Exact Solutions of Three Nonlinear Heat Transfer Problems

Mohammad Danish, Shashi Kumar and Surendra Kumar

***Abstract*— In this work, three nonlinear heat transfer problems namely, steady state heat conduction in a rod, unsteady cooling of a lumped system and steady state heat transfer from a rectangular fin into the free space by the radiation mechanism, have been solved analytically. Earlier these three problems were solved by various researchers by using homotopy perturbation, homotopy analysis and optimal homotopy analysis methods and the approximate series solutions were obtained. Here, we have obtained exact analytical solutions of these three problems in terms of a simple algebraic function, a Lambert *W* function and the Gauss's hypergeometric function, respectively. These exact solutions agree very well with those obtained by the numerical schemes and are better than the recent approximate solutions. Moreover, these can also serve as the yardsticks for future testing of the approximate solutions.**

***Index Terms*— Heat transfer, Conduction, Convection, Radiation**

I. INTRODUCTION

THIS research work mainly stresses on finding the exact analytical solution of three nonlinear heat transfer problems which have nonlinear temperature dependent terms. The first problem represents the steady state heat conduction process in a metallic rod and is described by a nonlinear BVP (boundary value problem) in a second order ODE (ordinary differential equation). Recently, Rajabi *et al.* [1] have solved this problem by using a well known approximate method i.e. HPM (homotopy perturbation method), whereas Sajid and Hayat [2] and Domairry and Nadim [3] have solved the same problem by using HPM and another very popular approximate scheme i.e. HAM (homotopy analysis method). These workers have obtained the results in the form of a finite. The second problem, considered by Ganji [4] using HPM, by Abbasbandy [5] using HAM and by Marinca and Herişanu [6] using OHAM (optimal HAM), depicts the unsteady heat convection from a lumped system.

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The related governing equation of this problem is expressed by a nonlinear initial value problem (IVP) in a first order ODE. The solutions were found in series form. The third problem describes the steady state radiative heat transfer from a rectangular fin into the free space and the model equation is give by a nonlinear BVP in a second order ODE. This problem has also been recently considered by Ganji [4], Abbasbandy [5] and Marinca and Herişanu [6] by using HPM, HAM and OHAM, respectively, and the solution were found in terms of the truncated series.

One should note that the series solutions have changeable degree of accuracy and radius of convergence, and are strongly dependent on the number of terms in the series as well as on the parameters' values. Because of this, there remains a region outside which the series solutions start deviating and their regular use becomes limited. Nonetheless, in such cases efforts are made either to obtain the exact analytical solutions or to solve the problem with the help of some suitable numerical technique. Fortunately, we have shown that all the above three mentioned problems are exactly solvable in terms of algebraic function, Lambert *W* function [7] and hypergeometric function, respectively. These solutions have been obtained by using simple mathematical manipulations e.g. assuming an implicit form of the solution or by reducing the equation into a simpler form by adding and subtracting certain terms, as elaborated in the following sections. Thus found analytical solutions are fairly helpful since:

(i) Better insight of the actual physical process is easily gained.
(ii) These can straightforwardly be utilized in finding the precise temperature profiles and temperature gradients for a whole range of parameters' values disparate to their approximate series counterparts which have convergence related issues for the entire range of parameters' values especially for the extreme values of parameters.
(iii) One can also be deploy them to validate the accuracy of other approximate solutions.

Physical description of the mentioned processes, derivation of respective model equations and the methods to find the exact solutions are discussed below.

II. PROBLEM 1: HEAT CONDUCTION IN A METALLIC ROD

This problem mainly portrays the steady conductive heat transfer in a metallic rod and practically arises in estimating the thermal conductivity of metals e.g. heat flow meters [8, 9]. In this problem, the two ends of the rod are kept at different but fixed temperatures and heat transfer takes place from higher temperature to the lower by the mechanism of conduction. In this conduction problem, we assume that the thermal conductivity varies linearly with temperature and there is no heat loss to the surrounding from the round surface of the rod.

We consider a rod of length, L and uniform cross sectional area, A_c with its end maintained at two different temperatures i.e. $T(x=0)=T_a$ and $T(x=L)=T_b$. For these stated assumptions, the steady state energy balance over the rod gives in the following dimensional equation and the associated BCs (boundary conditions):

$$\frac{d}{dx}\left(A_c k(T)\frac{dT}{dx}\right)=0 \tag{1a}$$

BCI: $T=T_a$ at $x=0$ (1b)

BCII: $T=T_b$ at $x=L$ (1c)

Where $k(T)=k_a\left(1+\beta\frac{T-T_a}{T_b-T_a}\right)$ is the temperature dependent thermal conductivity of the rod. With the introduction of the following dimensionless variables, the governing equation and the associated BCs i.e. (1a)-(1c), transform into the following equations i.e. (2a)-(2c):

$$\xi=\frac{x}{L},\ \theta=\frac{T-T_a}{T_b-T_a}$$

$$(1+\beta\theta)\theta''+\beta(\theta')^2=0 \tag{2a}$$

BCI: $\theta(0)=0$ (2b)

BCII: $\theta(1)=1$ (2c)

Where θ' & θ'' represents the first and second order derivatives of θ with respect to ξ, respectively. Following two different approaches can be adopted to obtain the exact solution of the above equation, as demonstrated below:

A. Approach I

A careful inspection of (2a) shows that it can conveniently be expressed in the following form:

$$((1+\beta\theta)\theta')'=0 \tag{3}$$

Integrating the above equation two times with respect to ξ, one obtains the following quadratic equation in θ:

$$\left(\theta+\beta\frac{\theta^2}{2}\right)=C_1\xi+C_2 \tag{4}$$

Where $C_1=\left(1+\beta/2\right)$ and $C_2(=0)$ are the constants of integration and have been found from the associated BCs i.e. (2b) & (2c). Substituting these values in (4) and solving for θ, one finds the following two explicit solutions; two solutions appear because of the nonlinear nature of the equation.

$$\theta=\frac{-1+\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \tag{5a}$$

$$\theta=\frac{-1-\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \tag{5b}$$

Since, second solution does not satisfy the BCs and is unrealistic it is, therefore, rejected. If one expands (5a) around $\beta=0$ using Taylor series the following approximate series is obtained.

$$\theta\approx\xi+\frac{1}{2}\beta\left(\xi-\xi^2\right)+\frac{1}{2}\beta^2\left(\xi^3-\xi^2\right)+\ldots$$

If one compares it with the approximate HPM solution [(47)] of Rajabi *et al.* [1] and approximate HAM solution of Domairry and Nadim [3] for the convergence control parameter $h=-1$ (used therein), an accurate compliance is observed.

However, we could not compare the results obtained by exact solution with the results of Sajid and Hayat [2] since no such solution term was provided. However, in this case the results were judged against those of Sajid and Hayat [2] by tabulating the values of temperature gradients at $\xi=0$ and $\xi=1$ (see Table 1). A close conformity is observed between these values.

The results obtained by the current solution i.e. (5a) have also been successfully verified against those obtained by (47) of Rajabi *et al.* [1] and those obtained by numerical methods, as shown in Fig. 1. In this figure it is clearly visible that the approximate temperature profile obtained by Rajabi *et. al.* [1] deviates appreciably even for moderate values of β and becomes redundant for larger values of β. Although not shown, the same characteristics can also be ascribed to the HAM solution of Domairry and Nadim [3] for the convergence control parameter $h=-1$. On the contrary, no deviation is observed in the present solution, even for higher values of β. One notes from Fig. 1 that as β varies from 0 to ∞, the temperature of the rod tends to reach the higher temperature ($\theta=1$) and thus ascertain the fact that with the increase in thermal conductivity the temperature of the rod also rises.

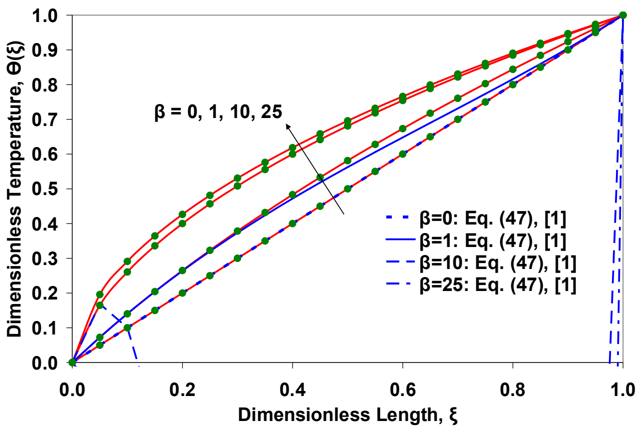

Fig. 1 Dimensionless temperature profiles along the length of the rod (problem 1), solid lines: exact solution; filled circle: numerical solution

B. Approach II

In this approach we assume that the derivative θ' is a function of θ only i.e. $\theta'=p(\theta)$, in other words, the solution of (2a) exhibits an implicit form i.e. $f(\theta)=\xi$. Consequently, $\theta''=\frac{1}{2}\frac{d\left(p^2\right)}{d\theta}$, where p (still unknown) is a function of θ, only. It is useful to mention that this approach is quite helpful whenever the independent variable ξ is absent in the concerned equation. Replacing

θ' & θ'' in (2a) by the above respective definitions, one obtains:

$$(1+\beta\theta)\frac{d(p^2)}{d\theta}+2\beta p^2=0 \quad (6)$$

Now, substituting $p^2=y$ and after little alterations the above equation reduces to the following first order linear ODE:

$$(1+\beta\theta)y'+2\beta y=0 \quad (7)$$

Solving the above first order linear ODE by integrating factor method one finds:

$$y=\frac{C_1}{(1+\beta\theta)^2} \quad (8)$$

Or

$$p(\theta)=\frac{d\theta}{d\xi}=\frac{\sqrt{C_1}}{(1+\beta\theta)} \quad (9)$$

Where C_1 is a constant of integration. Integrating the above (9) once more, one finds the expression for θ [note that the equation below is similar, in form, to the (4)]:

$$\left(\theta+\beta\frac{\theta^2}{2}\right)=\sqrt{C_1}\xi+C_2 \quad (10)$$

C_2 is another constant of integration and $C_1=\left(1+\beta/2\right)^2$ and $C_2(=0)$ are evaluated from the associated BCs, like in the first approach. Substituting the values of these constants in (10) and solving for θ, one arrives at the following two solutions which are exactly same as those given in (5a) & (5b).

$$\theta=\frac{-1+\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \quad (11a)$$

$$\theta=\frac{-1-\sqrt{1+2\beta\xi+\beta^2\xi}}{\beta} \quad (11b)$$

Second solution does not satisfy the BCs so discarded. Rest of the discussion remains same as presented in Approach I.

TABLE I
COMPARISON OF DIMENSIONLESS TEMPERATURE GRADIENT AT BOTH THE ENDS OF THE ROD (PROBLEM 1)

S. No.	β	$\theta'(1)$			$\theta'(0)$	
		Numerical solution	Sajid & Hayat [2]	Exact solution (5a)	Numerical solution	Exact solution (5a)
1	0.5	0.833333	0.833333	0.833333	1.250000	1.250000
2	2	0.666667	0.666667	0.666667	2.000000	2.000000
3	5	0.583333	0.583333	0.583333	3.500000	3.500000
4	25	27/52	-	27/52	27/2	27/2

III. PROBLEM 2: COOLING OF A LUMPED SYSTEM

This problem represents the temporary cooling of a lumped system the specific heat of which varies linearly with temperature. In real world, this problem arises in the cooling of heated stirred vessels and cooling of electronic components with high thermal conductivity etc [8]. HPM, HAM and OHAM solutions of this problem have been found by Ganji [4], Abbasbandy [5] and Marinca and Herișanu [6], respectively and the solutions were obtained in the form of series. The problem is stated as: at the outset of the experiment, a system with density ρ, volume V and heat transfer area A, is exposed to a surrounding at different temperature (T_a) and heat is transferred from the system to the surrounding by convection. The leading model equation is derived by applying the unsteady energy balance over the system and is described by the following nonlinear IVP (initial value problem) in first order ODE:

$$\rho Vc(T)\frac{dT}{dt}+hA(T-T_a)=0 \quad (12a)$$

$$\text{IC: } T(0)=T_b \quad (12b)$$

Where $c(T)=c_a\left(1+\beta\frac{T-T_a}{T_b-T_a}\right)$ is the heat capacity of the system showing linear dependency on temperature and h is the constant heat transfer coefficient. With the assistance of the following dimensionless quantities, (12a) & (12b) attain the dimensionless form given by (13a) & (13b), respectively.

$$\tau=\frac{hAt}{\rho Vc_a},\ \theta=\frac{T-T_a}{T_b-T_a}$$

$$(1+\beta\theta)\theta'+\theta=0 \quad (13a)$$

$$\text{IC: } \theta(0)=1 \quad (13b)$$

A simple rearrangement of the above (13a) yields:

$$\frac{\theta'}{\theta}+\beta\theta'=-1 \quad (14)$$

Integrating (14) with respect to τ results in:

$$Log[\theta]+\beta\theta=-\tau+C_1 \quad (15)$$

Where C_1 is the constant of integration and using IC, it is found to be $C_1=\beta$. Substituting back the so found value of C_1 in (15), provides the following exact analytical solution.

$$Log[\theta]+\beta\theta=\beta-\tau \quad (16)$$

Due to the above implicit form of θ, it has to be found for each and every τ by solving (16) with the help of some suitable iterative numerical scheme. This feature limits the repeated use of the above formula. Keeping this in view, we now develop, from (16), the explicit solution form. A constant term $Log[\beta]$ is added and subtracted in (16) and after performing a little modification, (17) is obtained.

$$Log\left[\beta\theta e^{\beta\theta}\right]=\beta-\tau+Log[\beta] \quad (17)$$

Equation (17) can be further expressed as:

$$(\beta\theta)e^{(\beta\theta)}=\beta e^{\beta-\tau} \quad (18)$$

The L.H.S. of (18) can be replaced by the Lambert *W* function (implemented as *ProductLog* function in some mathematical softwares e.g. Mathematica). A Lambert *W* function is basically the inverse function of $x=ye^y$ i.e. $y=Lambert(x)$ and is symbolized by $y=W(x)$. In general, the domain and range of the function is the set of complex values however, for $x\in[0,\infty)$ Lambert *W* function yields single real values. For $x\in(-\infty,-1/e)$, Lambert *W* function does not evaluate to any real value whereas, for $x\in[-1/e,0)$ it computes two real values. Now, with this function available, the transient dimensionless temperature profile is given by:

$$\theta = \frac{1}{\beta} ProductLog\left[\beta e^{\beta-\tau}\right] \quad (19)$$

Expanding θ around $\beta = 0$ by using Taylor series, yields the following expansion which harmonizes with the (18) of Ganji [4] and (9) of Abbasbandy [5] for $h = -1$.

$$\theta \approx e^{-\tau} + \beta\left(e^{-\tau} - e^{-2\tau}\right) + \frac{\beta^2}{2}\left(e^{-\tau} - 4e^{-2\tau} + 3e^{-3\tau}\right) + \ldots$$

Fig. 2 compares that the transient temperature profiles obtained by the present (19), HPM solution obtained by Ganji [4] and those obtained by numerical scheme. It is clear that the present solution match very well with the numerical solution whereas, the solutions obtained by Ganji [4] show considerable discrepancies except for $\beta = 0$ where the (13a) becomes linear. Fig. 2 also supports the fact that with the increase in β, the specific heat increases which in turn causes the decrease in temperature gradient.

Extending the comparison, the initial rates of temperature change, given by the following (20), have also been found using (19) and plotted in Fig. 3 along with those obtained by Abbasbandy [5].

$$\theta'(0) = \frac{-1}{1+\beta} \quad (20)$$

Accuracy is evident by the overlapping profiles. Similar comparisons with the OHAM solution of Marinca and Herişanu [6] have been avoided due to their more involved solution expression. However, it can be shown that our present solution, being exact in form, is superior to the approximate solution of Marinca and Herişanu [6].

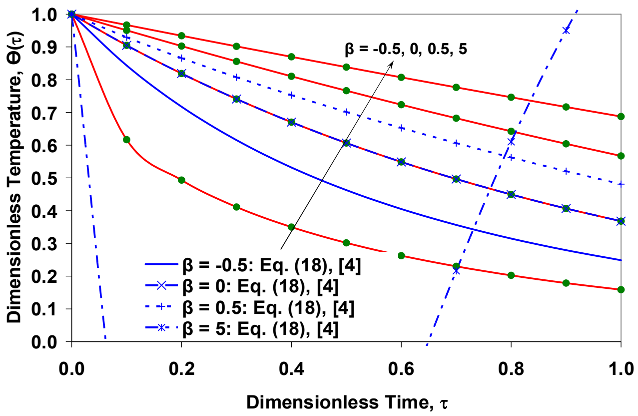

Fig. 2. Transient profile of the dimensionless temperature (problem 2), solid lines: exact solution: filled circle: numerical solution

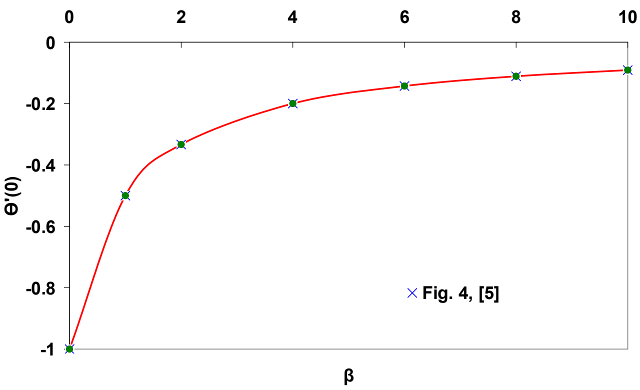

Fig. 3. Initial rate of change of dimensionless temperature vs. β (problem 2 solid lines: exact solution; filled circle: numerical solution

IV. PROBLEM 3: STEADY STATE RADIATIVE HEAT TRANSFER FROM A RECTANGULAR FIN

This problem represents the steady state heat transfer from a rectangular fin to the free space by the radiation mechanism. Such situations appear in the cooling of the heated parts of the space vehicles. This problem, too, has been tackled by Ganji [4], Abbasbandy [5] and Marinca and Herişanu [6] with the help of HPM, HAM and OHAM, respectively and the solutions were obtained in the form of series. We consider a rectangular fin having cross sectional area A_c, perimeter P, length L and the constant thermal conductivity and emissivity as k and $\in$, respectively. The fin base is maintained at a higher temperature T_b and the fin is transmitting the heat energy into the space by the mode of radiation. It is assumed that the steady state is prevailing and the negligible heat transfer takes place from fin end [10]. Keeping these assumptions in view, the governing model equation is derived by applying the steady energy balance over the fin element and is described by the following nonlinear BVP in second order ODE:

$$\frac{d}{dx}\left(A_c k \frac{dT}{dx}\right) = P\sigma \in (T^4 - T_s^4) \quad (21a)$$

BCI: $T = T_b$ at $x = L$ (at fin base) (21b)

BCII: $\frac{dT}{dx} = 0$ at $x = 0$ (at fin end) (21c)

It is worthwhile to note that the space temperature can very well be replaced by the absolute zero temperature i.e. $T_s = 0$ [4-6]. Taking this fact into account and defining the following dimensionless variables, the above equations are conveniently expressed into the dimensionless form given by (22a) - (22c).

$$\theta = \frac{T}{T_b}, \quad \xi = \frac{x}{L}, \quad \varepsilon = \frac{\sigma \in P T_b^3 L^2}{k A_c}$$

And the (21a) - (21c) become

$$\frac{d^2\theta}{d\xi^2} = \varepsilon\theta^4 \quad (22a)$$

BCI: $\theta = 1$ at $\xi = 1$ (at fin base) (22b)

BCII: $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$ (at fin end) (22c)

To solve the above BVP, the same approach has been followed as adopted previously for the solution of problem 1, and here also, it is assumed that the derivative $\frac{d\theta}{d\xi}$ is a function of θ only i.e. $\frac{d\theta}{d\xi} = p(\theta)$ where p is yet to be found. This assumption leads to $\theta'' = \frac{1}{2}\frac{d\left(p^2\right)}{d\theta}$. Replacing θ'' in (22a) by this relation, one obtains:

$$\frac{d\left(p^2\right)}{d\theta} = 2\varepsilon\theta^4 \quad (23)$$

Now, replacing p^2 with y, the (23) attains the following first order linear ODE:

$$\frac{dy}{d\theta} = 2\varepsilon\theta^4 \quad (24)$$

Integrating the above equation, one finds

$$y = \frac{2}{5}\varepsilon\theta^5 + C_1 \qquad (25)$$

C_1 is constant of integration and can be evaluated with the help of BCII i.e. (22c) and is found to be $C_1 = -\frac{2}{5}\varepsilon\theta_0^5$; where θ_0 is the unknown dimensionless temperature at the fin base. Substituting this value of C_1 in (25), one gets

$$y = \left(\frac{d\theta}{d\xi}\right)^2 = \frac{2}{5}\varepsilon\left(\theta^5 - \theta_0^5\right) \qquad (26)$$

A minor rearrangement of the above equation yields

$$\frac{d\theta}{\sqrt{\frac{2}{5}\varepsilon\left(\theta^5-\theta_0^5\right)}} = d\xi \qquad (27)$$

Integrating the above equation between the limits prescribed by the BCs I & II, following definite integral is found.

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\frac{2}{5}\varepsilon\left(\theta^5-\theta_0^5\right)}} = \int_0^{\xi} d\xi \qquad (28)$$

The integration of the above equation gives the following result.

$$\sqrt{\frac{5}{2\varepsilon\left(\theta^5-\theta_0^5\right)}}\,\theta\sqrt{1-\frac{\theta^5}{\theta_0^5}}\,HG_2F_1\left[\frac{1}{5},\frac{1}{2},\frac{6}{5},\frac{\theta^5}{\theta_0^5}\right] - i\sqrt{\frac{5\pi}{2\varepsilon}}\frac{1}{\theta_0^{3/2}}\frac{\Gamma\left[\frac{6}{5}\right]}{\Gamma\left[\frac{7}{10}\right]} = \xi \qquad (29)$$

The unknown θ_0 is computed by solving the following nonlinear equation which has been obtained by forcing (29) to satisfy the unutilized BCI i.e. $\theta = 1$ at $\xi = 1$.

$$\sqrt{\frac{5}{2\varepsilon\left(1-\theta_0^5\right)}}\sqrt{1-\frac{1}{\theta_0^5}}\,HG_2F_1\left[\frac{1}{5},\frac{1}{2},\frac{6}{5},\frac{1}{\theta_0^5}\right] - i\sqrt{\frac{5\pi}{2\varepsilon}}\frac{1}{\theta_0^{3/2}}\frac{\Gamma\left[\frac{6}{5}\right]}{\Gamma\left[\frac{7}{10}\right]} = 1 \qquad (30)$$

Where $\Gamma[z]$ and $HG_2F_1[a,b,c,z]$ are the well known Gamma and the Gauss' Hypergeometric functions, respectively and are defined as follows [11]:

$$\Gamma[z] = \int_0^{\infty} t^{z-1}e^{-t}dt$$

$$HG_2F_1[a,b,c,z] = \frac{\Gamma[c]}{\Gamma[b]\,\Gamma[c-b]}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

Ganji [4], Abbasbandy [5] and Marinca and Herişanu [6] have solved this problem by using HPM, HAM and OHAM, respectively and solutions are obtained in terms of the series. For comparison purposes, the two terms HPM and HAM solutions of Ganji [4] and Abbasbandy [5] are reproduced below, however, because of complexity in the expression of Marinca and Herişanu [6], it has not been considered here.

$$\theta_{Ganji} \cong 1 + \varepsilon\left(\frac{x^2-1}{2}\right) + \varepsilon^2\left(\frac{x^4-6x^2+5}{6}\right) \qquad (31)$$

$$\theta_{Abbasbandy} \cong 1 - \varepsilon h\left(\frac{x^2-1}{2}\right) - \varepsilon h(1+h)\left(\frac{x^2-1}{2}\right) + \varepsilon^2 h^2\left(\frac{x^4-6x^2+5}{6}\right) \qquad (32)$$

Figs. 4 & 5, plot the dimensionless temperature profiles obtained by the above approximate series solutions, the accurate numerical scheme as well as those obtained by the presently obtained exact solution i.e. (29) & (30). It can be noted that in Fig. 5 the same value of the parameter ε have been taken as those considered in [4] and [5] i.e. $\varepsilon = 0.7$. It can be seen in Fig. 4 that the profile obtained by Ganji [4] deviates to some extent with the numerical solution whereas the profile obtained by the exact analytical solution depicts an excellent matching with its numerical counterpart.

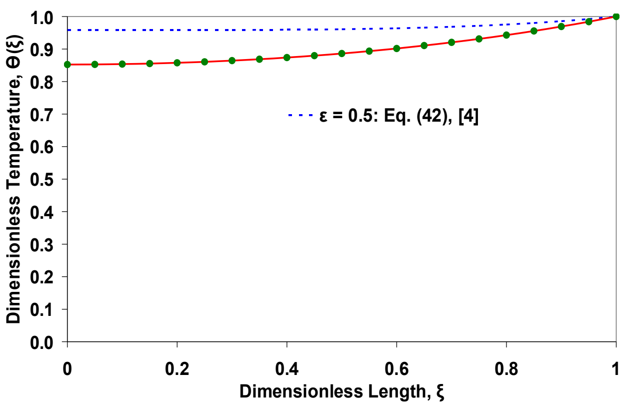

Fig. 4. Dimensionless temperature profiles along the length of the fin (problem 3), solid lines: exact solution; filled circle: numerical solution

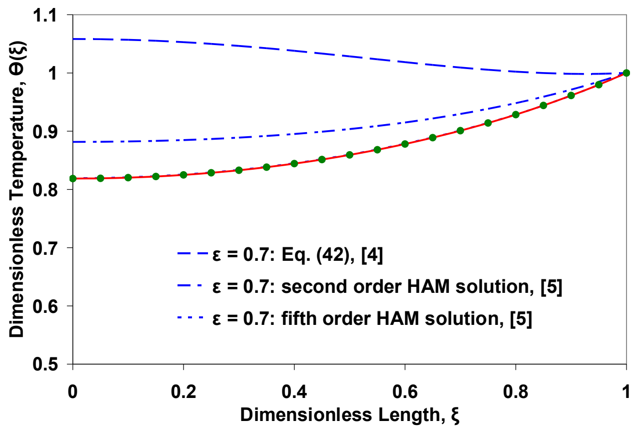

Fig. 5. Dimensionless temperature profiles along the length of the fin (problem 3), solid lines: exact solution; filled circle: numerical solution

Similarly, in Fig. 5, the two terms HPM solution of Ganji [4] yields divergent results whereas, the two term HAM solution of Abbasbandy [5] show minor deviations with the numerically obtained accurate profile. However, the five term HAM solution obtained by Abbasbandy [5] matches well with the numerical solution. In contrast to this, the exact analytical solution i.e. (29) & (30) are in complete agreement with the numerical solution. It can be verified that the deviations in the series solutions of Ganji [4] and Abbasbandy [5], will increase with the increase in the value of ε, however, this is not true for the currently derived exact solution. The true profiles signify the sharp decrease in temperature with the increase in the parameter ε. This observation is in compliance with the physics of the problem.

V. Conclusion

In this work, the three nonlinear heat transfer problems of practical interests have been solved in an exact manner and the solutions are found in terms of elementary algebraic and transcendental functions. These problems represent steady state heat conduction in a solid rod, the unsteady cooling of a lumped parameter system and the steady state radiative heat transfer from a rectangular fin to the space, respectively. The corresponding exact solutions have been obtained in terms of a simple algebraic function, Lambert W function and Gauss's hypergeometric function, respectively. These analytical solutions match well with their numerical counterparts and are found to be finer than the earlier obtained approximate solutions. From these exact solutions one can get a better picture of the physical process unlike their approximate alternatives; moreover, these can be pretty useful in judging the accuracy of other approximate solutions and are valid for all parameter ranges.

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Nomenclature

A	[m^2]	heat transfer area
A_c	[m^2]	cross-sectional area
a,b,c	[-]	constants
c_a	[J/kg.K]	specific heat at temperature T_a
$c(T)$	[J/kg.K]	specific heat at temperature T
C_1, C_2	[-]	constants of integration
h	[J/s.m^2.K]	heat transfer coefficient
k_a	[J/s.m.K]	thermal conductivity at temperature T_a
$k(T)$	[J/s.m.K]	thermal conductivity at temperature T
L	[m]	length of rod
p	[-]	function of θ
t	[s]	time
T	[K]	temperature
T_s	[K]	radiation sink temperature
u	[-]	dummy variable
V	[m^3]	volume
x	[m]	distance variable
y	[-]	function of θ
z	[-]	dummy variable

Greek letters

β	[-]	dimensionless parameter for $k(T)$ and $c(T)$
$\in$	[-]	emissivity
ε	[-]	conduction radiation parameter
θ	[-]	dimensionless temperature
ρ	[kg/m^3]	density
σ	[W/m^2.K^4]	Stephan-Boltzmann constant ($=5.669\times10^{-8}$)
τ	[-]	dimensionless time
ξ	[-]	dimensionless distance

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