensions growing from 1 to N) .

Thus to obtain some general formulae we propose to estimate the necessary characteristics from a simulation sample and then to built a regression model for the relation we are interested in.

III. Monte Carlo Simulations and Regression Models

Monte Carlo simulation has long served as an important tool in a wide variety of disciplines. Through computer simulation, one can study the features of real-life decision problems and/or formal-theoretic models that are too difficult to examine analytically, [3],[11],[13], [14]. In this section we describe the Monte Carlo experiment designed to produce data containing the information about the relations we want to investigate and then we present the results of the regression analysis applied to the data.

A. Monte Carlo Experiment

The idea of the Monte Carlo simulation is to draw a sample $\{Z_i\}_1^m$, i.e. a realization of the stochastic process $\{Z_1, Z_2, \ldots Z_m\}$ composed of independent and identically distributed random variables having the same distribution as the random phenomenon we want to study. Next, based on the sample, various characteristics of the unknown probability distribution can be estimated. Indeed, by the strong law of large numbers, for any Borel function f for which the expected value E$f(Z)$ exists, the average $\bar{f}_m = \frac{1}{m}\sum_{i=1}^m f(Z_i)$ will almost surely (*a.s.*) converge to E$f(Z)$. In particular, when the sample size m tends to infinity we have

$$\bar{Z}_m = \frac{1}{m}\sum_{i=1}^m Z_i \xrightarrow{a.s.} E(Z) = V_N$$

$$S_m^2 = \frac{1}{m}\sum_{i=1}^m (Z_i - V_N(y))^2 \xrightarrow{a.s.} \sigma_Z^2$$

and

$$\frac{M}{m} = \frac{1}{m}\sum_{i=1}^m \mathbf{1}_{(-\infty,a]}(Z_i) \xrightarrow{a.s.} \Pr(Z \le a)$$

In the latter expression M denotes the number of the values in the Monte Carlo sample which are not greater than a.

In our Monte Carlo simulations the random sample is composed of random variables having the same distribution as the optimal payoff $Z=W(Y_{\tau^*})$ which is generated according its definition with the help of the following procedure.

```
Set z=0;n=0;
While z≤t* and n<N Do Step 1 and Step 2
    Step 1. Set z=z+REX(λ)
    Step 2. Set n=n+1
If z<T Set z=B·z
Else Set z=0
Return z
```

In the above procedure the function REX(λ) returns a pseudorandom number generated according to the exponential probability distribution having the density function given by (11), t^* is given by (13), N is the upper bound for the number of possible steps given in the considered problem.

B. Monte Carlo Estimation of the Value of the Problem

In order to learn how accurate the Monte Carlo approximations are, we compare the theoretical values obtained for various setups of the BTP with the values estimated with the help of the Monte Carlo simulations. Table I shows the results obtained for the following design parameters: (T=50, K=10), (T=50, K=5), (T=150, K=10). The boundaries N for the number of steps are 6,8,10, and 12 in each case. The Monte Carlo sample size is m=10000.

TABLE I
COMPARISON OF THE THEORETICAL PROBLEM VALUES V_N AND THEIR MONTE CARLO ESTIMATES V_N^{MC}

N	T=50, K=10,		T=50, K=5		T=150, K=10	
	V_N	V_N^{MC}	V_N	V_N^{MC}	V_N	V_N^{MC}
6	27.83	27.96	31.38	31.35	83.49	83.18
8	33.45	33.61	32.00	32.10	100.38	100.50
10	36.40	36.39	32.08	32.07	109.19	108.99
12	37.57	37.62	32.08	32.17	112.67	112.43

We see that the Monte Carlo estimates are very close to the true problem values in each case. It can be easily verified that the relative error of the estimates is less than 0.5%. We could make it even smaller if we increased the sample size m.

Let the limit T=150 and the boundary for the number of steps N=10. We compute the values V_N^{MC} for values K_i=0.5+35i/300, i=1,…,300. Fig. 1 shows both, the data obtained from the Monte Carlo experiment, as well as the graph of a the theoretical value of V_{10} as a function of the parameter K, computed according the formula (14). We see that the theoretical values and their approximations can hardly be distinguished. To measure the quality of the approximations we also compute the average relative error ARE of the Monte Carlo approximations. The formula for ARE is as follows:

$$ARE = \frac{1}{m}\sum_{i=1}^{m}\left|V_{10}^{MC}(K_i) - V_{10}(K_i)\right| / V_{10}(K_i) \tag{15}$$

where m, as previously, is the Monte Carlo sample size.

The value of ARE obtained for our data is 0.00302 – it confirms that the Monte Carlo approximations are really good.

Next, based on the Monte Carlo sample, we built a regression model for the value V_{10} as a function of the parameter K. The estimated regression model, denoted by V_{10}^{MC}, is given by the following formula:

$$V_{10}^{MC}(K) = \begin{cases} \beta_0 + \beta_1/K + \beta_2 K + \beta_3 K^2 + \beta_4 K^3 & , \quad K \le g \\ \beta_5 + \beta_6/K + \beta_7 K + \beta_8 K^2 & , \quad K > g \end{cases} \tag{16}$$

We obtain the following *least squares* (*LS*) estimates b_i for the unknown coefficients β_i, i=0,…,8:
b_0=-8.50, b_1=14.59, b_2=-0447, b_3= 125.15, b_4= 68.70,
b_5=-1.27, b_6=0.0082, b_7= -0.000022 , b_8= -0.000022,
g = 16.4065

The graph of the estimated model for the value V_{10}^{MC} looks the same as a the graph of the function V_{10} presented in the Fig. 1 so we omit its presentation in this text.

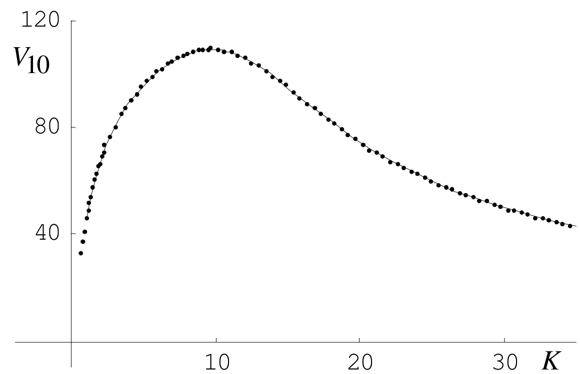

Fig. 1. Graph of a the value V_{10} as a function of K when T=150 (continuous line) and the Monte Carlo approximation of the values obtained for Kœ [1,35] (dots). The theoretical values and their approximations are almost the same – they can hardly be distinguished in the figure. The average relative error of the Monte Carlo approximations equals 0.0030.

To compare the functions V_{10} and V_{10}^{MC} more precisely one can compute an average relative error IRE for Kœ [a,b] as an integral

$$IRE(V_{10}, V_{10}^{MC}) = \frac{1}{b-a}\int_a^b \left|V_{10}^{MC}(x) - V_{10}(x)\right| / V_{10}(x)dx \tag{17}$$

The error computed for the interval [1, 35] equals 0.00323 which confirms that we have obtained really good approximation of the theoretical function V_{10}. One can also notice that the formula (16) obtained via our Monte Carlo studies is much simpler that the theoretical one, see (14). What is more, such formulae can be obtained directly for any given number N whereas the relation (14) must be computed with the help of the recursive equation (10), which appears to be very time consuming.

In the next part of the paper we adopt this approach to build models for the risk characteristics of the optimal stopping rule.

C. Monte Carlo Estimation of the Risk Characteristics

As we have already pointed out, the only design parameters both considered risk characteristics depend on are the boundary for the number of steps N and the ratio K=T/λ. Now, to illustrate the usage of the proposed approach in a case where the boundary N is known, we consider in details a problem with N=10. For such a problem we want to build a model describing the relation between the risk characteristics and the parameter K.

For 1000 values of the parameter K changing uniformly in the interval [1, 20] we estimate the risk characteristics SV, and

PrF. For this purpose we use the Monte Carlo estimators described in Section 3.A assuming m=10000. In this simulations for each number K the limit T was chosen uniformly at random from the interval [50, 200]. Figures 2 and 3 present the results.

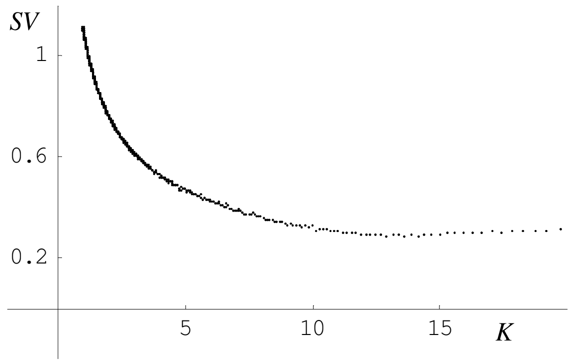

Fig. 2. Monte Carlo estimates for the characteristic *SV* when $K \in [1,20]$, N=10 and m=10000. The values of the parameter T are chosen at random from the interval [50,200]

The simulation results show that the smallest values of the characteristic *SV* are achieved for the values of K which are a bit greater than K^*=9.611 for which the expected payoff achieves its maximum, see Fig.1 and 2. One can also notice that for greater values of K the risk characteristic *SV* is almost unchanged. The greatest values of the characteristic *SV* are achieved for small values of K and this can be easily intuitively explained. As we know, the parameter K equals the average number of steps needed to cover the distance T. Thus small values of the parameter imply relatively high probability that the process will cross the border T even after the first step and that implies large variation of the payoff. At the same time for such values of K the expected payoff connected with the OSR is small, see Fig.1.

Fig. 3 illustrates the dependence between the probability of failure *PrF* and the parameter K. It shows that statistically the probability is a decreasing function of K. Based on the simulations it can be also estimated that the probability of failure for the K^* equals 7.2%. Thus if one wants to assure smaller probability of failure then, if possible, value greater than K^* should be chosen for K. On the other hand in such a case the expected payoff will be less than the maximal one, compare Fig. 1 and 3.

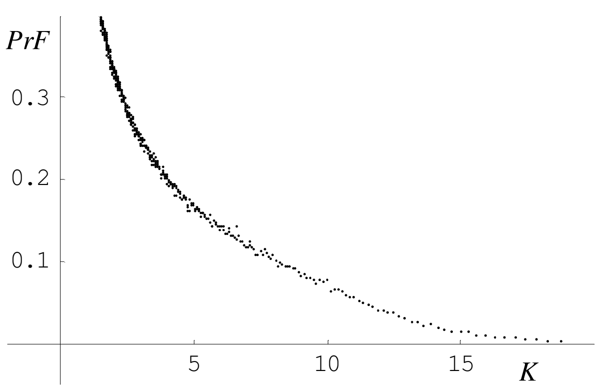

Fig. 3. Monte Carlo estimates for the characteristic *PrF* when $K \in [1,20]$, N=10 and m=10000. Parameter T was chosen at random from the interval [50,200]

It is worth emphasizing that the Monte Carlo simulations provide us with quite a precise knowledge about the investigated relations, and the knowledge could be very difficult to obtain with the help of any formal analytical tools.

D. *Regression Model for the Ratio SV*

Now we build the regression models of the relationship between the risk characteristic *SV* and the parameter K. After preliminary studies we assume the following form of the regression function:

$$SV(K) = \begin{cases} \beta_0 + \beta_1 / K + \beta_2 K + \beta_3 K^2, & K \le g \\ \beta_4 + \beta_5 / K + \beta_6 K, & K > g \end{cases} \tag{18}$$

Based on the Monte Carlo data presented in Fig. 2 we obtain the following LS estimates b_i for the unknown coefficients β_i, i=0,…,6:

b_0=0.7039, b_1=0.5049, b_2= - 0.1195, b_3= 0.0109, b_4= 0.1367, b_5=1.5445, b_6=0.0038, g = 4.4038

Fig 4. presents the graph of the estimated regression model for the characteristic *SV*.

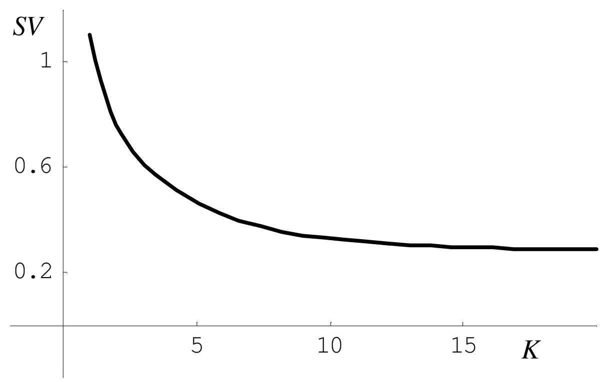

Fig. 4. Dependence between *SV* and K when N=10, the estimated regression model. The average prediction error computed for this model on the base of 1000 new data points equals 0.0035, the relative prediction error amounts to 0.63%

To check the usefulness of the model we generate *another* 1000 data points and compute the average absolute error of prediction. The error equals to 0.0035. The average relative absolute error equals 0.63% . Both these measures confirm practical usefulness of the regression model.

E. *Regression Model for the Probability of Failure.*

For the probability of failure we choose the logistic regression model having the following structure:

$$PrF(K) = \frac{\mathrm{Exp}(PF(K))}{1 + \mathrm{Exp}(PF(K))} \tag{19}$$

where

$$PF(K) = \beta_0 + \beta_1 / K + \beta_2 K + \beta_3 K^2$$

The LS estimates of the coefficients are the following:

b_0= -1.2932, b_1=1.4272, b_2=-0.05456, b_3= -0.0101

The graph of the function *PrF* is presented in Fig. 5. The average prediction error in this case (computed on the base of 1000 new data points) is 0.0081 and the relative prediction error amounts to 3.02%. We see that the regression model performs really well - such small errors in estimation of the probability are usually practically irrelevant.

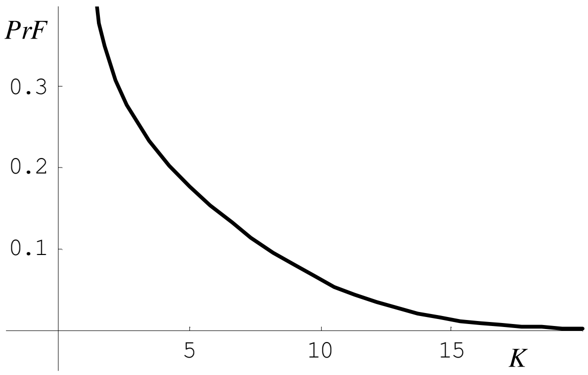

Fig. 5. Dependence between PrF and K when N=10, the estimated regression model. The average prediction error computed for this model on the base of 1000 new data points equals 0.0081, the relative prediction error amounts to 3.02%

F. Models Relating Risk Characteristics to both Design Parameters

In previous sections we have developed the regression models for the risk measures in the case where the boundary for the possible number of steps N was given, namely N=10. It is obvious that the same approach can be utilized to obtain similarly good models in case of any given value of N. On the other hand it can be important for some optimization and/or control tasks connected with the blackjack type stopping problems to know the relation between the risk characteristics and both design parameters; K and N. However, there are not even any theoretical expressions relating N and the value of the problem V_N. Thus the only way to cope with this problem is to make use of the Monte Carlo sample and build an appropriate regression model. Below we present the results obtained on the base of 3000 data points. In these simulations the parameter K was uniformly drawn at random from the interval [2,..,20]. Taking into account that the probability distribution of the steps X_i, i=1,2..., is an exponential one with $\lambda=T/K$, see (11), we can conclude that if $N<K/3$ then the probability $PrF=\mathrm{P}(Y_{\tau*} > T)$ is close to 0, whilst to study the values of the characteristic SV it is practically enough to perform theoretical studies of the moments of the Erlang distribution of the state Y_N. On the other hand, when $N>3K$ then the probability that $Y_N > T$ is close to 1 and the parameter N practically is not really a boundary for the possible number of steps, because the process will almost surely stop earlier. Thus in this case the parameter N practically does not influence the risk characteristics. Consequently in our simulations the parameter N was a natural number drawn uniformly at random from the interval [max(2,K/3) ,..., 3K].

Based on the Monte Carlo sample we obtain the following general regression model for the characteristic SV :

$$GSV(K,N) = \beta_0 + \beta_1 N + \beta_2\left(\frac{N}{K}\right) + \beta_3\frac{N}{K^2} + \beta_4\left(\frac{N}{K}\right)^2 + \beta_5\frac{1}{K} \qquad (20)$$

with the following LS estimates b_i for the unknown coefficients β_i, i=0,…,5:

b_0=0.316247 , b_1= -0.00850239, b_2=0.010147,
b_3= -0.0981012, b_4= 0.0130596, b_5= 0.997077

The model for the risk characteristic PrF is now proposed in the following logit form:

$$PrF(K,N) = \frac{\mathrm{Exp}(P(K,N))}{1+\mathrm{Exp}(P(K,N))}$$

where

$$P(K,N) = \beta_0 + \beta_1 N + \beta_2\frac{K}{N} + \beta_3\frac{K^2}{N} + \beta_4\frac{1}{K} + \beta_5 K + \beta_6 K^2 \qquad (21)$$

The LS estimates b_i obtained for the regression coefficients β_i, i=0,…,6, in this case are as follows:

b_0= -0.417676 , b_1= -0.0645557, b_2= -0.580249,
b_3= -0.147975, b_4= 0.911603, b_5= -0.0464804,
b_6= 0.00786099,

To check usefulness of the two above models, we generate 3000 new data points and compute the average errors and relative errors of prediction. For the model GSV given by (20) they are equal to 0.0132 and 0.0302, respectively. For the logit model (21) the errors amount to 0.00381 and 0.0628. So, the values of the errors again indicate very good performance of the models.

Obviously, as a special case, one may obtain from (20) another expression for the relation between SV and K in case where N=10 and compare it with the model for SV given by (18). It is interesting to observe that the relative difference between the two functions, measured by IRE given by (17), is very small and computed for the interval [3,18] equals 3.56%. Analogously measured relative difference between the model (19) and the one obtained from equation (21) in the case where N=10 amounts to 9.83%. However, the relations (18) and (19) are a bit more accurate because they were dedicated especially for the BTP with N=10, while the relations (20) and (21) are much more general.

IV. Final Remarks

Studying the risk characteristics of the decision rules is an important part of decision making. However results connected with the risk analysis can hardly be found in the optimal stopping literature because of the very complex nature of the formal-theoretical relations between the design parameters of the optimal stopping problem and the risk characteristics of the OSR. Thus in this paper we propose to make use of the Monte Carlo simulations to investigate the relations. However, the Monte Carlo experiments – as all computer simulations – are subject to similar weakness; the results may depend on specific experiment design. Thus we propose here to generate the Monte Carlo sample for a wide range of the design parameters values and next to make use of the regression analysis to obtain analytical expressions modeling the relations which we want to study. Following the approach we develop models for the blackjack type optimal stopping problems with linear payoff and exponential step. The models allow the decision maker to study the risk characteristics of the OSR for a wide range of the design parameters. Estimated prediction errors appear to be very small, which indicates that the approach results in the analytical models which are very good approximations of the true relationship.

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