A Wholesaler's Optimal Ordering and Quantity Discount Policies for Deteriorating Items

Hidefumi Kawakatsu

***Abstract*—This study analyses the seller's (wholesaler's) decision to offer quantity discounts to the buyer (retailer). The seller purchases products from an upper-leveled supplier (manufacturer) and then sells them to the buyer who faces her customers' demand. The seller attempts to increase her profit by controlling the buyer's order quantity through a quantity discount strategy. The buyer tries to maximize her profit considering the seller's proposal. We formulate the above problem for deteriorating items as a Stackelberg game between the seller and buyer to analyze the existence of the seller's optimal quantity discount pricing policy which maximizes her total profit per unit of time. The same problem is also formulated as a cooperative game. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed formulation.**

***Index Terms*—quantity discounts, deteriorating items, total profit, Stackelberg game, cooperative game.**

I. INTRODUCTION

THis paper presents a model for determining optimal all-unit quantity discount strategies in a channel of one seller (wholesaler) and one buyer (retailer). Many researchers have developed models to study the effectiveness of quantity discounts. Quantity discounts are widely used by the seller with the objective of inducing the buyer to order larger quantities in order to reduce their total transaction costs associated with ordering, shipment and inventorying. Monahan[1] formulated the transaction between the seller and the buyer (see also [2], [3]), and proposed a method for determining an optimal all-unit quantity discount policy with a fixed demand. Lee and Rosenblatt[4] generalized Monahan's model to obtain the "exact" discount rate offered by the seller, and to relax the implicit assumption of a lot-for-lot policy adopted by the seller. Parlar and Wang[5] proposed a model using a game theoretical approach to analyze the quantity discount problem as a perfect information game. For more work: see also Sarmah et al.[6]. These models assumed that both the seller's and the buyer's inventory policies can be described by classical economic order quantity (EOQ) models. The classical EOQ model is a cost-minimization inventory model with a constant demand rate. It is one of the most successful models in all the inventory theories due to its simplicity and easiness.

In many real-life situations, retailers deal with perishable products such as fresh fruits, food-stuffs and vegetables. The inventory of these products is depleted not only by demand but also deterioration. Yang[7] has developed the model to determine an optimal pricing and a ordering policy for deteriorating items with quantity discount which is offered by the vendor. However, his model assumed that the deterioration rate of the vendor's inventory is equal to its rate of the retailer's inventory, and focused on the case where both the buyer's and the vendor's total profits can be approximated using Taylor series expansion.

In this study, we discuss a quantity discount problem between a seller (wholesaler) and a buyer (retailer) under circumstances where both the wholesaler's and the retailer's inventory levels of the product are depleted not only by demand but also by deterioration. The wholesaler purchases products from an upper-leveled supplier (manufacturer) and then sells them to the retailer who faces her/his customers' demand. The shipment cost is characterized by economies of density[8]. The wholesaler is interested in increasing her/his profit by controlling the retailer's order quantity through the quantity discount strategy. The retailer attempts to maximize her/his profit considering the wholesaler's proposal. Our previous work has formulated the above problem as a Stackelberg game between the wholesaler and the retailer to show the existence of the wholesaler's optimal quantity discount pricing policy which maximizes her/his total profit per unit of time[9]. In this study, we also formulate the same problem as a cooperative game. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.

II. NOTATION AND ASSUMPTIONS

The wholesaler uses a quantity discount strategy in order to improve her/his profit. The wholesaler proposes, for the retailer, an order quantity per lot along with the corresponding discounted wholesale price, which induces the retailer to alter her/his replenishment policy. We consider the two options throughout the present study as follows:

Option V_1**:** The retailer does not adopt the quantity discount proposed by the wholesaler. When the retailer chooses this option, she/he purchases the products from the wholesaler at an initial price in the absence of the discount, and she/he determines her/himself an optimal order quantity which maximizes her/his own total profit per unit of time.

Option V_2**:** The retailer accepts the quantity discount proposed by the wholesaler.

The main notations used in this paper are listed below:

Q_i**:** the retailer's order quantity per lot under Option $V_i (i = 1, 2)$.

S_i**:** the wholesaler's order quantity per lot under Option $V_i (i = 1, 2)$.

T_i**:** the length of the retailer's order cycle under Option $V_i (i = 1, 2)$.

h_s, h_b**:** the wholesaler's and the retailer's inventory holding costs per item and unit of time, respectively.

a_s, a_b**:** the wholesaler's and the retailer's ordering costs per lot, respectively.

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H. Kawakatsu is with Department of Economics & Information Science, Onomichi University, 1600 Hisayamadacho, Onomichi 722-8506 Japan (corresponding author to provide phone: +81-848-22-8312 (ex.617); fax: +81-848-22-5460; e-mail: kawakatsu@onomichi-u.ac.jp).

$\xi(T_i)$**:** the shipment cost per shipment from the wholesaler to the retailer.
c_s**:** the wholesaler's unit acquisition cost (unit purchasing cost from the upper-leveled manufacturer).
p_s**:** the wholesaler's initial unit selling price, i.e., the retailer's unit acquisition cost in the absence of the discount.
y**:** the discount rate for the wholesale price proposed by the wholesaler, i.e., the wholesaler offers a unit discounted price of $(1-y)p_s$ $(0 \leq y < 1)$.
p_b**:** the retailer's unit selling price, i.e., unit purchasing price for her/his customers.
θ_s**,** θ_b **:** the deterioration rates of the wholesaler's inventory and of the retailer's inventory, respectively ($\theta_s < \theta_b$).
μ**:** the constant demand rate of the product.

The assumptions in this study are as follows:

1) The retailer's inventory level is continuously depleted due to the combined effects of its demand and deterioration. In contrast, the wholesaler's inventory is depleted by deterioration during $[jT_i, (j+1)T_i)$ $(j = 0, 1, 2, \cdots)$, but at time jT_i her/his inventory level decreases by Q_i because of shipment to the retailer.
2) The rate of replenishment is infinite and the delivery is instantaneous.
3) Backlogging and shortage are not allowed.
4) The quantity of the item can be treated as continuous for simplicity.
5) Both the wholesaler and the retailer are rational and use only pure strategies.
6) The shipment cost is characterized by economies of density[8], i.e., the shipment cost per shipment decreases as the retailer's lot size increases. We assume, for simplicity, that $\xi(T_i) \equiv \beta - \alpha Q_i(T_i)$ (> 0).
7) The length of the wholesaler's order cycle is given by $N_i T_i$ under Option V_i $(i = 1, 2)$, where N_i is a positive integer. This is because the wholesaler can possibly improve her/his total profit by increasing the length of her/his order cycle from T_i to $N_i T_i$. In this case, the wholesaler's lot size can be obtained by the sum of N_i times of the retailer's lot size and the cumulative quantity of the waste products to be discarded during $[0, N_i T_i)$.

III. RETAILER'S TOTAL PROFIT

This section formulates the retailer's total profit per unit of time for the Option V_1 and V_2 available to the retailer.

A. Under Option V_1

If the retailer chooses Option V_1, her/his order quantity per lot and her/his unit acquisition cost are respectively given by $Q_1 = Q(T_1)$ and p_s, where p_s is the unit initial price in the absence of the discount. In this case, she/he determines her/himself the optimal order quantity $Q_1 = Q_1^*$ which maximize her/his total profit per unit of time.

Since the inventory is depleted due to the combined effect of its demand and deterioration, the inventory level, $I^{(b)}(t)$, at time t during $[0, T_1)$ can be expressed by the following differential equation:

$$dI^{(b)}(t)/dt = -\theta_b I^{(b)}(t) - \mu. \tag{1}$$

By solving the differential equation in Eq. (1) with a boundary condition $I^{(b)}(T_1) = 0$, the retailer's inventory level at time t is given by

$$I^{(b)}(t) = \rho \left[e^{\theta_b (T_1 - t)} - 1 \right], \tag{2}$$

where $\rho = \mu/\theta_b$.

Therefore, the initial inventory level, $I^{(b)}(0)$ $(= Q_1 = Q$ (T_1), in the order cycle becomes

$$Q(T_1) = \rho \left(e^{\theta_b T_1} - 1 \right). \tag{3}$$

On the other hand, the cumulative inventory, $A(T_1)$, held during $[0, T_1)$ is expressed by

$$A(T_1) = \int_0^{T_1} I^{(b)}(t)dt = \rho \left[\frac{\left(e^{\theta_b T_1} - 1\right)}{\theta_b} - 1 \right]. \tag{4}$$

Hence, the retailer's total profit per unit of time under Option V_1 is given by

$$\begin{aligned} \pi_1(T_1) &= \frac{p_b \int_0^{T_1} \mu dt - p_s Q(T_1) - h_b A(T_1) - a_b}{T_1} \\ &= \rho(p_b \theta_b + h_b) - \frac{\left(p_s + \frac{h_b}{\theta_b}\right) Q(T_1) + a_b}{T_1}. \end{aligned} \tag{5}$$

In the following, the results of analysis are briefly summarized:

The proof is given in A.

There exists a unique finite $T_1 = T_1^*$ (> 0) which maximizes $\pi_1(T_1)$ in Eq. (5). The optimal order quantity is therefore given by

$$Q_1^* = \rho \left(e^{\theta_b T_1^*} - 1 \right). \tag{6}$$

The total profit per unit of time becomes

$$\pi_1(T_1^*) = \rho \left[(p_b \theta_b + h_b) - \theta_b \left(p_s + \frac{h_b}{\theta_b} \right) e^{\theta_b T_1^*} \right]. \tag{7}$$

B. Under Option V_2

If the retailer chooses Option V_2, the order quantity and unit discounted wholesale price are respectively given by $Q_2 = Q_2(T_2) = \rho \left(e^{\theta_b T_2} - 1 \right)$ and $(1-y)p_s$. The retailer's total profit per unit of time can therefore be expressed by

$$\begin{aligned} \pi_2(T_2, y) = {} & \rho(p_b \theta_b + h_b) \\ & - \frac{\left[(1-y)p_s + \frac{h_b}{\theta_b}\right] Q_2(T_2) + a_b}{T_2}. \end{aligned} \tag{8}$$

IV. WHOLESALER'S TOTAL PROFIT

This section formulates the wholesaler's total profit per unit of time, which depends on the retailer's decision. Figure 1 shows both the wholesaler's and the retailer's transitions of inventory level in the case of $N_i = 3$.

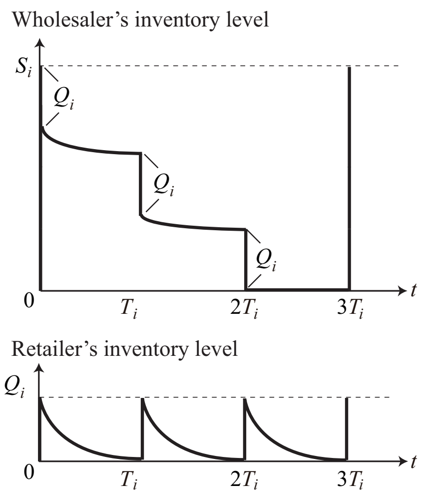

Fig. 1. Transition of Inventory Level ($N_i = 3$)

A. Total Profit under Option V_1

If the retailer chooses Option V_1, her/his order quantity per lot and unit acquisition cost are given by Q_1 and p_s, respectively. The length of the wholesaler's order cycle can be divided into N_1 shipping cycles ($N_1 = 1, 2, 3, \cdots$) as described in assumption 7), where N_1 is also a decision variable for the wholesaler.

The wholesaler's inventory is depleted only due to deterioration during $[(j-1)T_1, jT_1)$ in jth shipping cycle ($j = 1, 2, \cdots, N_1$). Therefore, the wholesaler's inventory level, $I^{(s)}(t)$, at time t can be expressed by the following differential equation:

$$dI^{(s)}(t)/dt = -\theta_s I^{(s)}(t), \tag{9}$$

with a boundary condition $I^{(s)}(jT_1) = z_j(T_1)$, where $z_j(T_1)$ denotes the remaining inventory at the end of the jth shipping cycle. By solving the differential equation in Eq. (9), the wholesaler's inventory level, $I^{(s)}(t) = I_j^{(s)}(t)$, at time t in jth shipment cycle is given by

$$I_j^{(s)}(t) = z_j(T_1) e^{\theta_s (jT_1 - t)}. \tag{10}$$

It can easily be confirmed that the inventory level at the end of the $(N_1 - 1)$th shipping cycle becomes Q_1, i.e. $z_{N_1-1}(T_1) = Q_1$, as also shown in Fig. 1. By induction, we have

$$z_j(T_1) = Q(T_1) \left[e^{(N_1 - j)\theta_s T_1} - 1 \right] / \left[e^{\theta_s T_1} - 1 \right]. \tag{11}$$

The wholesaler's order quantity, $S_1 = S(N_1, T_1)$ $(= z_0(T_1))$ per lot is then given by

$$S(N_1, T_1) = Q(T_1) \left[e^{N_1 \theta_s T_1} - 1 \right] / \left[e^{\theta_s T_1} - 1 \right]. \tag{12}$$

On the other hand, the wholesaler's cumulative inventory, $B_j(T_1)$, held during jth shipping cycle is expressed by

$$\begin{aligned} B_j(T_1) &= \int_{(j-1)T_1}^{jT_1} I_j^{(s)}(t) dt \\ &= \frac{Q(T_1)}{\theta_s} \left[e^{(N_i - j)\theta_s T_1} - 1 \right]. \end{aligned} \tag{13}$$

The wholesaler's cumulative inventory, held during $[0, N_1 T_1)$ becomes

$$\begin{aligned} B(N_1, T_1) &= \sum_{j=1}^{N_1 - 1} B_j(T_1) \\ &= \frac{Q_1(T_1)}{\theta_s} \left(\frac{e^{N_1 \theta_s T_1} - 1}{e^{\theta_s T_1} - 1} - N_1 \right). \end{aligned} \tag{14}$$

Hence, for a given N_1, the wholesaler's total profit per unit of time under Option V_1 is given by

$$\begin{aligned} P_1(N_1, T_1^*) &= \frac{1}{N_1 T_1^*} \cdot \big[p_s N_1 Q(T_1^*) - N_1 \xi(T_1) \\ &\qquad - c_s S(N_1, T_1^*) - h_s B(N_1, T_1^*) - a_s \big] \\ &= \frac{\left(p_s + \frac{h_s}{\theta_s} + \alpha \right) Q(T_1^*) - \beta}{T_1^*} \\ &\quad - \frac{\left(c_s + \frac{h_s}{\theta_s} \right) S(N_1, T_1^*) + a_s}{N_1 T_1^*}. \end{aligned} \tag{15}$$

B. Total Profit under Option V_2

When the retailer chooses Option V_2, she/he purchases $Q_2 = Q(T_2)$ units of the product at the unit discounted wholesale price $(1 - y)p_s$. In this case, the wholesaler's order quantity per lot under Option V_2 is expressed as $S_2 = S(N_2, T_2)$, accordingly the wholesaler's total profit per unit of time under Option V_2 is given by

$$\begin{aligned} P_2(N_2, T_2, y) &= \frac{1}{N_2 T_2} \cdot \big[(1 - y) p_s N_2 Q(T_2) - N_2 \xi(T_2) \\ &\qquad - c_s S(N_2, T_2) - h_s B(N_2, T_2) - a_s \big] \\ &= \frac{\left[(1 - y) p_s + \frac{h_s}{\theta_s} + \alpha \right] Q(T_2) - \beta}{T_2} \\ &\quad - \frac{\left(c_s + \frac{h_s}{\theta_s} \right) S(N_2, T_2) + a_s}{N_2 T_2}, \end{aligned} \tag{16}$$

where

$$Q(T_2) = \rho \left(e^{\theta_b T_2} - 1 \right), \tag{17}$$

$$S(N_2, T_2) = Q(T_2) \left[e^{N_2 \theta_s T_2} - 1 \right] / \left[e^{\theta_s T_2} - 1 \right]. \tag{18}$$

V. RETAILER'S OPTIMAL RESPONSE

This section discusses the retailer's optimal response. The retailer prefers Option V_1 over Option V_2 if $\pi_1^* > \pi_2(T_2, y)$, but when $\pi_1^* < \pi_2(T_2, y)$, she/he prefers V_2 to V_1. The retailer is indifferent between the two options if $\pi_1^* = \pi_2(T_2, y)$, which is equivalent to

$$y = \frac{\left(p_s + \frac{h_b}{\theta_b} \right) \left[Q(T_2) - \rho \theta_b T_2 e^{\theta_b T_1^*} \right] + a_b}{p_s Q(T_2)}. \tag{19}$$

Let us denote, by $\psi(T_2)$, the right-hand-side of Eq. (19). It can easily be shown from Eq. (19) that $\psi(T_2)$ is increasing in T_2 $(\geq T_1^*)$.

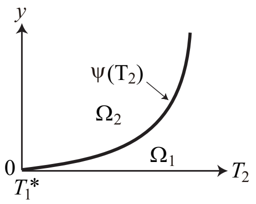

Fig. 2. Characterization of retailer's optimal responses

VI. WHOLESALER'S OPTIMAL POLICY UNDER THE NON-COOPERATIVE GAME

The wholesaler's optimal values for T_2 and y can be obtained by maximizing her/his total profit per unit of time considering the retailer's optimal response which was discussed in Section V. Henceforth, let Ω_i $(i = 1, 2)$ be defined by

$$\begin{aligned}\Omega_1 &= \{(T_2, y) \mid y \le \psi(T_2))\},\\ \Omega_2 &= \{(T_2, y) \mid y \ge \psi(T_2))\}.\end{aligned}$$

Figure 2 depicts the region of Ω_i $(i = 1, 2)$ on the (T_2, y) plane.

A. Under Option V_1

If $(T_2, y) \in \Omega_1 \setminus \Omega_2$ in Fig. 2, the retailer will naturally select Option V_1. In this case, the wholesaler can maximize her/his total profit per unit of time independently of T_2 and y on the condition of $(T_2, y) \in \Omega_1 \setminus \Omega_2$. Hence, the wholesaler's locally maximum total profit per unit of time in $\Omega_1 \setminus \Omega_2$ becomes

$$P_1^* = \max_{N_1 \in N} P_1(N_1, T_1^*), \tag{20}$$

where N signifies the set of positive integers.

B. Under Option V_2

On the other hand, if $(T_2, y) \in \Omega_2 \setminus \Omega_1$, the retailer's optimal response is to choose Option V_2. Then the wholesaler's locally maximum total profit per unit of time in $\Omega_2 \setminus \Omega_1$ is given by

$$P_2^* = \max_{N_2 \in N} \hat{P}_2(N_2), \tag{21}$$

where

$$\hat{P}_2(N_2) = \max_{(T_2, y) \in \Omega_2 \setminus \Omega_1} P_2(N_2, T_2, y). \tag{22}$$

More precisely, we should use "sup" instead of "max" in Eq. (22).

For a given N_2, we show below the existence of the wholesaler's optimal quantity discount pricing policy $(T_2, y) = (T_2^*, y^*)$ which attains Eq. (22). It can easily be proven that $P_2(N_2, T_2, y)$ in Eq. (16) is strictly decreasing in y, and consequently the wholesaler can attain $\hat{P}_2(N_2)$ in Eq. (22) by letting $y \to \psi(T_2) + 0$. By letting $y = \psi(T_2)$ in Eq. (16), the total profit per unit of time on $y = \psi(T_2)$ becomes

$$\begin{aligned}P_2(N_2, T_2) = {} & \rho\,(p_s + h_b/\theta_b)\,\theta_b e^{\theta_b T_1^*}\\ & - \frac{1}{N_2 T_2} \cdot \big[C \cdot S(N_2, T_2) - H(N_2) Q(T_2)\\ & + (a_b + \beta) N_2 + a_s\big],\end{aligned} \tag{23}$$

where

$$\begin{aligned}C &= (c_s + h_s/\theta_s), & (24)\\ H(N_2) &= (h_s/\theta_s - h_b/\theta_b + \alpha) N_2. & (25)\end{aligned}$$

Let us now define $L(T_2)$ as follows:

$$\begin{aligned}L(T_2) \equiv {} & C\theta_s T_2 Q(T_2)\\ & \times \frac{N_2 e^{N_2\theta_s T_2}(e^{\theta_s T_2} - 1) - e^{\theta_s T_2}(e^{N_2\theta_s T_2} - 1)}{(e^{\theta_s T_2} - 1)^2}\\ & + \big[\rho\theta_b e^{\theta_b T_2} T_2 - Q(T_2)\big]\\ & \times \left[C \frac{e^{N_2\theta_s T_2} - 1}{e^{\theta_2 T_2} - 1} - H(N_2)\right].\end{aligned} \tag{26}$$

We here summarize the results of analysis in relation to the optimal quantity discount policy which attains $\hat{P}_2(N_2)$ in Eq. (22) when N_2 is fixed to a suitable value.

The proofs are given in Appendix B.

1) $N_2 = 1$:

- $(c_s + h_b/\theta_b - \alpha) > 0$:

  In this subcase, there exists a unique finite $\tilde{T}_2$ $(> T_1^*)$ which maximizes $P_2(N_2, T_2)$ in Eq. (23), and therefore (T_2^*, y^*) is given by

  $$(T_2^*, y^*) \to (\tilde{T}_2, \tilde{y}), \tag{27}$$

  where $\tilde{y} = \psi(\tilde{T}_2)$.

  The wholesaler's total profit then becomes

  $$\begin{aligned}\hat{P}_2(N_2) = {} & \rho\theta_b\big[(p_s + h_b/\theta_b)\, e^{\theta_b T_1^*}\\ & - (c_s + h_b/\theta_b - \alpha)\, e^{\theta_b T_2^*}\big].\end{aligned} \tag{28}$$

- $(c_s + h_b/\theta_b - \alpha) \le 0$:

  In this subcase, the optimal policy can be expressed by

  $$(T_2^*, y^*) \to (\hat{T}_2, 1), \tag{29}$$

  where $\hat{T}_2$ $(> T_1^*)$ is the unique finite positive solution to $\psi(T_2) = 1$.

  The wholesaler's total profit is therefore given by

  $$\hat{P}_2(N_2) = -\frac{(c_2 - \alpha) Q(\hat{T}_2)}{\hat{T}_2} - \beta - a_s. \tag{30}$$

2) $N_2 \ge 2$:

Let us define $T_2 = \tilde{T}_2$ $(> T_1^*)$ as the unique solution (if it exists) to

$$L(T_2) = (a_b + \beta) N_2 + a_s. \tag{31}$$

In this case, the optimal quantity discount pricing policy is given by Eq. (27).

C. Under Option V_1 and V_2

In the case of $(T_2, y) \in \Omega_1 \cap \Omega_2$, the retailer is indifferent between Option V_1 and V_2. For this reason, this study confines itself to a situation where the wholesaler does not use a quantity discount policy $(T_2, y) \in \Omega_1 \cap \Omega_2$.

D. Optimal value for N_i

For a given T_i, we here derive a lower bound and an upper bound for the optimal value of $N_i = N_i^*$ ($N_i^* = 1, 2, 3, \cdots$) which maximizes $P_1(N_1, T_1^*)$ in Eq. (15) and $P_2(N_2, T_2, y))$ in Eq. (16).

Let $K(T_i)$ be defined by

$$K(T_i) \equiv (c_s + h_s/\theta_s)Q(T_i)/(e^{\theta_s T_i} - 1). \quad (32)$$

In the following, the results of analysis are briefly summarized.

The proofs are shown in Appendix C.

1) Lower bound $N_i = N_i^{(L)}(T_i)$ ($\leq N_i^*$):
   - $(e^{\theta_s T_i} - 1)^2 \geq a_s/K(T_i)$:
     $N_i^{(L)}(T_i) = 1$.
   - $(e^{\theta_s T_i} - 1)^2 < a_s/K(T_i)$:
     There exists a unique finite $N_i^{(L)}(T_i)$ (≥ 1) which is the solution to

$$N_i e^{N_i \theta_s T_i^*}(e^{\theta_s T_i} - 1) - (e^{N_i \theta_s T_i} - 1) = a_s/K(T_i). \quad (33)$$

2) Upper bound $N_i = N_i^{(U)}(T_i)$ ($\geq N_i^*$):
   There exists a unique finite $N_i^{(U)}(T_i)$ ($\geq N_i^{(L)}(T_i)$) which is the solution to

$$N_i e^{(N_i-1)\theta_s T_i}(e^{\theta_s T_i} - 1) - (e^{N_i \theta_s T_i} - 1) = a_s/K(T_i). \quad (34)$$

The above results indicate that the optimal N_i^* satisfies

$$1 \leq N_i^{(L)}(T_i) \leq N_i^* < N_i^{(U)}(T_i). \quad (35)$$

In the above, it should be reminded that we can use $T_1 = T_1^*$ under Option V_1.

VII. WHOLESALER'S OPTIMAL POLICY UNDER THE COOPERATIVE GAME

This section discusses a cooperative game between the wholesaler and the retailer. We focus on the case where the wholesaler and the retailer maximize their joint profit. We here introduce some more additional notations N_3, T_3 and Q_3, which correspond to N_2, T_2 and Q_2 respectively, under Option V_2 in the previous section.

Let $J(N_3, T_3, y)$ express the joint profit function per unit of time for the wholesaler and the retailer, i.e., let $J(N_3, T_3, y) = P2(N_3, T_3, y) + \pi_2(T_3, y)$, we have

$$J(N_3, T_3, y) = \rho(p_b\theta_b + h_b) - \frac{1}{N_3 T_3} \cdot \big[C \cdot S(N_3, T_3) - H(N_3)Q(T_3) + (a_b + \beta)N_3 + a_s\big]. \quad (36)$$

It can easily be proven from Eq. (36) that $J(N_3, T_3, y)$ is independent of y and we have $J(N_3, T_3, y) = P_2(N_3, T_3, \psi(T_3)) + \pi_1^*$. This signifies that the optimal quantity discount policy $(T_3, y) = (T_3^*, y^*)$ which maximizes $J(N_3, T_3, y)$ in Eq. (36) is given by (T_2^*, y^*) as shown in Section VI. This is simply because, in this study, the inventory holding cost is assumed to be independent of the value of the item.

TABLE I
SENSITIVITY ANALYSIS

(a) Under Option V_1

a_s	Q_1^*	p_1	$S_1^*(N_1^*)$	P_1^*
500	47.35	300.00	47.35(1)	910.75
1000	47.35	300.00	99.09(2)	861.24
2000	47.35	300.00	99.09(2)	804.78
3000	47.35	300.00	155.61(3)	752.21

(b) Under Option V_2

a_s	Q_2^*	p_2^*	$S_2^*(N_2^*)$	P_2^*
500	107.63	285.01	107.63(1)	998.56
1000	117.65	281.63	117.65(1)	972.8
2000	135.77	275.49	135.77(1)	926.23
3000	152.04	270.04	152.04(1)	884.43

VIII. NUMERICAL EXAMPLES

Table I reveals the results of sensitively analysis in reference to Q_1^*, p_1 ($= p_s$), S_1^* ($= S(N_1^*, T_1^*)$), N_1^*, P_1^*, Q_2^* ($= Q(T_2^*)$), p_2^* (=$(1 - y^*)p_s$), S_2^* ($= S(N_2^*, T_2^*)$), N_2^*, P_2^* for $(c_s, p_s, p_b, a_b, h_s, h_b, \theta_s, \theta_b, \mu, \alpha, \beta) = (100, 300, 600, 1200, 1, 1.1, 0.01, 0.015, 5, 2, 1000)$ when $a_s = 500$, 10 00, 2000 and 3000.

In Table I(a), we can observe that both S_1^* and N_1^* are non-decreasing in a_s. As mentioned in Section II, under Option V_1, the retailer does not adopt the quantity discount offered by the wholesaler, which signifies that the wholesaler cannot control the retailer's ordering schedule. In this case, the wholesaler's cost associated with ordering should be reduced by increasing her/his own length of order cycle and lot size by means of increasing N_1.

Table I(b) shows that, under Option V_2, S_2^* increases with a_s, in contrast, N_2^* takes a constant value, i.e., we have $N_2^* = 1$. Under Option V_2, the retailer accepts the quantity discount proposed by the wholesaler. The wholesaler's lot size can therefore be increased by stimulating the retailer to alter her/his order quantity per lot through the quantity discount strategy. If the wholesaler increases N_2 one step, her/his lot size also significantly jumps up since N_2 takes a positive integer. Under this option, the wholesaler should increase her/his lot size using the quantity discount rather than increasing N_2 when a_s takes larger values. We can also notice in Table I that we have $P_1^* < P_2^*$. This indicates that using the quantity discount strategy can increase the wholesaler's total profit per unit of time.

IX. CONCLUSION

In this study, we have discussed a quantity discount problem between a wholesaler and a retailer under circumstances where both the wholesaler's and the retailer's inventory levels of the product are depleted not only by demand but also by deterioration. The wholesaler is interested in increasing her/his profit by controlling the retailer's order quantity through the quantity discount strategy. The retailer attempts to maximize her/his profit considering the wholesaler's proposal. We have formulated the above problem as a Stackelberg game between the wholesaler and the retailer to show the existence of the wholesaler's optimal quantity discount policy that maximizes her/his total profit per unit of time in the same manner as our previous work[9]. In this study, we have also formulated the same problem as a cooperative game. The result of our analysis reveals that the wholesaler is indifferent between the cooperative and non-cooperative options. It should be pointed out that our results are obtained under the situation where the inventory holding cost is independent of the value of the item. The relaxation of such a restriction is an interesting extension.

APPENDIX A

This appendix shows the existence of a unique optimal order quantity which maximizes the retailer's total profit per unit of time under Option V_1.

By differentiating $\pi_1(T_1)$ in Eq. (5) with respect to T_1, we have

$$\frac{d}{dT_1}\pi_1(T_1) = -\frac{\rho\left(p_s+\frac{h_b}{\theta_b}\right)\left[\theta_b T_1 e^{\theta_b T_1}-(e^{\theta_b T_1}-1)\right]-a_b}{T_1^2}. \tag{37}$$

Then $\frac{d}{dT_1}\pi_1(T_1)\geq 0$ agrees with

$$\theta_b T_1 e^{\theta_b T_1}-(e^{\theta_b T_1}-1) \leq \frac{a_b}{\rho\left(p_s+\frac{h_b}{\theta_b}\right)}. \tag{38}$$

Let $L_b(T_1)$ express the left-hand-side of Inequality (38), and we have

$$L_b'(T_1) = \theta_b^2 T_1 e^{\theta_b T_1}\ (>0), \tag{39}$$

$$L_b(0) = 0\ \left(< \frac{a_b}{\rho\left(p_s+\frac{h_b}{\theta_b}\right)}\right), \tag{40}$$

$$\lim_{T_1\to+\infty} L_1(T_1) = +\infty. \tag{41}$$

On the basis of the above results, we can show that there exists a unique finite T_1^* (>0). The retailer's optimal order quantity per lot can therefore be given by Eq. (6).

APPENDIX B

In this appendix, we discuss the existence of the optimal quantity discount pricing policy which attains $\hat{P}_2(N_2)$ in Eq. (22) when N_2 is fixed to a suitable value.

1) $N_2=1$:

By differentiating $P_2(N_2,T_2)$ in Eq. (23) with respect to T_2, we have

$$\frac{\partial}{\partial T_2}P_2(N_2,T_2) = -\frac{1}{T_2^2}\Bigg\{\left[\rho\theta_b e^{\theta_b T_2}-Q(T_2)\right]\left(c_s+\frac{h_b}{\theta_b}-\alpha\right) -(a_b+a_s+\beta)\Bigg\}. \tag{42}$$

It can easily be shown from Eq. (42) that the sign of $\frac{\partial}{\partial T_2}P_2(N_2,T_2)$ is positive when $(c_s+h_b/\theta_b-\alpha)=0$. In contrast, in the case of $\left(c_s+\frac{h_b}{\theta_b}-\alpha\right)\gtrless 0$, $\frac{\partial}{\partial T_2}P_2(N_2,T_2)\geq 0$ agrees with

$$\theta_b T_2 e^{\theta_b T_2}-\left(\theta_b e^{\theta_b T_2}-1\right) \ \substack{\leq\\ \geq} \ \frac{a_b+a_s+\beta}{\rho(c_s+h_b/\theta_b-\alpha)}. \tag{43}$$

Let $L_1(T_2)$ express the left-hand-side of Inequality (43), we have

$$L_1'(T_2) = \theta_b^2 T_2 e^{\theta_b T_2}\ (>0), \tag{44}$$

$$L_1(T_1^*) = \frac{a_b}{\rho(p_s+h_b/\theta_b)}, \tag{45}$$

$$\lim_{T_2\to+\infty} L_1(T_2) = +\infty. \tag{46}$$

From Eqs. (44), (45) and (46), the existence of an optimal quantity discount pricing policy can be discussed for the following two subcases:

- $(c_s+h_b/\theta_b-\alpha)>0$:
  Equiation (45) yields
  $$L_1(T_1^*) < \frac{a_b+a_s+\beta}{\rho(c_s+h_b/\theta_b-\alpha)}. \tag{47}$$
  Equiations (44), (46) and (47) indicate that the sign of $\frac{\partial}{\partial T_2}P_2(N_2,T_2)$ changes from positive to negative only once. This signifies that $P_2(N_2,T_2)$ first increases and then decreases as T_2 increases, and thus there exists a unique finite $\tilde{T}_2$ $(>T_1^*)$ which maximizes $P_2(N_2,T_2)$ in Eq. (23). Hence, (T_2^*,y^*) is given by Eq. (27).
- $(c_s+h_b/\theta_b-\alpha)\leq 0$:
  In this subcase, $P_2(N_2,T_2)$ is increasing in T_2, and consequently the optimal policy can be expressed by Eq. (29).

2) $N_2\geq 2$:

By differentiating $P_2(N_2,T_2)$ in Eq. (23) with respect to T_2, we have

$$\frac{\partial}{\partial T_2}P_2(N_2,T_2) = -\frac{L(T_2)-(a_b+\beta)N_2-a_s}{N_2T_2^2}, \tag{48}$$

where $L(T_2)$ is defined by Eq. (26).

Then $\frac{\partial}{\partial T_2}P_2(N_2,T_2)\geq 0$ agrees with

$$L(T_2) \leq (a_b+\beta)N_2+a_s. \tag{49}$$

If we assume that there exists a unique solution to Eq. (31), the optimal quantity discount pricing policy can be given by Eq. (27).

APPENDIX C

For a given T_i, this appendix shows the existence of a lower bound and an upper bound for the optimal value of $N_i=N_i^*$ $(N_i^*=1,2,3,\cdots)$ which maximizes $P_1(N_1,T_1^*)$ in Eq. (15) and $P_2(N_2,T_2,y)$ in Eq. (16).

Let $G(T_i)$ be defined by

$$G(T_i) \equiv \frac{(p_i+h_s/\theta_s+\alpha)Q(T_i)-\beta}{T_i}, \tag{50}$$

then $P_1(N_1,T_1^*)$ and $P_2(N_2,T_2,y)$ can be rewritten as

$$P_i(N_i) = G(T_i)-\frac{(e^{\theta_s T_i}-1)+a_s/K(T_i)}{N_iT_i/K(T_i)}, \tag{51}$$

where $p_1=p_s$, $p_2=(1-y)p_s$ and $K(T_i)$ is defined by Eq. (32).

1) Lower bound $N_i=N_i^{(L)}(T_i)$ $(\leq N_i^*)$:

By calculating the difference between $P_i(N_i+1)$ and $P_i(N_i)$, we have

$$\Delta P_i^{(L)} = P_i(N_1+1) - P_i(N_i) = \frac{1}{(N_i+1)N_iT_i/K(T_i)} \cdot \Big[(\gamma^{N_i}-1) - N_i\gamma^{N_i}(\gamma-1) + a_s/K(T_i)\Big], \quad (52)$$

where $\gamma = e^{\theta_s T_i}$.
Then $\Delta P_i^{(L)} \geq 0$ agrees with

$$(\gamma^{N_i}-1) - N_i\gamma^{N_i}(\gamma-1) \geq -a_s/K(T_i). \quad (53)$$

Let us denote, by $L^{(L)}(N_i)$, the left-hand-side of Inequality (53), and we have

$$\Delta L^{(L)} = L^{(L)}(N_i+1) - L^{(L)}(N_i) = -(N_i+1) \times\gamma^{N_i}(\gamma-1)^2 \ (<0), \quad (54)$$

$$L^{(L)}(1) = -(\gamma-1)^2, \quad (55)$$

$$\lim_{N_i\to+\infty} L^{(L)}(N_i) = -\infty \ (< -a_s/K(T_i)). \quad (56)$$

From Eqs. (54), (55) and (56), we can clarify the conditions where a lower bound $N_i^{(L)}(T_i)$ exists as shown below.

- $(e^{\theta_s T_i}-1)^2 \geq a_s/K(T_i)$:
  In this subcase, $P_i(N_i)$ is non-increasing in N_i, and consequently $N_i^{(L)}(T_i) = 1$.
- $(e^{\theta_s T_i}-1)^2 < a_s/K(T_i)$:
  In this subcase, the sign of $\Delta P_i^{(L)}$ changes from positive to negative only once, and thus there exists a unique finite $N_i^{(L)}(T_i)$ (≥ 1) which is the solution to

$$N_i e^{N_i\theta_s T_i}(e^{\theta_s T_i}-1) - (e^{N_i\theta_s T_i}-1) = a_s/K(T_i). \quad (57)$$

2) Upper bound $N_i = N_i^{(U)}(T_i)$ $(\geq N_i^*)$:
By calculating the difference between $P_i(N_i)$ and $P_i(N_i-1)$, we have

$$\Delta P_i^{(U)} = P_i(N_1) - P_i(N_i-1) = \frac{1}{(N_i-1)N_iT_i/K(T_i)} \times\Big[(\gamma^{N_i}-1) - N_i\gamma^{N_i-1}(\gamma-1) + a_s/K(T_i)\Big]. \quad (58)$$

Then $\Delta P_i^{(U)} \geq 0$ agrees with

$$(\gamma^{N_i}-1) - N_i\gamma^{N_i-1}(\gamma-1) \geq -a_s/K(T_i). \quad (59)$$

Let $L^{(U)}(N_i)$ express the left-hand-side of Inequality (59), we have

$$\Delta L^{(U)} = L^{(U)}(N_i) - L^{(U)}(N_i-1) = -(N_i-1) \times\gamma^{N_i-2}(\gamma-1)^2 \ (<0), \quad (60)$$

$$L^{(U)}(1) = 0 \ (> -a_s/K(T_i)), \quad (61)$$

$$\lim_{N_i\to+\infty} L^{(U)}(N_i) = -\infty \ (< -a_s/K(T_i)). \quad (62)$$

These observations can clarify that there exists a unique finite $N_i^{(U)}(T_i)$ $(\geq N_i^{(L)}(T_i))$ which is the solution to

$$N_i e^{(N_i-1)\theta_s T_i}(e^{\theta_s T_i}-1) - (e^{N_i\theta_s T_i}-1) = a_s/K(T_i). \quad (63)$$

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