The Initial Boundary Value Problem of the Generalized Wigner System

Bin Li, and Jie-qiong Shen

Abstract—**This paper is concerned over the initial boundary value problem of the generalized Wigner system, which models quantum mechanical (charged) particles transport under the influence of a generalized Hartree potential field. Existence and uniqueness of the mild solution for one and three-dimensional problem are established on weighted-L^2 space by semigroup theory. The main difficulties in establishing mild solutions are to derive a priori estimates on the generalized Hartree potential field.**

Index Terms—**mild solution, semigroup theory, generalized Wigner system, initial boundary value problem, Hartree potential field.**

I. INTRODUCTION

IN the recent years, the following Hartree (Schrödinger) equation

$$i\hbar\psi_t(t,x) = -\frac{\hbar^2}{2}\Delta\psi(t,x) + V(t,x)\psi(t,x),\ x\in R^n \quad (1)$$

coupled the generalized Hartree potential

$$V(t,x) = \frac{\lambda}{|x|^\alpha} *_x \rho(t,x) \quad (2)$$

has been studied by many researchers in the quantum mechanics, see [1], [2] and the references therein for more details. This model describes the time-evolution of a complex-valued wave function $\psi(t,x)$, under the influence of a generalized Hartree potential (see [3], [4], [19] for a broader introduction). Where $\frac{\lambda}{|x|^\alpha}$ denotes a given real-valued interaction potential kernel, the symbol $*$ is a convolution, $\hbar$ is a Planck constant, and $\lambda\in R$.

The macroscopic density $\rho=\rho(t,x)$ is now defined by the zeroth order moment in the kinetic variable v, i.e., by the physical observables from both the wave function ψ and the Winger function w, namely,

$$\rho(t,x) = |\psi|^2 = \int_{R^n} w(t,x,v)dv. \quad (3)$$

The Wigner transform of $\psi(t,x)$ is

$$w(t,x,v) = \frac{1}{(2\pi)^n}\int_{R^n}\psi(t,x-\frac{\hbar}{2}y)\overline{\psi}(t,x+\frac{\hbar}{2}y)\exp(iv\cdot y)dy \quad (4)$$

where $\overline{\psi}$ denotes the complex conjugate of ψ. A straightforward calculation by applying the Wigner transform (4) to the Hartree equation (1) shows that $w(t,x,v)$ satisfies the so-called Wigner-like equation, see e.g. [5], [6],

$$w_t + (v\cdot\nabla_x)w - \Theta_\hbar[V]w = 0, \quad (5)$$

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where the Wigner function $w=w(t,x,v)$ is a probabilistic quasi-distribution function of particles at time $t\geq 0$, located at $x\in R^n$ with velocity $v\in R^n$. The operator $\Theta_\hbar[V]w$ in the equation (5) is a pseudo-differential operator, as in [7], [13], [14], formally defined by

$$\Theta_\hbar[V]w(t,x,v) = \frac{i}{(2\pi)^n\hbar}\int_{R^n}\int_{R^n}\delta[V](t,x,\eta)w(t,x,v')e^{i(v-v')\eta}dv'd\eta, \quad (6)$$

$$\delta[V](t,x,\eta) = V\left(t,x+\frac{\hbar\eta}{2}\right) - V\left(t,x-\frac{\hbar\eta}{2}\right).$$

It is clear that the equations (5)-(6) coupled eqution (2) contain the Wigner-Poisson equation (e.g., $n=3,\alpha=1$ and $n=1,\alpha=-1$). Over the past years, there have been many mathematical studies of mild or classical solution for the WP system, which models the charge transport in a semiconductor device under the Poisson potential. For instance, it has been studied in the whole space $R_x^3\times R_v^3$ (see [8] and the references therein), in a bounded spatial domain with periodic [9], or absorbing [10], or time-dependent inflow [13], [14], boundary conditions, and on a discrete lattice [11], [12]. Some different aspects regarding the derivation of the model have been reviewed in [20], [21].

The present paper is devoted to investigating the equations (5)-(6) coupled equation (2) and establishing certain mathematical results on the existence and uniqueness of the local mild solution, with the initial boundary conditions, which is very difficult: first, the function $V(t,x)$ does not satisfy $\Delta_x V=\rho$ as in WP problem (see [14] and so on); and the second is to derive priori estimates on the nonlinear operators $\Theta_\hbar[V]w$. Therefore, the mathematical analysis must be done on the some new methods such as splitting the singular kernel $\frac{1}{|x|^\alpha}$ with

$$\begin{cases} \alpha\in(0,1],\ n=3; \\ \alpha\in[-1,-\frac{1}{2}),\ n=1. \end{cases}$$

On the other hand, the natural choice of the functional setting for the study of the WP problem is the Hilbert space $L^2(R_x^n\times R_v^n)$, see [15], [5]. However, it can be immediately observed that the density $\rho(t,x)$, given by (3), is not well-defined for any $w(t,x,v)$ belonging to this space. In other words, the nonlinear term $\Theta_\hbar[V]w$ is not defined point-wise in t on the state space of the Wigner function. Therefore, in Section II we introduce two Hilbert spaces

$$X = L^2([0,l]^3\times R_v^3,(1+|v|^2)^2dxdv)$$

and

$$X_1 = L^2([0,l]\times R_v,(1+|v|^2)dxdv),$$

see also [17], [9], [14], such that, the existence of the density $\rho(t,x)$ is granted for any $w\in X$ or $w\in X_1$, respectively.

With the aforementioned notations, the main results of this paper can be described as the following theorems:

Theorem 1 Let $0 < \alpha \leq 1$, for every $w_0 \in X$, the equations (5)-(6) coupled with the equation (2), with initial boundary value conditions

$$\begin{aligned} w(t,0,x_2,x_3,v) &= w(t,v,l,x_2,x_3), \\ w(t,x_1,0,x_3,v) &= w(t,x_1,l,x_3,v), \\ w(t,x_1,x_2,0,v) &= w(t,x_1,x_2,l,v), \\ w(t=0,x,v) &= w_0(x,v),\ 1 < l < \infty, \end{aligned}$$

has a unique mild solution $w \in C([0,t_{\max}),X)$.

Theorem 2 Let $-1 \leq \alpha < -\frac{1}{2}$, for every $w_0 \in X_1$, the equations (5)-(6) coupled the equation (2), with initial boundary value conditions

$$w(t,0,v) = w(t,l,v),\ w(t=0,x,v) = w_0(x,v)$$

has a unique mild solution $w \in C([0,t_{\max}),X_1)$.

Remark 1 This result extends the previous result by Li, Shen [16] and generalizes the part previous result by Manzini [13].

Remark 2 It is straightforward to extend our results to the high dimensional case ($n > 3$) with $0 < \alpha \leq n-2$.

Our paper is structured as follows: In section II we introduce two weighted spaces X and X_1 for the Wigner function w that allows to define the nonlinear term $\Theta_\hbar[V]w$ in one and three dimensions. In section III, we obtain a local-in-time mild solution on the weighted L^2 space X, X_1 via the the Lumer-Phillips theorem [18] in one and three-dimensional, respectively.

II. The functional setting & preliminaries

In this section we shall discuss the functional analytic preliminaries for studying the nonlinear Wigner problem (5)-(6) coupled equation (2). Or more accurately, we shall introduce two appropriate state spaces for the Wigner function w which allows to control the nonlinear term $\Theta_\hbar[V]w$, which will be considered as a perturbation of the generator A defined in (23), respectively. This is one of the key ingredients for proving the theorems.

In the ensuing, we show indeed that the nonlinear operator $\Theta_\hbar[V]w$ is respectively (local) bounded in the weighted L^2 spaces, in symbols:

$$X := L^2(I_x \times R_v^3, (1+|v|^2)^2 dxdv),\ I_x = [0,l]_x^3, \quad (7)$$

$$X_1 := L^2(E \times R_v, (1+|v|^2) dxdv),\ E = [0,l]_x, \quad (8)$$

endowed with the following scalar products

$$\langle f,g\rangle_X := \int_I \int_{R_v^3} f(x,v)\cdot\overline{g(x,v)}(1+v^2)^2 dvdx, \quad (9)$$

$$\langle f,g\rangle_{X_1} := \int_I \int_{R_v} f(x,v)\cdot\overline{g(x,v)}(1+v^2) dvdx, \quad (10)$$

for $f,g \in X, X_1$. In our calculations, we shall use the following equivalent norms:

$$||f||_X^2 := ||f||_{L^2}^2 + \sum_{i=1}^3 ||v_i^2 f||_{L^2}^2, \quad (11)$$

$$||f||_{X_1}^2 := ||f||_{L^2}^2 + ||vf||_{L^2}^2. \quad (12)$$

The following proposition motivates our choice of the space X, X_1 for the analysis.

Lemma 1 For $n = 3$, let $w \in X$ and $\rho(t,x)$ defined in (3), for all $x \in I$, then ρ belongs to $L^2(I)$ and satisfies

$$||\rho||_{L^2(I)} \leq C||w||_X, \quad (13)$$

ρ also belongs to $L^1(I)$ and satisfies

$$||\rho||_{L^1(I)} \leq C||w||_X. \quad (14)$$

Moreover, for every $p \in [1,2]$, ρ belongs to $L^p(I)$ and satisfies

$$||\rho||_{L^p(I)} \leq C||w||_X. \quad (15)$$

Proof: The first assertion follows directly by using Cauchy-Schwartz inequality in v-integral, see also [13], [14]. On the other hand, by Hölder inequality, we have

$$\|\rho\|_{L^1(I)} \leq \int_I \left|\int_{R_v^3} w(t,x,v)dv\right| dx \leq$$

$$\left[\int_I 1^2 dx\right]^{\frac{1}{2}} \left[\int_0^l \left|\int_{R_v^3} w(t,x,v)dv\right|^2 dx\right]^{\frac{1}{2}} \leq C\|w\|_X.$$

Using the interpolation inequality, we get

$$\|\rho\|_{L^p(I)} \quad \leq \quad \|\rho\|_{L^2(I)}^\theta \|\rho\|_{L^1(I)}^{1-\theta} \leq C\|w\|_X.$$

Lemma 2 For $n = 1$, let $w \in X_1$ and $\rho(t,x)$ defined in (3), for all $x \in E$, then ρ belongs to $L^2(I)$ and satisfies

$$||\rho||_{L^2(E)} \leq C||w||_{X_1}, \quad (16)$$

ρ also belongs to $L^1(E)$ and satisfies

$$||\rho||_{L^1(E)} \leq C||w||_{X_1}. \quad (17)$$

Moreover, for every $p \in [1,2]$, ρ belongs to $L^p(E)$ and satisfies

$$||\rho||_{L^p(E)} \leq C||w||_{X_1}. \quad (18)$$

Proof: See the proof of the Lemma 1.

Next, we consider the Lipschitz properties of the pseudo-differential operator $\Theta_\hbar[V]w$ defined by (6). But by the definition of it, the w have to be 0-extended to R_x^3 and R_x. We will show indeed that this operator is well defined from the space X, X_1 to themselves. Moreover, we can state the following results:

Lemma 3 For $n = 3$, let $0 < \alpha \leq 1$, for all $w \in X$, the operator $\Theta_\hbar\,[V]\,w$ maps X into itself and there exists $C > 0$ such that

$$||\Theta_\hbar\,[V]\,w||_X \leq C||w||_X^2. \quad (19)$$

Proof: Indeed, the operator $\Theta_\hbar[V]w$ can be rewritten in a more compact form as,

$$\mathcal{F}_{v\to\eta}(\Theta_\hbar\,[V]\,w)(x,\eta) = \frac{i}{\hbar}\delta V(x,\eta)\mathcal{F}_{\eta\to v}^{-1} w(x,\eta), \quad (20)$$

where the symbol $*_v$ is the partial convolution with respect to the variable v, $\mathcal{F}_{v\to\eta}$ is the Fourier transformation with respect to the variable v and $\mathcal{F}_{\eta\to v}^{-1}$ its inverse:

$$\mathcal{F}_{v\to\eta}[f(x,\cdot)](\eta) = \int_{R^n} f(x,v)e^{iv\cdot\eta}dv,$$

$$\mathcal{F}_{\eta\to v}^{-1}[g(x,\cdot)](v) = \frac{1}{(2\pi)^n}\int_{R^n} g(x,v)e^{-iv\cdot\eta}d\eta$$

for suitable functions f and g. Then one has

$$\|\Theta_\hbar [V] w\|_{L^2} \le C\|\delta V(x,\eta)\mathcal{F}^{-1}_{\eta\to v} w\|_{L^2} \le C\|V\|_{L^\infty}\|\mathcal{F}^{-1}_{\eta\to v} w\|_{L^2} \le C\|V\|_{L^\infty}\|w\|_{L^2}.$$

Let $k(\cdot) = \frac{1}{|\cdot|^\alpha}$, $k_1 = k(\cdot)|_{|\cdot|\le 1}$ and $k_2 = k(\cdot)|_{|\cdot|>1}$, so, $k(\cdot) = k_1 + k_2$ with $k_1 \in L^p(R^3)$ for all $p \in [1, \frac{3}{\alpha})$ and $k_2 \in L^q(R^3)$ for all $q \in (\frac{3}{\alpha}, +\infty]$. On the other hand, since $V = \frac{1}{|\cdot|^\alpha} * \rho$, using Hölder's inequality we have

$$\|k_1 * \rho\|_{L^\infty(B)} \le C\|k_1\|_{L^2(B)}\|\rho\|_{L^2(B)} \le C\|\rho\|_{L^2(B)} \le C\|\rho\|_{L^2(I)},$$

where B is the three dimensional unit ball. Likewise, outside B we get

$$\|k_2 * \rho\|_{L^\infty(R^3\setminus B)} \le C\|k_2\|_{L^\infty(R^3\setminus B)}\|\rho\|_{L^1(I\setminus B)} \le C\|\rho\|_{L^1(I\setminus B)} \le C\|\rho\|_{L^1(I)}.$$

By Lemma 1, we can get

$$\|\Theta_\hbar [V] w\|_{L^2} \le C\|V\|_{L^\infty}\|w\|_{L^2} \le C\|w\|_X^2.$$

On the other hand, by [22], we have

$$v_i^2\Theta_\hbar [V] w = \frac{1}{4}\Theta_\hbar \left[\partial_i^2 V\right] w + \Theta_\hbar [V] v_i^2 w + \Omega_\hbar [\partial_i V] w$$

with the pseudo-differential operator

$$\Omega_\hbar[\varphi]w = \frac{i}{(2\pi)^n\hbar}\int_{R^n}\int_{R^n}\kappa[\varphi]w(t,x,v')e^{i(v-v')\eta}dv'd\eta,$$
$$\kappa[\varphi] = \varphi\left(t, x+\frac{\hbar\eta}{2}\right) + \varphi\left(t, x-\frac{\hbar\eta}{2}\right).$$

In the sequel we use the abreviation $\partial_i = \partial_{x_i}$, and get

$$\|v_i^2\Theta_\hbar [V] w\|_{L^2} \le \frac{1}{4}\|\Theta_\hbar \left[\partial_i^2 V\right] w\|_{L^2} + \|\Omega_\hbar [\partial_i V] w\|_{L^2} + \|\Theta_\hbar [V] v_i^2 w\|_{L^2}.$$

The first two terms can be estimated as follows:

$$\begin{aligned}\|\Theta_\hbar \left[\partial_i^2 V\right] w\|_{L^2} &\le C\|\delta(\partial_i^2 V)\mathcal{F}_{v\to\eta} w\|_{L^2} \le C\|\partial_i^2 V\|_{L^2(R^3)}\|\mathcal{F}_{v\to\eta} w\|_{L^2(I;L^\infty(R^3_\eta))} \le \\ &C\||x|^{-2-\alpha} *_x \rho\|_{L^2(R^3)}\|(1+|v|^2)w\|_{L^2} \le \\ &C(\|\frac{1}{|x|^{2+\alpha}} *_x \rho\|_{L^2(B)} + \|\frac{1}{|x|^{2+\alpha}} *_x \rho\|_{L^2(R^3\setminus B)})\|w\|_X \le \\ &C(\|\rho\|_{L^2(I)} + \|\rho\|_{L^1(I)})\|w\|_X \le C\|w\|_X^2\end{aligned}$$

by applying Hölder's inequality, $\frac{3}{2+\alpha} < 2$ with $0 < \alpha \le 1$ and the Sobolev imbedding $\mathcal{F}_{v\to\eta} w \in W^{2,2}(R^3_\eta) \hookrightarrow L^\infty(R^3_\eta)$.

$$\begin{aligned}\|\Omega_\hbar [\partial_i V] w\|_{L^2} &\le C\|\delta(\partial_i V)\partial_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2} \le C\|\partial_i V\|_{L^4(R^3)}\|\partial_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2(I;L^4(R^3_\eta))} \le \\ &C\||x|^{-1-\alpha} *_x \rho\|_{L^4(R^3)}\|(1+|v_i|^2)w\|_{L^2} \le \\ &C(\|\frac{1}{|x|^{1+\alpha}} *_x \rho\|_{L^4(B)} + \|\frac{1}{|x|^{1+\alpha}} *_x \rho\|_{L^4(R^3\setminus B)})\|w\|_X \le \\ &C\|\frac{1}{|x|^{1+\alpha}}\|_{L^{\frac{7}{5}}(B)}\|\rho\|_{L^{\frac{28}{15}}(B)}\|w\|_X + C\|\frac{1}{|x|^{1+\alpha}}\|_{L^4(R^3\setminus B)}\|\rho\|_{L^1(I\setminus B)}\|w\|_X \le \\ &C(\|\rho\|_{L^{\frac{28}{15}}(B)} + \|\rho\|_{L^1(I\setminus B)})\|w\|_X \le C\|w\|_X^2\end{aligned}$$

by the Sobolev imbedding $\partial_{\eta_i}\mathcal{F}_{v\to\eta} w \in W^{1,2}(R^3_\eta) \hookrightarrow L^4(R^3_\eta)$, $\frac{3}{1+\alpha} > \frac{7}{5}$ and $\frac{3}{1+\alpha} < 4$ with $0 < \alpha \le 1$, and Lemma 1. We also get

$$\begin{aligned}\|\Theta_\hbar [V] v_i^2 w\|_{L^2} &\le C\|V\partial^2_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2} \le C\|V\|_{L^\infty(R^3)}\|\partial^2_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2} \le \\ &C(\|\frac{1}{|x|^\alpha} *_x \rho\|_{L^\infty(B)} + \|\frac{1}{|x|^\alpha} *_x \rho\|_{L^\infty(R^3\setminus B)})\|w\|_X \le \\ &C\|\frac{1}{|x|^\alpha}\|_{L^2(B)}\|\rho\|_{L^2(B)}\|w\|_X + C\|\frac{1}{|x|^\alpha}\|_{L^\infty(R^3\setminus B)}\|\rho\|_{L^1(I\setminus B)}\|w\|_X \le \\ &C(\|\rho\|_{L^2(B)} + \|\rho\|_{L^1(I\setminus B)})\|w\|_X \le C\|w\|_X^2\end{aligned}$$

by applying Hölder's inequality and Lemma 1. This concludes the proof of result.

Lemma 4 For $n = 1$, let $-1 \le \alpha < -\frac{1}{2}$, for all $w \in X_1$, the operator $\Theta_\hbar [V] w$ maps X_1 into itself and there exists $C > 0$ such that

$$||\Theta_\hbar [V] w||_{X_1} \le C||w||^2_{X_1}.$$

Proof: Indeed, the solution of Poisson equation V and $\widetilde{V}$, its extension with value zero outside [0,l], are essentially bounded. Moreover, we can get

$$\|\widetilde{V}\|_{L^\infty(R)} \le C\|\rho\|_{L^1(E)},$$

where $E = [0, l]$, see also [14]. Moreover, using Lemmas 2-3, we have

$$\|\Theta_\hbar [V] w\|_{L^2} \le C\|V\|_{L^\infty}\|w\|_{L^2} \le C\|\rho\|_{L^1(E)}\|w\|_{L^2} \le C\|w\|^2_{L^2}.$$

On the other hand, by [23], we get

$$v\Theta_\hbar [V] w = \Theta_\hbar [V] vw - \Omega_\hbar [V_x] w$$

with the pseudo-differential operator

$$\Omega_\hbar[V_x]w = \frac{i}{2\pi\hbar}\int_R\int_R\kappa[V_x]w(t,x,v')e^{i(v-v')\eta}dv'd\eta,$$
$$\kappa[V_x] = V_x\left(t, x+\frac{\hbar\eta}{2}\right) + V_x\left(t, x-\frac{\hbar\eta}{2}\right).$$

Hence,

$$\|v\Theta_\hbar [V] w\|_{L^2} \le \|\Omega_\hbar [V_x] w\|_{L^2} + \|\Theta_\hbar [V] vw\|_{L^2}.$$

Setting $r = -\alpha$ then $\frac{1}{2} < r \le 1$, and the first term can be estimated as follows:

$$\|\Omega_\hbar [V_x] w\|_{L^2} \le C\|\frac{1}{|x|^{1-r}} * \rho\|_{L^\infty(E)}\|w\|_{L^2} \le C(\|\rho\|_{L^2(B)} + \|\rho\|_{L^1(E\setminus B)})\|w\|_{L^2} \le C\|w\|^2_{L^2}.$$

For the last term, we have

$$\|\Theta_\hbar [V] vw\|_{L^2} \le C\|V\|_{L^\infty(E)}\|vw\|_{L^2} \le C\|\rho\|_{L^1(E)}\|vw\|_{L^2} \le C\|w\|_{L^2}\|vw\|_{L^2}.$$

The combination of these estimates yields the result.

Lemma 5 For $n = 3$, let $0 < \alpha \le 1$, for all $w \in X$, the operator $\Theta_\hbar [V] w$ is of class C^∞ in X, and satisfies

$$\|\Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_2] w_2\|_X \le C(\|w_1\|_X + \|w_2\|_X)\|w_1 - w_2\|_X.$$

Proof: For all $w_i \in X, i=1,2$, setting

$$\Pi = \Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_2] w_2,$$
$$\Pi_1 = \Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_1] w_2,$$
$$\Pi_2 = \Theta_\hbar [V_1] w_2 - \Theta_\hbar [V_2] w_2,$$

we have

$$\|\Pi\|_X \le \|\Pi_1\|_X + \|\Pi_2\|_X$$

with

$$\begin{aligned}
&\|\Pi_1\|_X = \|\Theta_\hbar [V_1](w_1-w_2)\|_X \le \\
&\|\Theta_\hbar [V_1](w_1-w_2)\|_{L^2} + \sum_{i=1}^3 \|v_i^2 \Theta_\hbar [V_1](w_1-w_2)\|_{L^2} \le \\
&C\|\delta V[w_1]\mathcal{F}_{v\to\eta}(w_1-w_2)\|_{L^2} + \\
&C\sum_{i=1}^3 \|\delta \partial_i^2 V[w_1])\mathcal{F}_{v\to\eta}(w_1-w_2)\|_{L^2} + \\
&C\sum_{i=1}^3 \|\delta(\partial_i V[w_1])\partial_{\eta_i}\mathcal{F}_{v\to\eta}(w_1-w_2)\|_{L^2} + \\
&C\sum_{i=1}^3 \|\delta V[w_1]\partial_{\eta_i}^2 \mathcal{F}_{v\to\eta}(w_1-w_2)\|_{L^2} \le \\
&C\|V[w_1]\|_{L^\infty}\|w_1-w_2\|_{L^2} + \\
&C\sum_{i=1}^3 \|\partial_i^2 V[w_1]\|_{L^2}\|\mathcal{F}[w_1-w_2]\|_{L^2(I;L^\infty(R_\eta^3))} + \\
&C\sum_{i=1}^3 \|\partial_i V[w_1]\|_{L^4}\|\partial_{\eta_i}\mathcal{F}[w_1-w_2]\|_{L^2(I;L^4(R_\eta^3))} + \\
&C\sum_{i=1}^3 \|V[w_1]\|_{L^\infty}\|\partial_{\eta_i}^2\mathcal{F}[w_1-w_2]\|_{L^2} \le \\
&C\|w_1\|_{L^2}\|w_1-w_2\|_X; \\
&\|\Pi_2\|_X = \|\Theta_\hbar [V_1-V_2] w_2\|_X \le \\
&\|\Theta_\hbar [V_1-V_2] w_2\|_{L^2} + \\
&\sum_{i=1}^3 \|v_i^2 \Theta_\hbar [V_1-V_2] w_2\|_{L^2} \le \\
&C\|\delta V[w_1-w_2]\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\
&C\sum_{i=1}^3 \|\delta\partial_i^2 V[w_1-w_2])\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\
&C\sum_{i=1}^3 \|\delta(\partial_i V[w_1-w_2])\partial_{\eta_i}\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\
&C\sum_{i=1}^3 \|\delta V[w_1-w_2]\partial_{\eta_i}^2\mathcal{F}_{v\to\eta} w_2\|_{L^2} \le \\
&C\|V[w_1-w_2]\|_{L^\infty}\|w_2\|_{L^2} + \\
&C\sum_{i=1}^3 \|\partial_i^2 V[w_1-w_2]\|_{L^2}\|\mathcal{F}w_2\|_{L^2(I;L^\infty(R_\eta^3))} + \\
&C\sum_{i=1}^3 \|\partial_i V[w_1-w_2]\|_{L^4}\|\partial_{\eta_i}\mathcal{F}w_2\|_{L^2(I;L^4(R_\eta^3))} + \\
&C\sum_{i=1}^3 \|V[w_1-w_2]\|_{L^\infty}\|\partial_{\eta_i}^2\mathcal{F}_{v\to\eta} w_2\|_{L^2} \le \\
&C\|w_2\|_X\|w_1-w_2\|_{L^2},
\end{aligned}$$

and the assertion is proved.

Lemma 6 For $n=1$, let $-1 \le \alpha < -\frac{1}{2}$, for all $w \in X_1$, the operator $\Theta_\hbar [V] w$ is of class C^∞ in X_1, and satisfies

$$\|\Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_2] w_2\|_{X_1} \le C(\|w_1\|_{X_1} + \|w_2\|_{X_1})\|w_1-w_2\|_{X_1}.$$

Proof: For all $w_i \in X, i=1,2$, by Lemmas 4-5, we have

$$\begin{aligned}
&\|\Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_2] w_2\|_{X_1} \le \\
&\|\Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_1] w_2\|_{X_1} + \\
&\|\Theta_\hbar [V_1] w_2 - \Theta_\hbar [V_2] w_2\|_{X_1}
\end{aligned}$$

with

$$\begin{aligned}
&\|\Theta_\hbar [V_1](w_1-w_2)\|_{X_1} \le \\
&\|\Theta_\hbar [V_1](w_1-w_2)\|_{L^2} + \|v\Theta_\hbar [V_1](w_1-w_2)\|_{L^2} \le \\
&C\|\delta V[w_1]\mathcal{F}_{v\to\eta}(w_1-w_2)\|_{L^2} + \\
&C\|\Omega_\hbar [V_x[w_1]](w_1-w_2)\|_{L^2} + \\
&C\|\Theta_\hbar [V[w_1]] v(w_1-w_2)\|_{L^2} \le \\
&C\|V[w_1]\|_{L^\infty}\|w_1-w_2\|_{L^2} + \\
&C\|V_x[w_1]\|_{L^\infty}\|w_1-w_2\|_{L^2} + \\
&C\|V[w_1]\|_{L^\infty}\|v(w_1-w_2)\|_{L^2} \le \\
&C\|w_1\|_{L^2}\|w_1-w_2\|_{X_1}; \\
&\|\Theta_\hbar [V_1-V_2] w_2\|_{X_1} \le \\
&\|\Theta_\hbar [V_1-V_2] w_2\|_{L^2} + \|v\Theta_\hbar [V_1-V_2] w_2\|_{L^2} \le \\
&C\|\delta V[w_1-w_2]\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\
&C\|\Omega_\hbar [V_x[w_1-w_2]] w_2\|_{L^2} + \\
&C\|\Theta_\hbar [V[w_1-w_2]] v w_2\|_{L^2} \le \\
&C\|V[w_1-w_2]\|_{L^\infty}\|w_2\|_{L^2} + \\
&C\|V_x[w_1-w_2]\|_{L^\infty}\|w_2\|_{L^2} + \\
&C\|V w_1-w_2]\|_{L^\infty}\|v w_2\|_{L^2} \le \\
&C\|w_2\|_{X_1}\|w_1-w_2\|_{L^2},
\end{aligned}$$

and the assertion is proved.

III. Proof of the Main Results

In this section, we will prove the main results of this paper by semigroup theory. Let we rewrite the Wigner equation as

$$w_t = Aw + \Theta_\hbar[V]w, \ t > 0, \tag{21}$$
$$w(t=0) = w_0, \tag{22}$$

where linear operator $A : D(A) \to X$ or $A : D_1(A) \to X_1$ by

$$Af = -v \cdot \nabla_x w \tag{23}$$

and theirs domain

$$\begin{aligned}
D(A) = \{&w \in X | v \cdot \nabla_x w \in X, \\
&w(0,x_2,x_3) = w(l,x_2,x_3), \\
&w(x_1,0,x_3) = w(x_1,l,x_3), \\
&w(x_1,x_2,0) = w(x_1,x_2,l), l > 1\}; \\
D_1(A) = \{&w \in X_1 | v \cdot \nabla_x w \in X_1, \\
&w(t,0,v) = w(t,l,v), l > 1\}.
\end{aligned}$$

Proof of Theorem 1: Indeed, the A (defined in (23)) generates a C_0 group of isometries $\{S(t), t \in R\}$ on X, given by $S(t)w(x,v) = w(x-vt,v)$, see also [17]. Next, we consider $\Theta_\hbar[V]w$ as a bounded perturbation of the generator A. On the other hand, since $\Theta_\hbar[V]w$ is locally Lipschitz continuous (see Lemmas 3, 5 for detail), Theorem 6.1.4 of [18] shows that the problem (21)-(22) coupled equation (2) has a unique mild solution for every $w_0 \in X$ on some time interval $[0, t_{\max})$, where $t_{\max}$ denotes the maximal existence time of the mild solution. Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \to t_{\max}} \|w\|_X = \infty.$$

This concludes the proof of result.

Proof of Theorem 2: In fact, we can get the assertion by repeating the analogous strategies in proof of Theorem 1.

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