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On the Lowest Unique Bid Auction with Multiple Bids

Nawapon Nakharutai, and Parkpoom Phetpradap*

Abstract—**This study focuses on the lowest unique sealed-bid auctions in which the winning bidder is the one who places the unique bid that has the lowest value where exactly m bids per bidder are allowed. The problem can be seen as injecting a minimum into a random subset of a larger subset. By assuming that bids are identical and independently placed according to a given probability distribution, we obtain various exact probabilities for the auctions, both as a bidder and an observer, for $m = 1, 2$. The results are obtained via the inclusion-exclusion principle. The computational results and algorithms to calculate the probabilities are also given.**

Index Terms—**Unique minimum, inclusion-exclusion principle, combinations, partitions, identical and independence, game theory, Internet auctions, reverse auctions, sealed unique-bid auctions**

I. Introduction

Auction is a process of buying or selling goods or services. For a classical auction, the winning bidder is determined by the bidder who makes the highest value bid. Nowadays, there are many types of auctions ([7], [8]), such as open ascending price auctions (English auctions), open descending price auctions (Dutch auctions), first-price sealed-bid auctions, second-price sealed-bid auctions (Vickrey auctions) and unique bid auctions. In this article, we consider the lowest unique bid auctions(LUBAs), type of auctions which are recently introduced in the last few years. Moreover, we introduce the lowest unique bid auctions where multiple bids from each bidder are compulsory. We summarised the basic rules of m-bid LUBAs as follow:

1) Each bidder makes exactly m different sealed bids, assumed to be discrete and positive value.
2) The winning bidder is the bidder who makes the bid that is unique, and has the lowest value amongst the unique bids.

We call the auction *classical LUBA* when $m = 1$, which means that each bidder makes exactly one bid. We demonstrate LUBAs by using an example here: In the LUBA with ten participated bidders where each bidder places two bids according to Table 1.

- For the classical case where Bids 1 from each bidder are considered, we can see that the unique bids are 1.75, 2.25 and 4.00. Hence 1.75 is the lowest unique bid, and Bidder 4 is the winner of the classical LUBA.
- For $m = 2$ case where both bids from each bidder are considered, we can see that unique bids are 2.75, 3.00, 3.25, 4.00 and 5.00. Hence 2.75 is the lowest unique bid and Bidder 9 is the winner of the 2-bid LUBA.

TABLE I
Example of LUBA with 10 bidders

Bidder	Bid 1	Bid 2
1	1.00	1.50
2	1.50	2.50
3	2.00	2.25
4	1.75	3.00
5	4.00	5.00
6	1.00	1.75
7	1.50	2.50
8	2.25	3.25
9	2.00	2.75
10	1.00	1.75

It is clear, for $m \geq 2$, that each bidder must place non-identical bids for his own, otherwise he will reduce his chance of winning the auction.

We can see that, to win LUBAs depends on several factors such as a number of bidders involved and strategies of participated bidders. The auction winners do not necessarily have to place low bids. This somehow creates increased attention from general participants, especially in the auctions over the internet. In general, bidders in LUBAs need to pay participation fees since the winner is justified in the opposite way as most of the other auctions where the winning bidders normally need to pay high fees to obtain the auctioned items. Additional rules may be added to attract further interests, for instant the winning bidder has to pay the exact amount of money he bids. Such rules may effect strategies of bidders.

The LUBA problems can be modelled and studied in many aspects including Mathematics, Statistics, Engineering, Economics and Computer Sciences. However, all previous results are for the classical LUBA problems. Bruss et al.[3] considered the classical LUBA models where the bidder's strategies and the number of participated bidders, which could be a random variable, were assumed to be known. The main aim of their article was to study the chance of winning the auctions, and their approximated solutions were obtained via Poisson approximation theory. Pigolotti et al. [11] and Ostling et al. [9], in different approaches, studied equilibrium strategies of the auctions and compare their developed results with the selected data set from the internet auctions. Note that, urn models can also be used for the problems where balls represent bids from participated bidders and urns represent values of bids([2], [5]) and the lowest unique bid is represented by the left-most urn that contains exactly one ball. However, we do not use such models here.

In this article, when we participate in a LUBA, we refer *winning probability at bid k* as the probability that we are the winning bidder given that we place bid k. We refer *bid*

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distribution or *bid strategy* of Bidder i as the probability distribution of the value(s) of bid(s) placed by Bidder i. When we do not participate in a LUBA, we refer *winner probability at bid* k as the probability that the lowest unique bid occurs at the bid k, and we refer *winner distribution* or *equilibrium distribution* as the probability distribution of the value of lowest unique bid.

The main aim of this article is to derive *exact* winning probabilities for classical LUBAs and 2-bid LUBAs under the assumptions that the number of participated bidders is fixed and the bid distribution of each bidder is independent and identically distributed. Our results are obtained via the inclusion-exclusion principle. The result and the proof of the classical LUBA had been shown in Phetpradap [10]. However, we repeat those arguments again to ease readers to understand other results in this article. For each result, we propose the algorithm that generate winning probabilities along with the comments. The discussion on the algorithms, as well as the validity of our LUBA models, will be discussed in the conclusion. Furthermore, we extend our approaches to obtain winner distributions when we do not participate in such auctions. An example of the use of the results of this article is in dynamic task allocation in robot teams [13].

The rest of this article are presented as follow: In Section 2, we propose main notations and assumptions that are used throughout the article. The results and proofs of the winning probabilities in classical LUBAs and 2-bid LUBAs are displayed in Section 3 and 4 respectively. For both sections, we give examples and calculation of winning probabilities along with comments on algorithm. In Section 5, we act as an observer in the LUBAs and obtain the winner distributions. Finally, we discuss pros and cons of our methods in Section 6. The validity of the models and future study are also interpreted.

II. LUBA Models and Main Notations

In this section, we propose Mathematical models and assumptions for classical LUBAs and 2-bid LUBAs. For both models, we assume the following

Assumption 1. (1) *The number of participated bidders,* excluding us, *is fixed and equals to* n.
(2) *Bids from each bidder are independent and identically distributed according to a given distribution.*

Also, without loss of generality, we may assume that bids are integers.

For a *classical LUBA*, and for an integer k, *let* $\mathbb{P}_1(k)$ *be the probability that our bid* k *is the lowest unique bid* when we participate in the auction. Also, *let* $\hat{\mathbb{P}}_1(k)$ *be the probability that the bid* k *is the lowest unique bid* when we do not participate in the auction.

Similarly for a *2-bid LUBA*, and for integers k_1, k_2, *let* $\mathbb{P}_2(k_1, k_2)$ *be the probability of winning the auction when the bids* k_1 *and* k_2 *are placed* when we participate in the auction. Also, *let* $\hat{\mathbb{P}}_2(k)$ *be the probability that the bid* k *is the lowest unique bid* when we do not participate in the auction.

Last but not least, in order to be able to calculate the probabilities by using computers, we need to restrict an upper price limit of the auction. This can be seen as support of the set of bids.

III. Classical LUBAs

In this section, we display exact winning probabilities in classical LUBAs, which was proved in [10]. Then, we give an example and use the result to compute the winning probabilities. We also give comments on the algorithm that generates the probabilities.

Recall from Assumption 1 that the number of bidders *excluding us* is n and $\mathbb{P}_1(k)$ is the probability that our bid k is the lowest unique bid. Let X_i be the bid placed by Bidder i and define $p_k := P(X_i = k)$ to be the probability that Bidder i places the bid k

Theorem 2. *(Winning probabilities in a classical LUBA) With Assumption 1. For* $k \in \mathbb{N}$,

$$\mathbb{P}_1(k) = (1-p_k)^n \Big(1 - \sum_{i=1}^{k-1} \Big[(-1)^{i-1} \sum_{\substack{I \subset \{1,\dots,k-1\} \\ |I|=i}} A(I)\Big]\Big), \tag{1}$$

where,

$$A(I) = \mathcal{P}(n,i)\Big(\prod_{j\in I}\frac{p_j}{1-p_k}\Big)\Big(1-\sum_{j\in I}\frac{p_j}{1-p_k}\Big)^{n-i},$$

and $\mathcal{P}(k_1,k_2) = \frac{k_1!}{(k_1-k_2)!}$ *with* $\mathcal{P}(k_1,k_2) = 0$ *for* $k_2 > k_1$.

Proof: We re-write winning probability in terms of events of bids. For $i = 1, 2, \dots$, let

- E_i be the event that no bidder places the bid i,
- U_i be the event that exactly one bidder places the bid i,

with E_i^C and U_i^C are their complement events. Now, it can be seen that

$$\begin{aligned}\mathbb{P}_1(k) &= P(E_k \cap U_1^C \cap U_2^C \cap \cdots \cap U_{k-1}^C) \\ &= P(E_k)P(\bigcap_{i=1}^{k-1} U_i^C | E_k).\end{aligned} \tag{2}$$

Obviously $P(E_k) = (1-p_k)^n$, while

$$P(\bigcap_{i=1}^{k-1} U_i^C | E_k) = 1 - P(\bigcup_{i=1}^{k-1} U_i | E_k). \tag{3}$$

By the inclusion-exclusion principle, we get

$$P(\bigcup_{i=1}^{k-1} U_i | E_k) = \sum_{i=1}^{k-1}(-1)^{i-1} \sum_{\substack{I \subset \{1,\dots,k-1\} \\ |I|=i}} P(U_I|E_k), \tag{4}$$

where $U_I = \bigcap_{j\in I} U_j$. Note that $P(U_I|E_k)$ is the probability that unique bids occur at bids $j \in I$ given that no one bids on the bid k. Hence, we get, for $I \subset \{1, 2, \dots, k-1\}$,

$$\begin{aligned}P(U_I|E_k) &= P(\bigcap_{j\in I} U_i|E_k) \\ &= \mathcal{P}(n,|I|)\Big(\prod_{j\in I}\frac{p_j}{1-p_k}\Big)\Big(1-\sum_{j\in I}\frac{p_j}{1-p_k}\Big)^{n-|I|}.\end{aligned} \tag{5}$$

Combining (2) - (5), Theorem 2 follows. ■

Example 3. Define the probability mass functions of geometric(p) and Poisson(λ) random variables, Y and Z respectively, as

$$P(Y=i) = (1-p)^{i-1}p, \qquad \forall i \in \mathbb{Z}^+,$$
$$P(Z=k) = \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}, \qquad \forall k \in \mathbb{Z}^+.$$

The plots of the winning probabilities of LUBA with 50 bidders, where bid distributions are geometric(1/2), geometric(1/3), Poisson(2) and the price limits are 200, are given in Figure 1. The optimum places to make the bid should be at 5, 7 and 7 respectively. For large k, the winning probabilities seems to converge to a constant. This is because, by the choice of bid distributions we choose, high bids are rarely occur.

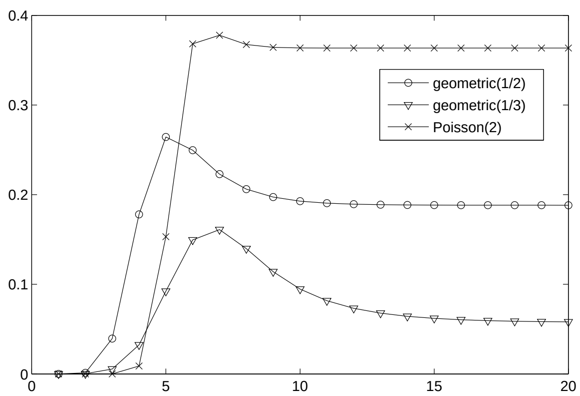

Fig. 1. The winning probabilities of the classical LUBAs with $n = 50$. The price limits are 200.

Algorithm and comments To generate winning probabilities from Theorem 2, the input required are the number of participated bidders, the bid distribution and the price limit. The algorithm to generate (1) is straightforward.

IV. Two-bid LUBAs

In this section, we derive exact winning probabilities in 2-bid LUBAs where each bidder makes two (different) bids. Recall from Assumption 1 that the number of bidders *excluding us* is n, which means that the total number of bids is $2n$. Then, we give an example and use the result to compute the winning probabilities, We also give comments on the algorithm that generates the probabilities.

We start the section by defining the following notations:

1) Let $X_i = (i_1, i_2) \in \mathbb{N}^2$ be the bids made by Bidder i. Without loss of generality, we assume that $i_1 < i_2$.
2) For $i_1 < i_2$, we define
$$p_{i_1,i_2} := P(X_i = (i_1, i_2))$$
to be the probability that Bidder i places bid i_1 and i_2. We set $p_{i_1,i_2} = 0$ for $i_1 \geq i_2$.
3) For $j \in \mathbb{N}$, define $\tilde{p}_j$ to be the probability that exactly one bid of X_i is j. In other words,
$$\tilde{p}_j = \sum_{k\in\mathbb{N}} (p_{jk} + p_{kj}).$$
4) For an index set $I \subset \mathbb{N}$, we define $\tilde{p}_{j\setminus I}$ as the probability that exactly one bid is j and the other bid must *not* be from the index set I. That is
$$\tilde{p}_{j\setminus I} = \sum_{\substack{k\in\mathbb{N}\\ k\notin I}} (p_{jk} + p_{kj}).$$

We can see that price limits have the role to make the probabilities computable. To give an example of these notations, suppose the bids distribution of each bidder is, for $i_1, i_2 \in \mathbb{N}$,

$$p_{i_1,i_2} = \begin{cases} (2-q)q^2(1-q)^{i_1+i_2-3}, & i_1 < i_2; \\ 0, & i_1 \geq i_2. \end{cases} \tag{6}$$

We call the random variables with mass functions defined in (6) as *geometric*(q,q). Note that (6) refers to the situation where each bidder places the joint geometric distribution with the same parameter q and with the restriction that both bids cannot be the same. We can work out from the equation that

$$\tilde{p}_j = (2-q)q(1-q)^{j-2}[1 - q(1-q)^{j-1}]. \tag{7}$$

5) For an index set $I \subset \mathbb{N}$, let $\mathfrak{J}(I)$ be a collection of disjoint subsets where each subset contains exactly two elements from I. For $k = 0, 1, \ldots, \lfloor \frac{|I|}{2} \rfloor$, define $\mathfrak{J}^k := \mathfrak{J}^k(I)$ to be the set of the disjoint subsets of $\mathfrak{J}(I)$ that have exactly k subsets. Clearly,
$$\mathfrak{J}(I) = \mathfrak{J}^0 \cup \mathfrak{J}^1 \cup \ldots \cup \mathfrak{J}^{\lfloor |I|/2 \rfloor}.$$
Moreover, we define n_k as the number of distinct elements in $\mathfrak{J}^k$. It can be proved that
$$n_k = \frac{|I|!}{(|I|-2k)!k!2^k}.$$
We may write the collection $\mathfrak{J}^k$ as
$$\mathfrak{J}^k = \{J_1^k, J_2^k, \ldots, J_{n_k}^k\}.$$
That is, for $j = 1, \ldots, n_k$, we have J_j^k as sets of k disjoint subsets where each subset contains exactly two elements from I. In other words,
$$J_j^k = \{J_{j,1}^k, J_{j,2}^k, \ldots, J_{j,k}^k\}.$$
Now, for $l = 1, \ldots, k$, note that $J_{j,l}^k$ is the set which has two elements and $J_{j,l}^k \subset I$. We set
$$\tilde{J}_j^k = \bigcup_{l=1}^{k} J_{j,l}^k.$$

For example, if $I = \{1, 2, 4, 5\}$, then

$$\mathfrak{J}^0 = \Big\{\emptyset\Big\},$$
$$\mathfrak{J}^1 = \Big\{\{\{1,2\}\}, \{\{1,4\}\}, \{\{1,5\}\}, \{\{2,4\}\}, \{\{2,5\}\}, \{\{4,5\}\}\Big\},$$
$$\mathfrak{J}^2 = \Big\{\{\{1,2\},\{4,5\}\}, \{\{1,4\},\{2,5\}\}, \{\{1,5\},\{2,4\}\}\Big\}.$$

Now, we are ready for the main results of this section. For $k_1 < k_2$, recall that $\mathbb{P}_2(k_1, k_2)$ is the probability of winning a 2-bid LUBA when bids k_1 and k_2 are placed. To derive the winning probability, we express it in terms of events of bids. With the similar set up as in the proof of Theorem 2, we remind that E_i is the event that no bidder places the bid

i and U_i is the event that exactly one bidder places the bid i.

Lemma 4. *(Inclusion-exclusion principle of winning probabilities)*
With Assumption 1, for $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$,

$$\mathbb{P}_2(k_1, k_2) = \mathrm{P}(k_1, *) + \mathrm{P}(*, k_2) - \mathrm{P}(k_1, k_2), \quad (8)$$

where,

- $\mathrm{P}(k_1, *)$ *is the probability that our bid k_1 is the winning bid,*
- $\mathrm{P}(*, k_2)$ *is the probability that our bid k_2 is the winning bid* ignoring what happens at k_1,
- $\mathrm{P}(k_1, k_2)$ *is the probability that both of our bids k_1 and k_2 can be the winning bid.*

Proof: Note that to win a 2-bid LUBA, we either win at the bid k_1 or at the bid k_2. Hence,

$$\mathbb{P}_2(k_1, k_2) = P\big(E_{k_1} \cap \bigcap_{i=1}^{k_1-1} U_i^C\big) + P\big(E_{k_1}^C \cap E_{k_2} \cap \bigcap_{i=1, i\neq k_1}^{k_2-1} U_i^C\big). \quad (9)$$

Note that the second term in (9) can be written as

$$P\big(E_{k_1}^C \cap E_{k_2} \cap \bigcap_{i=1, i\neq k_1}^{k_2-1} U_i^C\big) = P\big(E_{k_2} \cap \bigcap_{i=1, i\neq k_1}^{k_2-1} U_i^C\big) - P\big(E_{k_1} \cap E_{k_2} \cap \bigcap_{i=1, i\neq k_1}^{k_2-1} U_i^C\big),$$

as required in the lemma. ■

Theorem 5. *(Winning probabilities in a 2-bid LUBA)*
With Assumption 1. For $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, define

$$q_{k_1} = 1 - \tilde{p}_{k_1}, \quad q_{k_1,k_2} = 1 - \tilde{p}_{k_1} - \tilde{p}_{k_2} + p_{k_1,k_2},$$

and

$$I_1 = \{1, 2, \dots k_1 - 1\}, \quad I_2 = \{k_1 + 1, k_1 + 2, \dots k_2 - 1\}.$$

Then,

$$\mathbb{P}_2(k_1, k_2) = q_{k_1}^n \mathbf{P}_{I_1}^{(1)}(k_1) + q_{k_2}^n \mathbf{P}_{I_1 \cup I_2}^{(1)}(k_2) - q_{k_1,k_2}^n \mathbf{P}_{I_1 \cup I_2}^{(2)}(k_1, k_2), \quad (10)$$

where, for $J \subset \mathbb{N}$ and $(c_d(1), \dots, c_d(d)) := c_d \in \mathbb{N}^d$, for $d = 1, 2$,

$$\mathbf{P}_J^{(d)}(c_d) = 1 - \sum_{i=1}^{|J|} (-1)^{i-1} \sum_{\substack{I \subset J \\ |I|=i}} A^*(U_I), \quad (11)$$

and

$$A^*(U_I) = \sum_{k=0}^{\lfloor \frac{|I|}{2} \rfloor} \sum_{j=1}^{n_k} \Big[\mathcal{P}(n, |I| - 2k)\Big(\prod_{\{r_1,r_2\} \in J_j^k, r_1<r_2} \frac{p_{r_1,r_2}}{q_{c_d}}\Big) \Big(\prod_{s \in I \setminus \tilde{J}_j^k} \frac{\tilde{p}_{s \setminus I \cup \bigcup_{l=1}^{d} c_d(l)}}{q_{c_d}}\Big)\Big(\big(1 - \sum_{r\in I} \frac{\tilde{p}_r}{q_{c_d}} + \sum_{r_1,r_2 \in I, r_1<r_2} \frac{p_{r_1,r_2}}{q_{c_d}}\big)^{n-|I|+k}\Big)\Big]. \quad (12)$$

Proof: To apply a similar technique as in Theorem 2, we write all the terms in Lemma 4 in the conditional probabilities form:

$$\mathrm{P}(k_1, *) = P(E_{k_1}) P\big(\bigcap_{i=1}^{k_1-1} U_i^C | E_{k_1}\big) \quad (13)$$

$$\mathrm{P}(*, k_2) = P(E_{k_2}) P\big(\bigcap_{i=1, i\neq k_1}^{k_2-1} U_i^C | E_{k_2}\big) \quad (14)$$

$$\mathrm{P}(k_1, k_2) = P(E_{k_1} \cap E_{k_2}) P\big(\bigcap_{i=1, i\neq k_1}^{k_2-1} U_i^C | E_{k_1} \cap E_{k_2}\big) \quad (15)$$

Note that $(13) - (15)$ can be determined in a similar way with

$$P(E_i) = (1 - \tilde{p}_i)^n, \text{ for } i = k_1, k_2, \quad (16)$$

and,

$$P(E_{k_1} \cap E_{k_2}) = (1 - \tilde{p}_{k_1} - \tilde{p}_{k_2} + p_{k_1,k_2})^n, \quad (17)$$

therefore, we only show the calculation of the second term on the right hand side of (13). By the similar arguments as in Theorem 2, and (4), we get

$$P\big(\bigcap_{i=1}^{k_1-1} U_i^C | E_{k_1}\big) = 1 - \sum_{i=1}^{k_1-1} (-1)^{i-1} \sum_{\substack{I \subset \{1,\dots,k_1-1\} \\ |I|=i}} P(U_I | E_{k_1}). \quad (18)$$

where $U_I = \bigcap_{j \in I} U_j$. Finally, we derive $P(U_I | E_{k_1})$. Note that we can not apply the result in(5) immediately since the situations on unique bids are more complicated than on the classical LUBA. However, a similar approach can still be applied.
For the index set I, since all bids in the set I must be unique, we therefore categorise the bidders into three types:

(1) Bidders who make exactly two unique bids amongst the set I,
(2) Bidders who make exactly one unique bid amongst the set I,
(3) Bidders who make no unique bid amongst the set I.

Now, by using notations defined earlier, we calculate the conditional probabilities by considering all possible J_j^k for $k = 0, 1, \dots, \lfloor \frac{|I|}{2} \rfloor$ and $j = 1, \dots, n_k$. We may think of the elements in the set $\tilde{J}_j^k$ as the unique bids made by type 1 bidders, the elements in the set $I \setminus \tilde{J}_j^k$ as the bids of type 2 bidders, while type 3 bidders make all of their bid on the elements in the set I^C. With this idea, we get

$$
\begin{aligned}
&P(U_I|E_{k_1}) \\
&= \sum_{k=0}^{\lfloor\frac{|I|}{2}\rfloor}\sum_{j=1}^{n_k}\Bigg[\Bigg(\mathcal{P}(n,k)\prod_{\{r_1,r_2\}\in J_j^k, r_1<r_2}\frac{p_{r_1,r_2}}{1-\tilde{p}_{k_1}}\Bigg) \\
&\quad \Bigg(\mathcal{P}(n-k,|I|-2k)\prod_{s\in I\setminus\tilde{J}_j^k}\frac{\tilde{p}_{s\setminus I\cup\{k_1\}}}{1-\tilde{p}_{k_1}}\Bigg)\Bigg((1- \\
&\quad \sum_{r\in I}\frac{\tilde{p}_r}{1-\tilde{p}_{k_1}}+\sum_{r_1,r_2\in I,r_1<r_2}\frac{p_{r_1,r_2}}{1-\tilde{p}_{k_1}})^{n-|I|+k}\Bigg)\Bigg] \\
&= \sum_{k=0}^{\lfloor\frac{|I|}{2}\rfloor}\sum_{j=1}^{n_k}\Bigg[\mathcal{P}(n,|I|-2k)\Bigg(\prod_{\{r_1,r_2\}\in J_j^k, r_1<r_2}\frac{p_{r_1,r_2}}{1-\tilde{p}_{k_1}}\Bigg) \\
&\quad \Bigg(\prod_{s\in I\setminus\tilde{J}_j^k}\frac{\tilde{p}_{s\setminus I\cup\{k_1\}}}{1-\tilde{p}_{k_1}}\Bigg)\Bigg((1-\sum_{r\in I}\frac{\tilde{p}_r}{1-\tilde{p}_{k_1}} \\
&\qquad +\sum_{r_1,r_2\in I,r_1<r_2}\frac{p_{r_1,r_2}}{1-\tilde{p}_{k_1}})^{n-|I|+k}\Bigg)\Bigg].
\end{aligned}
$$

■

Example 6. The plot of winning probabilities with $n = 40$, where each bidder places bids according to geometric(1/2, 1/2) defined in (6) and the price limit is 100, is given in Figure 2. The highest winning probability occurs at $k_1 = 7$ and $k_2 = 8$ with 0.4982 chance of winning.

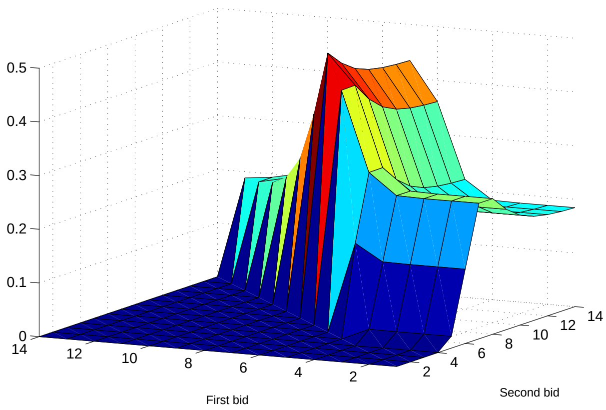

Fig. 2. The winning probabilities in the 2-bid LUBA with $n = 40$ and bids distribution is geometric(1/2, 1/2). The price limit is 100.

Algorithm and comments

1) The algorithm that generates results from Theorem 5 is far more complicated than in Theorem 2. This is because every possible combinations in (12) is needed to be considered, which has no easy way to deal with. We therefore need an algorithm that generates the combinations of subsets in the way of our set up. The algorithm to generated required combinations works as follow Then, the algorithm to generate winning probabilities in (10) can be done as follow:
   a) Declare the number of participated bidders, the bids joint distribution and the price limit.

Algorithm 1: Creating combinations of index set

Input: Index set I and integer value k (must be less than $\lfloor I/2\rfloor$)
Output: $\mathfrak{J}^k$, set of disjoint subsets
Create C, set of all possible subsets of I that contain exactly $2k$ elements.
Label each subset of C as $C_1, C_2, \ldots, C_m$ where $m = \frac{|I|!}{(|I|-2k)!2k!}$.
Set $j = 1$
while $j \leq n_k$ **do**
  for *i from 1 to m* **do**
    call function *subpartition*$(C_i, \emptyset, j)$
  end
end
For the function *subpartition*, label $S_1, \ldots, S_{2k}$ as elements of (subset) S
function *subpartition*(S, P, j)
if $|S| = 2$ **then**
  $J_j^k = P \cup (S_1, S_2)$ and $j = j + 1$
else
  for *l from 2 to 2k* **do**
    subpartition$(S \setminus \{S_1, S_l\}, P \cup (S_1, S_l), j)$
  end
end

   b) Calculate the terms in (12), (11) and (10) respectively. Algorithm 1 is required in order to calculate (12). Note that the three terms in (10) can be calculated using similar commands.
2) Since the rule of 2-bid LUBAs forces each bidder to make two bids, we therefore need to obtain winning probabilities in two dimensional space. However, we may think of an alternative way to find the bid that maximise the chance of winning by calculating winning probabilities where we make one bid, while the others still make two bids. The probabilities are actually $\mathbf{P}_{I_1}^{(1)}(k)$ for all k. We calculate winning probabilities using the same model as in Example 6. The first two optimum bids are 7 and 8 with the chance of winning 0.2527 and 0.2525 respectively, agrees with the calculation in Example 6. The plot of winning probabilities is given in Figure 3.

V. Winning Probabilities as an Observer

In this section, we derive winner distributions in classical LUBAs and 2-bid LUBAs when we do not participate in the auctions. We use similar ideas of the proofs of Theorem 2 and Theorem 5. Recall that $\hat{\mathbb{P}}_1(k)$ and $\hat{\mathbb{P}}_2(k)$ are the probability that the winner bid occurs at the bid k given that n bidders are involved in a classical LUBA and a 2-bid LUBA respectively.

Corollary 7. *(Winner probabilities as an observer)*

(a) *For the classical LUBA with Assumption 1, and $k \in \mathbb{N}$*

$$\hat{\mathbb{P}}_1(k) = \sum_{i=1}^{k-1}(-1)^i \sum_{\substack{I\subset\{1,\ldots,k-1\} \\ |I|=i}} B(I), \qquad (19)$$

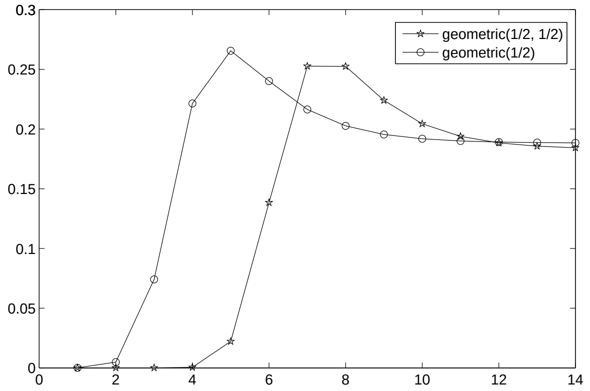

Fig. 3. The winning probabilities by placing one bid in 2-bid LUBA compared with the winning probabilities in the classical LUBAs with $n = 40$. The price limits are 100.

where,

$$B(I) = \mathcal{P}(n, i+1)\Big(\prod_{j\in I\cup\{k\}} p_j\Big)\Big(1 - \sum_{j\in I\cup\{k\}} p_j\Big)^{n-(i+1)}.$$

(b) *For the 2-bid LUBA with Assumption 1, and $k \in \mathbb{N}$*

$$\hat{\mathbb{P}}_2(k) = \sum_{i=0}^{k-1}(-1)^i \sum_{\substack{J=I\cup\{K\}\\ I\subset\{1,\ldots,K-1\}\\ |I|=i}} P(U_J), \qquad (20)$$

where

$$P(U_J) = \sum_{k=0}^{\lfloor\frac{|J|}{2}\rfloor}\sum_{j=1}^{n_k}\Big[\mathcal{P}(n, |J|-2k) \Big(\prod_{\{r_1,r_2\}\in J_j^k, r_1<r_2} p_{r_1,r_2}\Big)\Big(\prod_{s\in J\setminus \tilde{J}_j^k}\tilde{p}_{s\setminus J}\Big) \Big(\big(1 - \sum_{r\in J}\tilde{p}_r + \sum_{r_1,r_2\in J, r_1<r_2} p_{r_1,r_2}\big)^{n-|J|+k}\Big)\Big].$$

Proof:

(a) We display the probability in terms of events as

$$\hat{\mathbb{P}}_1(k) = P\big(U_k \cap \bigcap_{i=1}^{k-1} U_i^C\big) = P\big(\bigcap_{i=1}^{k-1} U_i^C\big) - P\big(\bigcap_{i=1}^{k} U_i^C\big) = P\big(\bigcup_{i=1}^{k} U_i\big) - P\big(\bigcup_{i=1}^{k-1} U_i\big).$$

Now, by using the idea of the inclusion-exclusion principle similar to the proof of Theorem 2, we can deduce that

$$P\big(\bigcup_{i=1}^{k} U_i\big) - P\big(\bigcup_{i=1}^{k-1} U_i\big) = \sum_{i=0}^{k-1}(-1)^i \sum_{\substack{I\subset\{1,\ldots,k-1\}\\ |I|=i}} P(U_I \cap U_k)$$
$$= \sum_{i=0}^{k-1}(-1)^i \sum_{\substack{I\subset\{1,\ldots,k-1\}\\ |I|=i}} \Big[\mathcal{P}(n, i+1)\Big(\prod_{j\in I\cup\{k\}} p_j\Big) \Big(1 - \sum_{j\in I\cup\{k\}} p_j\Big)^{n-(i+1)}\Big].$$

(b) With a similar idea as in the proof of Corollary 7(a), we can see that

$$\hat{\mathbb{P}}_2(k) = \sum_{i=0}^{k-1}(-1)^i \sum_{\substack{I\subset\{1,\ldots,k-1\}\\ |I|=i}} P(U_I \cap U_k) = \sum_{i=0}^{k-1}(-1)^i \sum_{\substack{I\subset\{1,\ldots,k-1\}\\ |I|=i}} P(U_J),$$

where $J = I \cup \{k\}$. Finally, by a similar approach as in Theorem 5, we get

$$P(U_J) = \sum_{k=0}^{\lfloor\frac{|J|}{2}\rfloor}\sum_{j=1}^{n_k}\Big[\mathcal{P}(n, |J|-2k) \Big(\prod_{\{r_1,r_2\}\in J_j^k, r_1<r_2} p_{r_1,r_2}\Big)\Big(\prod_{s\in J\setminus \tilde{J}_j^k}\tilde{p}_{s\setminus J}\Big) \Big(\big(1 - \sum_{r\in J}\tilde{p}_r + \sum_{r_1,r_2\in J, r_1<r_2} p_{r_1,r_2}\big)^{n-|J|+k}\Big)\Big].$$

■

Example 8. We derive winner distributions for the classical LUBA and the 2-bid LUBA where bids distributions are geometric(1/2) and geometric(1/2, 1/2) respectively with $n = 40$ and price limits 100. We can see that the most likely winner bids are 4 and 6 respectively. The plots of winner distributions are given in Figure 4.

VI. Conclusion and Future Work

In this article, we introduce one of the ways to obtain interesting probabilities of LUBAs. We also propose computer algorithms to obtain these probabilities, at which upper price limits of the auctions are required.The limits are acted as the supports of the random set of bids.

From the results in previous sections, it is clear that the number of participated bidders as well as the bid strategies effect the winning probabilities. Therefore, we believe that bid distribution should not be fixed as it can be changed due to human factors. As long as we gain further information of bid distributions, we can update the data to get more accurate results. One choice of the distribution that can be used is

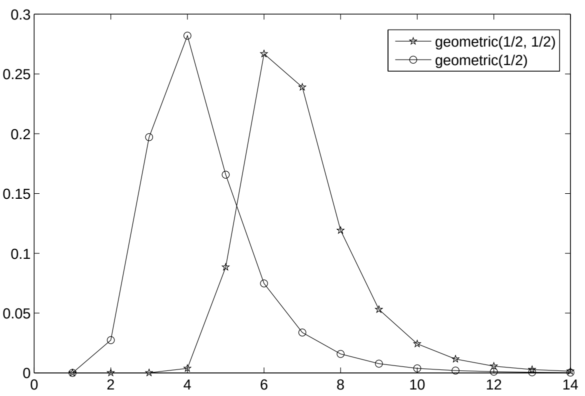

Fig. 4. Winner distributions with $n = 40$ and the price limit 100.

an empirical distribution from real world data set over the internet.

One important assumption from our model is that the number of bidders and the bid strategy need to be fixed, which is, in general, unlikely to happen in the real world. However, it does not affect our models on the condition that we have information on the number of bidders and bid strategies. Indeed, suppose that the number of participated bidders, N, follows a probability distribution with density g and define $g_n = \mathbb{P}(N = n)$. In a classical LUBA, let $\mathbb{P}_1^N(k)$ be the probability that we win the auction when we place the bid k and with N participated bidders, and define $\hat{\mathbb{P}}_1^N(k)$ similarly for the winner probability. Then,

$$\mathbb{P}_1^N(k) = \sum_n g_n \mathbb{P}_1(k), \qquad \hat{\mathbb{P}}_1^N(k) = \sum_n g_n \hat{\mathbb{P}}_1(k).$$

Similarly for the bid strategies, let $s_1, s_2, \ldots, s_d$ be possible bid strategies with, respectively, $h_1, h_2, \ldots, h_d$ probabilities of being chosen. Then, we can again apply the law of total probability to find a represented bid distribution.

The main advantage of this work is to get exact calculations of winning probabilities, which are obtained via inclusion-exclusion principle technique. However, the drawback of our results is the use of all possible combinations of unique bids that come out during the calculation of the probabilities, which make the problem uncomputable by hands. The algorithm to calculate results, especially Algorithm 1, take a considerable time to process. Nevertheless, we believe that the idea of the proofs and the use of the inclusion-exclusion principle will benefit the readers who want to tackle similar kind of problems.

Last but not least, we would like to introduce the idea to obtain the winning probabilities for the m-bid LUBAs when $m \geq 3$. By defining similar notations as in the beginning of Section 3 properly, we expect similar results and proofs as in Theorem 5. Lemma 9 describe a result in the similar way as in Lemma 4. The lemma can be proved via Mathematical induction. Let $\mathbb{P}_m(k_1, k_2, \ldots, k_m)$ be the probability of winning the bid when bids $k_1, k_2, \ldots, k_m$ are placed.

Lemma 9. *(Inclusion-exclusion principle for m-bid LUBA)*

$$\mathbb{P}_m(k_1, k_2, \ldots, k_m) = \sum_{i=1}^{m} (-1)^{i+1} \sum_{\substack{I \in (\prod_{j=1}^{m} \{*, k_j\}) \\ |I| = i}} \mathrm{P}(I),$$

where

- $I = (x_1, x_2, \ldots, x_m)$ *is an* m*-dimensional vector, and* x_i *must be either* $*$ *or* k_i *for* $i = 1, \ldots, m$,
- $|I|$ *is the number of* x_i*'s which are not* $*$.

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