Two-Period Inventory Clearance Problem with Reference Price Effect of Demand

Takeshi Koide, Hiroaki Sandoh, *Member, IAENG*

Abstract— **The target products in this paper is perishable and cannot be sold in the following periods. The decision maker can sell the products at a discount price to stimulate demand when he/she judges the demand in the period is less than originally expected. The discount might increase the revenue of the period but it decreases the reference price of consumers. The demands in the following periods are declined caused by the decreased reference price. The decision maker should take the reference price effect into account so as to increase long-term profit. We analyzed an expected profit function in a single period with consideration for the reference price effects of consumers in our previous study. This paper proposes a procedure to explore an optimal pricing in two periods based on our previous results. Numerical experiments show the importance of the reference price effects.**

Index Terms— **clearance, inventory, optimal pricing, reference price effect.**

I. Introduction

A range of very perishable prepared foods, such as sushi, sliced raw fish, fried meals, salad, are sold in retail stores as package products in many countries. Some of them are cooked and packed before opening time of the day and expires at the end of the day due to deterioration of freshness. Firms try to predict the demand of the day to supply appropriate amount of products. When the predicted demand is greater than in reality, some products are unsold and then they are disposed or salvaged with an extra operating cost.

Firms sometimes discount the products during the day if a decision maker of the firms judges that some products will be unsold. The discount sale stimulates demand and the number of unsold products is reduced. An appropriated discount increases the revenue and profit of the day, but it also drops consumers' reference price for the products. The reference price roles as an anchor price to judge whether a sales price is a gain or a loss for the consumers. The reference price is well-known as the reference point in the prospect theory proposed by Kahneman and Tversky [1, 2, 3]. The falling-off of the reference price makes some consumers feel unwilling to purchase the products at a regular price in the future. The decreasing of demand caused by the decline of reference prices is called as the reference price effect on demand. The discount sales one day has a risk to reduce profit in the long run. From a long-term business perspective, firms should discount sales prices advisedly.

There are some studies discussing promotional planning problems with the reference price effect to derive optimal pricing policies to maximize long-term revenues [4, 5, 6]. In their models, the promotional discount aims to stimulate demand temporarily and not to decrease the disposals of unsold products. The inventory level, namely the quantity of the products, is not integrated into their models. Petruzzi and Dada reviewed past literature discussing both discount pricing and inventory control, and they also proposed a general model [7]. Their model derives both an optimal price and an optimal inventory level, but it does not consider the reference price effect.

Our previous work [8] analyzed an expected profit function analytically to treat stochastic demand and inventory level in a single period model as a fundamental study for multi-period optimal pricings. It has been shown that the profit function for loss-neutral consumers is concave and the optimal clearance price is derived through a first order condition. For loss-averse and loss-seeking consumers, the function has been proved to be concave or bimodal. In our latest work [9], the profit functions are investigated in more detail to propose a procedure to explore optimal discount pricings.

In this study, we extend our previous model to apply to a discount pricing problem in two periods. Mathematical analysis reveals that a profit function in two periods is also concave if target consumers are loss-neutral. A procedure is proposed to explore optimal pricing in two periods for loss-averse and loss-seeking consumers. Through numerical experiments, the sensitivity analysis of the optimal price and the maximum profit is conducted with respect to initial reference price and inventory level. The results in the numerical experiments appeals that the reference price and inventory level have a major influence on an expected profit.

Section II introduces the backgrounds of this study and defines the model discussed in this study. Section III shows some properties on an optimal discount pricing in a single period, which are summary of the results derived in our previous studies [8, 9]. Section IV extends the context in Section III to an optimal discount pricing in two periods. Numerical studies are conducted in Section V to investigate the sensitivity of optimal prices and an objective value against some parameters such as average inventory level and initial reference price.

Manuscript received July 06, 2012. This work was partly supported by Grant-in-Aid for Scientific Research(C), 2011, No.22510147 and MEXT, Japan.

Takeshi Koide is with Department of Intelligence and Informatics, Konan University, Kobe 658-8501, Japan (corresponding author, e-mail: koide@konan-u.ac.jp).

Hiroaki Sandoh is with Department of Management Science and Business Administration, Graduate School of Economics, Osaka University, Toyonaka 560-0043, Japan (e-mail: sandoh@econ.osaka-u.ac.jp)

(Advance online publication: 27 August 2012)

II. Backgrounds and Definitions

A. Optimal Pricing and Inventory Level

Consider a price-setting firm which deals in a single type of perishable products. The firm cannot be sold the unsold products in the following periods. The firm determines the sales price p and the inventory level q before opening hour to maximize the expected profit. In a single period case, the optimum price p^* and the optimum inventory level q^* can be solved within a framework of the famous newsvendor problem [7, 10].

Let $D(p) = \beta_0 - \beta_1 p + \varepsilon_d$ be the stochastic fundamental demand function with respect to p, where β_0 , $\beta_1 > 0$ and ε_d is a random variable with mean 0 and range $[d_L, d_H]$. In accordance with Petruzzi and Dada [7], define new variables $z = q - E[D(p)]$, so-called stocking factor, the optimal price p^* which maximizes the expected profit $\Pi(p, q)$ is given by the following equation:

$$p^* = p^0 - \frac{\Theta(z)}{2\beta_1}, \tag{1}$$

where $\Theta(z)$ is the expected amounts of shortages and p^0 is the optimal price which maximizes the riskless expected profit $E[(p - c)D(p)]$. From (1), the optimum p^* only depends on z. With letting $p = p^*(z)$, the expected profit $\Pi(p^*(z), z)$ becomes just a function with respect to z, then the optimal stocking factor z^* can be derived, then both p^* and q^* are also derived.

B. Reference Price Effect

The demand function comprising the reference price effects is modeled as follows:

$$D(p, r) = \beta_0 - \beta_1 p + \beta_{2G}[r - p]^+ - \beta_{2L}[p - r]^+ + \varepsilon_d, \tag{2}$$

where β_{2G}, $\beta_{2L} > 0$ and $[x]^+ = \max(x, 0)$. Consumers perceive a sold product as a gain if the sales price p is less than their reference prices r, and the demand is increased by $\beta_{2G}(r - p)$ from the fundamental demand $D(p)$. If the sales price p is above the reference price r, consumers perceive the price as a loss and the demand is decreased by $\beta_{2L}(p - r)$. When it holds $\beta_{2G} < \beta_{2L}$, $\beta_{2G} = \beta_{2L}$, and $\beta_{2G} > \beta_{2L}$, the consumers are respectively referred as to loss-averse (LA), loss-neutral (LN), and loss-seeking (LS) [1].

The consumers update their reference prices depending on the sales prices in successive periods. It is assumed that the reference price r_{t+1} on period $t + 1$ is determined by the reference price r_t and the sales price p_t on the previous period t, namely

$$r_{t+1} = \alpha r_t + (1 - \alpha) p_t. \tag{3}$$

The exponential smoothing represented by (3) is the most commonly adopted in the literature [4, 5, 6]. The smoothing parameter α implies how strongly the reference price is affected by past prices, where $0 \leq \alpha \leq 1$. The consumers with lower α have a short-term memory, and they are strongly influenced by recent sales prices.

C. Problem Definitions

Our aim is to derive an optimal discount pricing in multiple periods with taking the reference price effect into account. A two-period model is discussed in this study as the simplest case. At a prescheduled time during the operation hours, a firm can discount the products to stimulate demand. We refer to the time slot from the prescheduled time to closing hour as discount time slot. This study focuses on the optimal pricing to maximize the profit in the discount time slot in two periods.

The decision variables are the sales price p_t during the time slot in period t where $t = 1, 2$ and $p_t \in [p_L, p_H]$. Since p_t is a discount price, the upper limit of settable price p_H implies the original regular sales price of the products. Assume that p_H in period 1 and 2 are common value, but the assumption can be relaxed in the procedures proposed in the following. During the discount time slot, the products are sold at price p_t and they are not discounted additionally.

The demand in the discount time slot is given by (2), which means that the demand function is common in both periods. The reference price r_t in period t exists in the range $[p_L, p_H]$. The reference price is updated in accordance with (3). Assume that $D(p, r) > 0$ for any $p, r \in [p_L, p_H]$.

Let Q_t be the inventory at the beginning of the discount time slot in period t, not the one replenished before opening hour. The inventory level Q_t is assumed to be a random variable and given by $Q_t = q_t + \varepsilon_q$, where q_t is the average of Q_t and ε_q is a random factor. Assume that ε_q is common in two periods, its mean is 0, and its range is $[q_L, q_H]$. When it holds $D(p_t, r_t) \leq Q_t$, the unsold $Q_t - D(p_t, r_t)$ products are disposed or reused at the unit cost h, where h means the disposal cost if $h > 0$, and the salvage cost if $h < 0$. On the other hand, if it holds that $D(p_t, r_t) > Q_t$, the unsatisfied $D(p_t, r_t) - Q_t$ demands are estimated a penalty at the unit cost $s > 0$. Let c be the unit procurement cost of the products. Assume $-h < c < p_L$.

The firm aims to determine the discount prices to maximize the present value of expected profit during two periods. The objective function V is expressed as follows:

$$V(\boldsymbol{p}, \boldsymbol{q}, r_1) = \Pi(p_1, q_1, r_1) + \gamma \Pi(p_2, q_2, r_2), \tag{4}$$

where $\Pi(p_t, q_t, r_t)$ is the expected profit in period t and γ denotes the discount factor for present value. The vector $\boldsymbol{p}$ and $\boldsymbol{q}$ represent (p_1, p_2) and (q_1, q_2), respectively.

III. Optimal Pricing In a Single Period

A. Optimal Pricing for LN Consumers

This subsection discusses the optimal pricing for LN consumers. Let $\beta_{2G} = \beta_{2L} = \beta_2$, then the expected demand function $d(p, r) = E[D(p, r)]$ is given by the following equation:

$$d(p, r) = \beta_0 - \beta_1 p + \beta_2 (r - p) \equiv B_0(r) - B_1 p. \tag{5}$$

The both parameters $B_0(r)$ and B_1 are positive.

Let $\varepsilon = \varepsilon_q - \varepsilon_d$, then it holds

$$Q - D(p, r) = z + \varepsilon. \tag{6}$$

The average of ε is 0 and the range of ε is $[z_L, z_H]$ where $z_L = q_L - d_H$ and $z_H = q_H - d_L$. The profit $\pi(p, q, r)$ is expressed

$$\pi(p, q, r) = \begin{cases} pD(p, r) - cQ - h(z + \varepsilon) & if\ z + \varepsilon \geq 0, \\ pD(p, r) - cQ + (p + s)(z + \varepsilon) & otherwise. \end{cases} \tag{7}$$

Let $f(\cdot)$ and $F(\cdot)$ be the probabilistic density function and the distribution function of the variable ε. Define

$\bar{F}(\cdot) = 1 - F(\cdot)$. The expected profit $\Pi(p, q, r)$, hence, can be expressed by

$$\Pi(p, q, r) = \Psi(p, r) - L(p, z), \tag{8}$$

$$\Psi(p, r) = (p - c)d(p, r), \tag{9}$$

$$L(p, z) = (c + h)\Lambda(z) + (p - c + s)\Theta(z), \tag{10}$$

$$\Lambda(z) = \int_{-z}^{z_H} (z + u)f(u)du, \tag{11}$$

$$\Theta(z) = -\int_{z_L}^{-z} (z + u)f(u)du. \tag{12}$$

In (8), $\Psi(p, r)$ and $L(p, z)$ respectively imply the profit for $Q = D(p, r)$ and the expected cost incurred by excess and deficiency of inventory. The expected volumes of excess and deficiency of inventory are denoted by $\Lambda(z)$ and $\Theta(z)$ defined in (11) and (12), respectively.

By differentiating (8) with respect to p, we obtain

$$\frac{\partial^2 \Pi(p,q,r)}{\partial p^2} = -2B_1\bar{F}(-z) - (p + h + s)B_1^2 f(-z) < 0. \tag{13}$$

The assumption of $h < p_L$ proves that $\Pi(p, q, r)$ is concave with respect to p and has a unique maximum. Hence, the following lemma has obtained.

Lemma 1 (Theorem 3 in [8])**.** *For LN consumers, the expected profit* $\Pi(p, q, r)$ *in a single period is concave with respect to p. The optimal price* $p^*(q, r)$ *which maximizes* $\Pi(p, q, r)$ *is derived by the following equation:*

$$p^*(q, r) = \min[\max\{\hat{p}(q, r), p_L\}, p_H], \tag{14}$$

where $\hat{p}(q, r)$ *is the unique solution of* $g(p, q, r) = 0$*:*

$$g(p, q, r) = B_0(r) - (2p + h)B_1 + (p + h + s)B_1F(-z) - \Theta(-z). \tag{15}$$

By differentiating (8) with respect to β_2, we obtain

$$\frac{\partial \Pi(p,q,r)}{\partial \beta_2} = -(p - r)\{(p + h + s)\bar{F}(-z) - s\}, \tag{16}$$

$$\frac{\partial^2 \Pi(p,q,r)}{\partial \beta_2} = -(p - r)^2(p + h + s)f(-z) < 0. \tag{17}$$

The expected profit $\Pi(p, q, r)$ is proved to be concave with respect to β_2 from (17). When the following inequality holds

$$(p + h + s)\bar{F}(d(p, r) - q) > s\,, \tag{18}$$

Equation (16) introduces that the expect profit $\Pi(p, q, r)$ is increasing, constant, and decreasing with respect to β_2 for $p < r$, $p = r$, and $p > r$, respectively. Hence, the next lemma has been obtained.

Lemma 2 (Lemma 2 in [8])**.** *For LN consumers, the expected profit* $\Pi(p, q, r)$ *is concave with respect to* β_2*. Furthermore, in case that (18) holds,* $\hat{p}(q, r)$ *is decreasing with respect to* β_2*.*

Similarly, by differentiating (8) respectively with respect to r and q, the following lemma has been introduced.

Lemma 3. *For LN consumers, the expected profit* $\Pi(p, q, r)$ *is concave with respect to r and q, respectively.*

B. Optimal Pricing for Asymmetric Consumers

Since $\beta_{2G} \neq \beta_{2L}$, LA and LS consumers are called asymmetric consumers. For the asymmetric consumers, the expected profit function $\Pi(p, q, r)$ is expressed as

$$\Pi(p, q, r) = \begin{cases} \Pi_G(p, q, r) & \text{if } p < r, \\ \Pi_L(p, q, r) & \text{if } p > r, \end{cases} \tag{19}$$

where $\Pi_G(p, q, r)$ and $\Pi_L(p, q, r)$ are respectively the profit $\Pi(p, q, r)$ with $\beta_2 = \beta_{2G}$ and $\beta_2 = \beta_{2L}$. The two functions $\Pi_G(p, q, r)$ and $\Pi_L(p, q, r)$ have a common point on $p = r$. Let $\hat{p}_G(r)$ and $\hat{p}_L(r)$ be respectively the prices to maximize $\Pi_G(p, q, r)$ and $\Pi_L(p, q, r)$.

Lemmas 1 and 2 restrict the possibility of the shape of the expected profit functions $\Pi(p, q, r)$ for LA and LS consumers, represented in Fig. 1. The function $\Pi(p, q, r)$ is concave except in the case of $\hat{p}_G(r) < r < \hat{p}_L(r)$ represented in Fig. 1(c), when the function is bimodal. This discussion introduces the following theorem as a procedure to derive the optimal price for the asymmetry consumers.

Theorem 1. *For LA and LS consumers, the optimal price* $p^*(q, r)$ *which maximizes the expected profit* $\Pi(p, q, r)$ *is derived by the following equations:*

$$P_1^* = \begin{cases} \{\hat{p}_G(r), \hat{p}_L(r)\} & \text{if } \hat{p}_G(r) \leq r \leq \hat{p}_L(r), \\ \{\hat{p}_G(r)\} & \text{if} \max\{\hat{p}_G(r), \hat{p}_L(r)\} \leq r, \\ \{\hat{p}_L(r)\} & \text{if } \min\{\hat{p}_G(r), \hat{p}_L(r)\} \geq r, \\ \{r\} & \text{otherwise,} \end{cases} \tag{20}$$

$$P_2^* = \{\min[\max\{p, \bar{p}(q, r), p_L\}, p_H] \mid p \in P_1^*\}, \tag{21}$$

$$p^*(q, r) = \text{argmax}_{p \in P_2^*}\ \Pi(p, q, r). \tag{22}$$

Note that the cardinality of P_2^* is two in the case of $\hat{p}_G(r) \leq r \leq \hat{p}_L(r)$. In the other cases, the cardinality is one and (21) derives the optimal price without (22).

If (18) holds, it holds $\hat{p}_L(r) < \hat{p}_G(r)$ for LA consumers and $\hat{p}_G(r) < \hat{p}_L(r)$ for LS consumers. Then the procedure given by Theorem 1 can be simplified as follows.

Corollary 1. *When consumers are LA and* (18) *holds for* $p = p_L$, (20) *can be replaced with the following equation:*

$$P_1^* = \begin{cases} \{\hat{p}_G(r)\} & \text{if } \hat{p}_G(r) \leq r, \\ \{r\} & \text{if } \hat{p}_L(r) \leq r \leq \hat{p}_G(r), \\ \{\hat{p}_L(r)\} & \text{otherwise.} \end{cases} \tag{23}$$

Corollary 2. *When consumers are LS and* (18) *holds for* $p = p_L$, (20) *can be replaced with the following equation:*

$$P_1^* = \begin{cases} \{\hat{p}_G(r)\} & \text{if } \hat{p}_L(r) \leq r, \\ \{\hat{p}_G(r), \hat{p}_L(r)\} & \text{if } \hat{p}_G(r) \leq r \leq \hat{p}_L(r), \\ \{\hat{p}_L(r)\} & \text{otherwise.} \end{cases} \tag{24}$$

It is noticeable that (18) can be written as follows:

$$\Pr(Q > D(p, r)) > \frac{s}{p+h+s}, \tag{25}$$

since $\bar{F}(z) = \Pr(\varepsilon > z)$. Inequality (25) has the same form as the well-known critical fractile for newsvendor problems.

IV. Optimal Pricing In Two Periods

A. Optimal Pricing for LN Consumers

In two periods case, the objective is to find optimal prices $\boldsymbol{p}$ to maximize $V(\boldsymbol{p}, \boldsymbol{q}, r_1)$ given by (4). While $\Pi(p_1, q_1, r_1)$ is independent of p_2, $\Pi(p_2, q_2, r_2)$ is dependent on p_1 as well as p_2. From Lemma 1, $\Pi(p_2, q_2, r_2)$ is concave with respect to p_2 and, hence, $V(\boldsymbol{p}, \boldsymbol{q}, r_1)$ is also concave with respect to p_2.

Next, we differentiate $V(\boldsymbol{p}, \boldsymbol{q}, r_1)$ with respect to p_1, Since it holds that

$$\frac{\partial r_2}{\partial p_1} = 1 - \alpha, \tag{26}$$

$$\frac{\partial z_2}{\partial r_2} = \frac{\partial}{\partial r_2}\{q_2 - d(p_2, r_2)\} = -\beta_2, \tag{27}$$

$$\frac{\partial \Psi(p_2, r_2)}{\partial r_2} = \frac{\partial}{\partial r_2}\{(p_2 - c)d(p_2, r_2)\} = \beta_2(p_2 - c), \tag{28}$$

$$\frac{\partial L(p_2, z_2)}{\partial r_2} = -\beta_2\{c + h - (p_2 + s + h)F(-z_2)\}, \tag{29}$$

$$\frac{\partial \Pi(p_2, q_2, r_2)}{\partial r_2} = \beta_2\{(p_2 + s + h)\bar{F}(-z_2) - s\}, \tag{30}$$

$$\frac{\partial^2 \Pi(p_2, q_2, r_2)}{\partial {r_2}^2} = -{\beta_2}^2(p_2 + s + h)f(-z_2) < 0, \tag{31}$$

then it holds

$$\frac{\partial V(p,q,r_1)}{\partial p_1} = \frac{\partial \Pi(p_1,q_1,r_1)}{\partial p_1} + \frac{\partial r_2}{\partial p_1}\frac{\partial \Pi(p_1,q_1,r_1)}{\partial r_2}$$

$$= B_0(r_1) - (p_1 - s)B_1 + (p_1 + h + s)B_1\bar{F}(-z_1) - \Theta(z_1)$$
$$+(1-\alpha)\,\beta_2\{(p_2 + s + h)\bar{F}(-z_2) - s\}, \tag{32}$$

$$\frac{\partial^2 V(p,q,r_1)}{\partial {p_1}^2} = -2B_1\bar{F}(-z_2) - (p_1 + h + s){B_1}^2 f(-z_1)$$
$$-(1-\alpha){\beta_2}^2(p_2 + s + h)f(-z_2) < 0. \tag{33}$$

The above discussion introduces the following theorem.

Theorem 2. *For LN consumers, the present value of the expected profit in the two periods $V(\boldsymbol{p}, \boldsymbol{q}, r_1)$ is concave with respect to both p_1 and p_2.*

Theorem 2 guarantees that some solving algorithms, such as steepest descent method, in the convex programming problem can be applied to the problem in this study to find optimal pricings if target consumers are LN.

B. Optimal Pricing for Asymmetric Consumers

As discussed in the previous section, the expected profit $\Pi(p, q, r)$ is not always concave if target consumers are LA and LS. The objective function $V(\boldsymbol{p}, \boldsymbol{q}, r_1)$, hence, is naturally not always concave for the asymmetric consumers. Since the expected profit $\Pi(p, q, r)$, however, is piecewise convex in a single period and from Theorem 2, the present value $V(\boldsymbol{p}, \boldsymbol{q}, r_1)$ is also piecewise convex. We propose a procedure to find optimal discount pricings in two periods for asymmetric consumers by using the property on the present value.

The target problem is named as MP, which means Main Problem, and it can be expressed in a optimization formulation as follows:

Problem MP:

$$\max._{\boldsymbol{p}}\, V(\boldsymbol{p}, \boldsymbol{q}, r_1) = \Pi(p_1, q_1, r_1) + \gamma\Pi(p_2, q_2, r_2) \tag{34}$$

s. t.

$$\Pi(p_t, q_t, r_t) = \begin{cases}\Pi_G(p_t, q_t, r_t) & if\ p_t < r_t \\ \Pi_L(p_t, q_t, r_t) & otherwise\end{cases} \quad (t = 1, 2), \tag{35}$$

$$r_2 = \alpha r_1 + (1 - \alpha)p_1, \tag{36}$$

$$p_L \le p_t \le p_H \ \ (t = 1, 2). \tag{37}$$

Problem MP can be divided into four partial problems as follows:

Problem $\mathbf{P_{GG}}$:

$$\max._{\boldsymbol{p}}\, V_{GG}(\boldsymbol{p}, \boldsymbol{q}, r_1) = \Pi_G(p_1, q_1, r_1) + \gamma\Pi_G(p_2, q_2, r_2) \tag{38}$$

s. t.

$$r_2 = \alpha r_1 + (1 - \alpha)p_1, \tag{39}$$

$$p_L \le p_t \le r_t \ \ (t = 1, 2). \tag{40}$$

Problem $\mathbf{P_{GL}}$:

$$\max._{\boldsymbol{p}}\, V_{GL}(\boldsymbol{p}, \boldsymbol{q}, r_1) = \Pi_G(p_1, q_1, r_1) + \gamma\Pi_L(p_2, q_2, r_2) \tag{41}$$

s. t.

$$r_2 = \alpha r_1 + (1 - \alpha)p_1, \tag{42}$$

$$p_L \le p_1 \le r_1, \tag{43}$$

$$r_2 \le p_2 \le p_H. \tag{44}$$

Problem $\mathbf{P_{LG}}$:

$$\max._{\boldsymbol{p}}\, V_{LG}(\boldsymbol{p}, \boldsymbol{q}, r_1) = \Pi_L(p_1, q_1, r_1) + \gamma\Pi_G(p_2, q_2, r_2) \tag{45}$$

s. t.

$$r_2 = \alpha r_1 + (1 - \alpha)p_1, \tag{46}$$

$$r_1 \le p_1 \le p_H, \tag{47}$$

$$p_L \le p_2 \le r_2. \tag{48}$$

Problem $\mathbf{P_{LL}}$:

$$\max._{\boldsymbol{p}}\, V_{LL}(\boldsymbol{p}, \boldsymbol{q}, r_1) = \Pi_L(p_1, q_1, r_1) + \gamma\Pi_L(p_2, q_2, r_2) \tag{49}$$

s. t.

$$r_2 = \alpha r_1 + (1 - \alpha)p_1, \tag{50}$$

$$r_t \le p_t \le p_H \ \ (t = 1, 2). \tag{51}$$

Since the objective function (34) is convex while β_2 is constant from Theorem 2, the objective functions in the four partial problems are all convex. In other words, the search region in Problem MP is divided into four partial regions illustrated in Fig. 2. The optimal solution in Problem MP is the best one among the optimal solutions in the four problems P_{GG}, P_{GL}, P_{LG}, and P_{LL}.

V. Numerical Experiments

This section discusses optimal discount pricings through numerical experiments. Some parameters are set in common throughout all subsections as follows: $\beta_0 = 100$, $\beta_1 = 0.1$, $c = 250$, $s = 50$, $h = -50$. The uniform distribution within the range [–20, 20] is adopted as the probabilistic density function of the variable ε.

A. Expected Profit in a Single Period

This subsection ascertains how expected profit function $\Pi(p, q, r)$ in a single period depends on the consumer's attitude toward sales prices, namely LN, LA, and LS. The average inventory level q is set to several values from 50 to 70.

Figure 3 represents $\Pi(p, q, r)$ of sales price p, where $\beta_{2G} = \beta_{2L} = 0.05$ and $r = 480$. As Lemma 1 has proved, $\Pi(p, q, r)$ is convex with respect to p. Additionally, $\Pi(p, q, r)$ is convex regarding q as shown in Lemma 3. In Fig. 3, the optimal price $p^*(q, r) = \hat{p}(q,r)$ and it is monotonically decreasing with respect to q under the given setting.

Both the optimal price $p^*(q, r)$ and the maximum expected profit $\Pi(p^*(q, r), q, r)$ are computed for cases with several values of r in order to investigate their sensitivity with respect to q and r. The search range $[p_L, p_H]$ is set so that $p_L = c = 250$ and $p_H = 500$ which means the regular price. When consumers' reference price $r = p_H$, the average demand for products with the regular price $d(p_H, p_H)$ is equal to 50.

Figure 4 shows the optimal prices $p^*(q, r)$ for LN consumers. The dashed line in Fig. 4 denotes $p^*(q, r) = r$ and the regions above and below the dashed line represent loss pricings and gain pricings, respectively. Figure 6 represents that optimal prices decrease with respect to both q and r. In the cases of $q = 50$ and $q = 55$, no discount sale is optimal. In the other cases, the firm should make discount according to the volume of q and r. The maximum expected profits $\Pi(p^*(q, r), q, r)$ for LN consumers is shown in Fig. 5. It is ascertained that $\Pi(p^*(q, r), q, r)$ increases with respect to r. Moreover, it is concave with respect to q, which means that extra inventory might increase maximum expected profit but excess inventory declines it.

Next, similar experiments are conducted for LA consumers where $\beta_{2G} = 0.05$ and $\beta_{2L} = 0.2$. The expected profit function $\Pi(p, q, r)$ is illustrated in Fig. 6. In the range of $p \leq r$, the shapes of the functions are the same as those in Fig. 3. In the range of $p > r$, $\Pi(p, q, r)$ decreases sharply in comparison with Fig. 3 since LA consumers strongly react loss sales prices $p > r$ and demand declines substantially. The functions $\Pi(p, q, r)$ are still concave with respect to p for all values of q in Fig. 6. The optimal price $p^*(q,r) = \hat{p}_G(r)$ when $q = 70$ and $p^*(q,r) = r$ otherwise.

Figure 7 indicates the optimal pricing $p^*(q, r)$ for LA consumers in the same parameters setting as in Fig. 4 except for β_{2L}. In comparison with Fig. 4, all the optimal prices in the loss region $p > r$ in Fig. 4 are declined down to the boundary line $p^*(q, r) = r$ and the other optimal prices in the region $p \leq r$ stay in the same way in Fig. 4. The price decline implies that LA consumers react sensitively to sales prices greater than their reference price and firms should not set the sales prices greater than the reference price inadvertently. The optimal prices are monotonically decreasing with both q and r in Fig. 7.

Finally, we show the experimental results for LS consumers where $\beta_{2G} = 0.2$ and $\beta_{2L} = 0.05$. The expected profit function $\Pi(p, q, r)$ illustrated in Fig. 8. In contradiction to Fig. 6, the shape of $\Pi(p, q, r)$ is the same as those in Fig. 3 in the range of $p \geq r$. In the range of $p < r$, demand is increased by reference price effect and the expect profit is also increased as a result. The functions $\Pi(p, q, r)$ with all values of q are not concave but bimodal in Fig. 8. The optimal price $p^*(q,r) = \hat{p}_L(r)$ when $q = 50$ and 55, and $p^*(q,r) = \hat{p}_G(r)$ otherwise.

The optimal pricing $p^*(q, r)$ for LS consumers and the maximum expected profit $\Pi(p^*(q, r), q, r)$ are shown in Fig. 9 and Fig. 10, respectively. In comparison with Fig. 4, most of the optimal prices are declined not only in the gain region but also in the loss region. While the optimal price $p^*(q, r)$ is non-decreasing with r for LN and LA consumers in Fig. 4 and Fig. 7, $p^*(q, r)$ is sometimes decreasing regarding r in the LS consumers case. The decreasing is occurred from the bimodality of the expected profit function shown in Fig. 8. Within the loss region or the gain region in Fig. 9, the optimal price is non-decreasing with r. These observations indicate that discount sale is effective if the amount of q is greater and the LS consumers have greater reference price r. Firms sometimes should sell products at the price $p > r$ even for LS consumers when there are fewer inventory or the reference price of LS consumers is lower.

In Figure 10, the maximum expect profit function $\Pi(p^*(q, r), q, r)$ is not always concave with respect to q. To be more precise, $\Pi(p^*(q, r), q, r)$ is bimodal for $r = 470$. The bimodality of the function makes it complicated to estimate its maximum intuitionally. For $r \geq 480$, $\Pi(p^*(q, r), q, r)$ with $q = 65$ is greater than that with the other values of q, which means that extra inventory contributes to increase the expected profit. For $r \leq 470$, the greatest of $\Pi(p^*(q, r), q, r)$ is by $q = 50$, which means that extra inventory is no use for the profit increase.

B. Optimal Price in Two Periods

The optimal prices and the maximum expected profit in two periods are computed. Additional to the parameter settings in a single period model discussed in the previous subsection, the following parameters are settled: $\alpha = 0.5$, $q_2 = 50$, and $\gamma = 0.95$. The given situation in the first period is the same as that in the previous subsection. In the second period, roughly speaking, firms should sell products at the regular price $p_H = 500$ when consumer's reference price r is high since $d(p_H, p_H) = 50$. If the firms make discount substantially in the first period, the reference price r is heavily down and the demand in the second period is also down. In this experiment, the optimal price in the first period $p_1^*(\boldsymbol{q}, r_1)$ is focused to compare with the results in the previous subsection.

The optimal prices in the first period for LN, LA, and LS consumers are illustrated in Fig. 11, Fig. 12, and Fig. 13, respectively. Figure 11 shows that the optimal prices increase linearly with respect to r in the same manner in the single period case shown in Fig. 4. The optimal prices are greater than those in Fig. 4, which means that firms should be careful not to decline consumer's reference price substantially so that the firms gain enough demand in the second period. A similar tendency is observed in Fig. 12 for LA consumers compared with the single period case in Fig. 7. While firms should sell products at $p \leq r$ under most settings in Fig. 7, the firms should sell products at $p \geq r$ under most settings in Fig. 12. For LS consumers, in Fig. 13, the increased amount of sale prices depends significantly on parameter settings. While the optimal prices in the gain region in Fig. 13 are slightly increased from the optimal prices in Fig. 9, the optimal price with $r = 470$ and $q = 60$ is highly increased from the gain region to the regular price $p_H = 500$. Since LS consumers are permissive to loss pricings, firms generally should emphasize the increase of profit gained by discounting in the first period than the decrease of profit caused by the reference effect in the second period. The firms, however, sometimes should stop making discount to raise consumer's reference price, which is occurred by the bimodality of expected function.

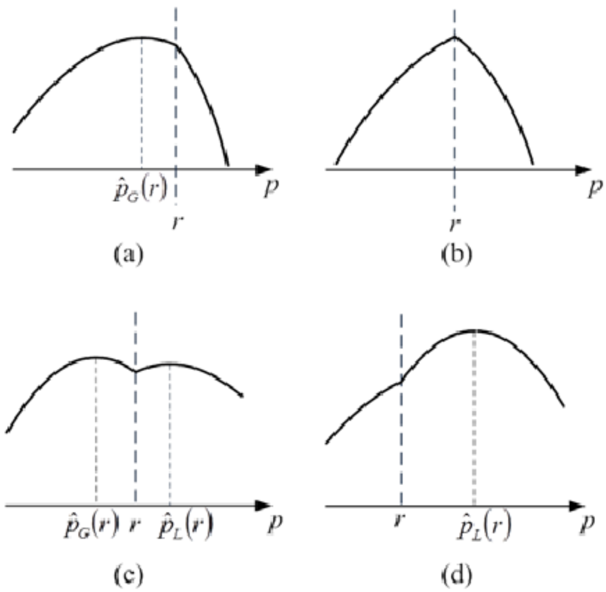

Figure 1. The feasible shapes of the function $\Pi(p, q, r)$. (a) $\hat{p}_G(r) \le r$ and $\hat{p}_L(r) \le r$, (b) $\hat{p}_G(r) \ge r$ and $\hat{p}_L(r) \le r$, (c) $\hat{p}_G(r) \le r \le \hat{p}_L(r)$, (d) $\hat{p}_G(r) \ge r$ and $\hat{p}_L(r) \ge r$.

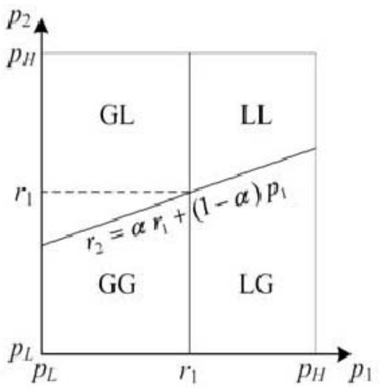

Figure 2. Divided search region for Problem MP

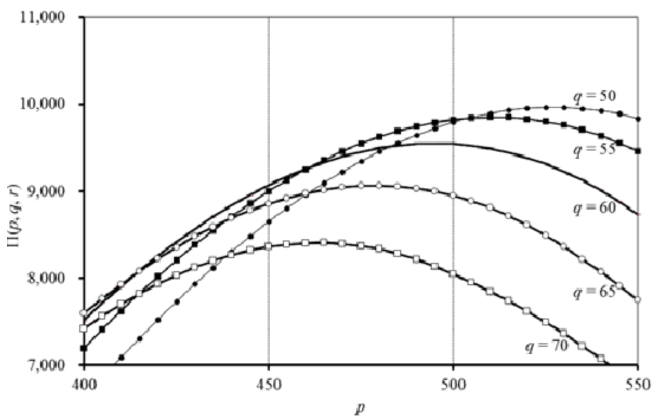

Figure 3. Expected profit functions for LN consumers

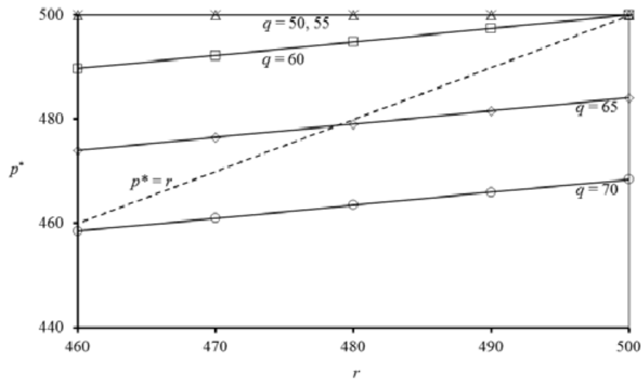

Figure 4. Optimal price for LN consumers

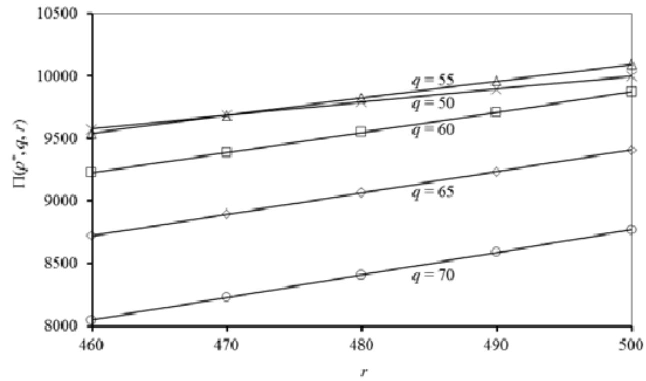

Figure 5. Maximum expected profit from LN consumers

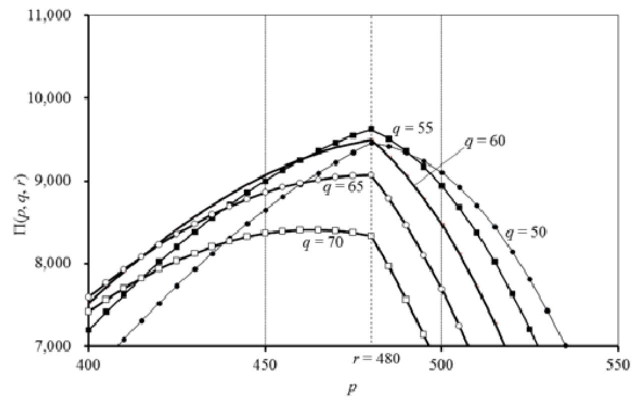

Figure 6. Expected profit functions for LA consumers

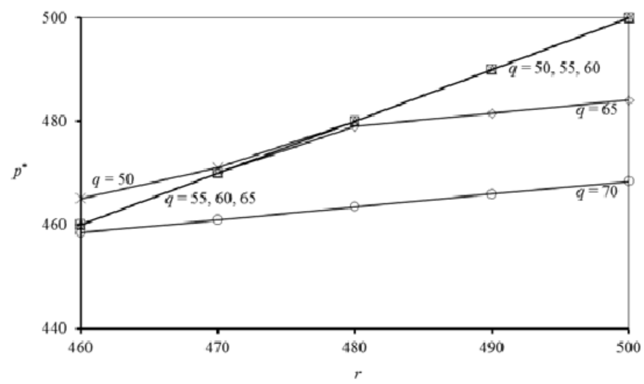

Figure 7. Optimal price for LA consumers

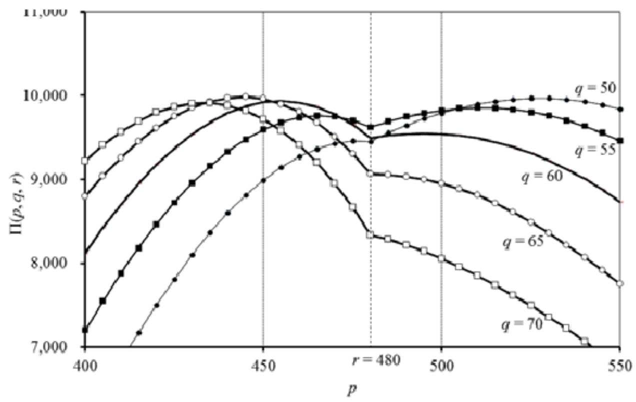

Figure 8. Expected profit functions for LS consumers

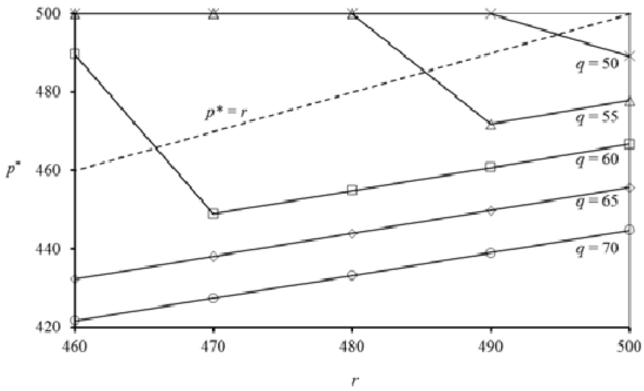

Figure 9. Optimal price for LS consumers

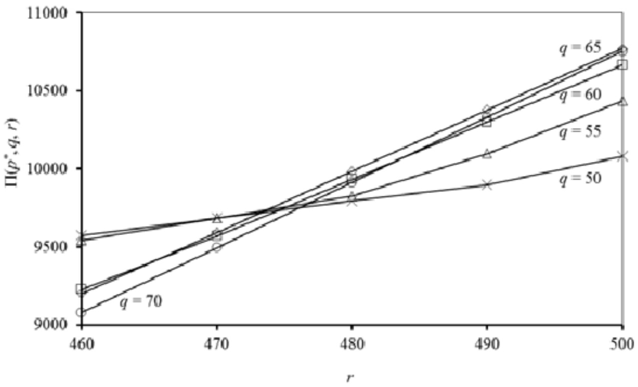

Figure 10. Maximum expected profit from LS consumers

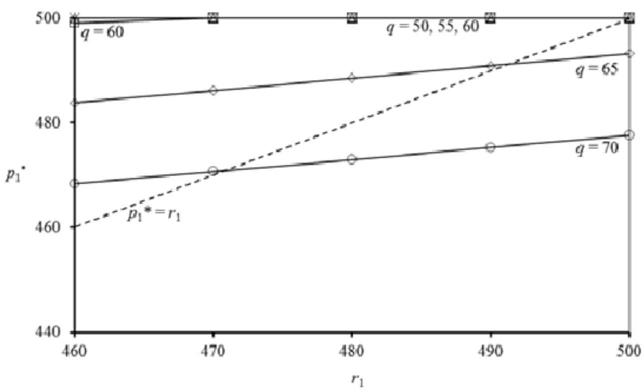

Figure 11. Optimal price in period 1 for LN consumers

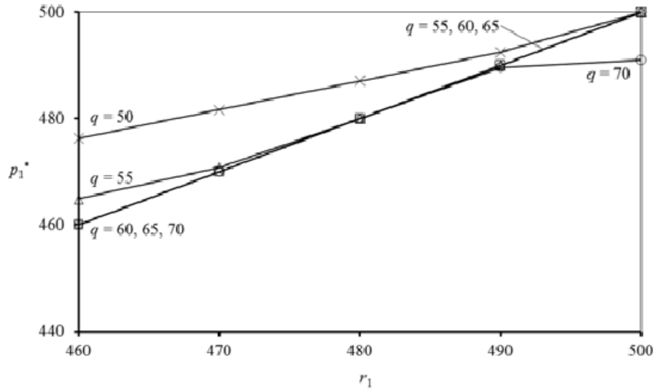

Figure 12. Optimal price in period 1 for LA consumers

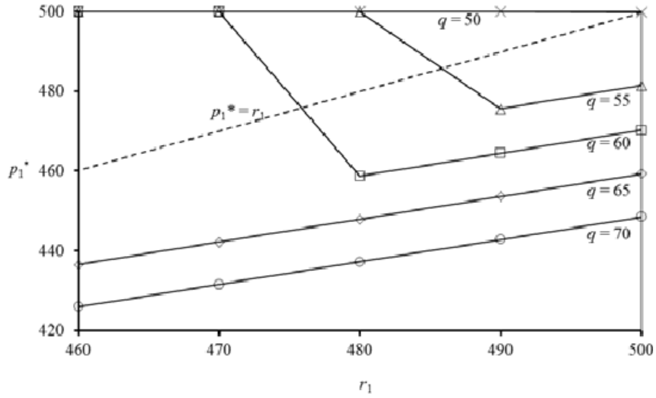

Figure 13. Optimal price in period 1 for LS consumers

VI. Conclusions

This study discussed a clearance pricing optimization in two periods analytically and numerically considering consumer's reference price effect. The objective function, the present value of the total expected profit, is concave if target consumers are LN. The objective function for LA and LS consumers is not always concave but piecewise concave. Optimal prices can be computed through the proposed procedure which searches divided four regions where the objective function is concave. Numerical studies indicate that firms should emphasize the reference price effect to maximize expected profit in two periods. Even though discount sales introduce maximum expected profit in the first period, declined reference price of consumers sometimes prevent the firms from maximizing total profit.

The target problem can be extended to multi-period version. When target consumer is LN, the objective function is still concave and optimal prices can be computed easily. The analysis on how the fluctuation of predicted inventory level in the future influences on optimal prices is currently under investigation. For asymmetric consumer, the proposed scheme to explore optimal prices can be extended. The number of divided search regions, however, increases exponentially and the simply extended procedure requires tremendous computing time if target periods are greater. Some resulting properties in our studies could serve to reduce the computational time. The extension to multi-period case will be discussed in our forthcoming paper.

REFERENCES

[1] D. Kahneman and A. Tversky, "Prospect theory: An analysis of decision under risk," *Econometrica*, vol. 47, 1979, pp. 263–291.
[2] G. Kalyanaram and R. S. Winer, "Empirical generalizations from reference price and asymmetric price response research," *Marketing Science*, vol. 14, pp. G161-G169, 1995.
[3] B. L. Fredrickson and D. Kahneman, "Duration neglect in retrospective evaluations of affective episodes," *Journal of Personality and Social Psychology*, vol. 65, pp. 45–55, 1993.
[4] E. A. Greenleaf, "The impact of reference price effects on the profitability of price promotions," *Marketing Science*, vol. 14, 1995, pp. 82–104.
[5] P. K. Kopalle, A. G. Rao, and J. L. Assuncao, "Asymmetric reference price effects and dynamic pricing policies," *Marketing Science*, vol. 15, 1996, pp. 60–85.
[6] I. Popescu and Y. Wu, "Dynamic pricing strategies with reference effects," *Operations Research*, vol. 55, 2007, pp. 413–429.
[7] N. C. Petruzzi and M. Dada, "Pricing and the newsvendor problem: A review with extensions," *Operations Research*, vol. 47, 1999, pp. 183–194.
[8] T. Koide and H. Sandoh, "A mathematical analysis on daily inventory clearance pricing with consumer's reference price," *Industrial Engineering & Management Systems*, vol. 11, no. 1, 2012, pp. 30–38.
[9] T. Koide and H. Sandoh, "An optimal daily inventory clearance operation with consumer's reference price," *Lecture Notes in Engineering and Computer Science: Proceedings of The International multiConference of Engineers and Computer Scientists 2012, IMECS 2012*, 14–16 March, 2012, Hong Kong, pp. 1587–1591.
[10] K. Arrow, T. Harris, and J. Marschak, "Optimal inventory policy,", *Econometrika*, vol. 19, 1951, pp. 250 – 272.