Two-grid Methods for Characteristic Finite Volume Element Approximations of Semi-linear Sobolev Equations

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Abstract—In this paper, two-grid methods for characteristic finite volume element solutions are presented for the semi-linear Sobolev equation. The method is based on the methods of characteristics, two-grid method and the finite volume element method. The nonsymmetric and nonlinear iterations are only executed on the coarse grid (with grid size $H$). And the fine-grid solution (with grid size $h$) can be obtained by a single symmetric and linear step. It is proved that the coarse grid can be much coarser than the fine grid. The two-grid methods achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = O(H^3)$. As a result, solving such a large semi-linear Sobolev equations will not be much more difficult than solving one single linearized equation.

Index Terms—Characteristics, Two-grid method, Finite volume element method, Sobolev equation, Error estimates.

I. INTRODUCTION

This paper consider the following semi-linear Sobolev equations:

$$
\begin{cases}
  c(x)u_t + d(x) \cdot \nabla u - \nabla \cdot (a(x) \nabla u_t) - \nabla \cdot (b(x) \nabla u) = f(u, x, t), \quad (x, t) \in \Omega \times I,
  u(x, 0) = u_0(x), \quad x \in \Omega,
  u(x, t) = 0, \quad (x, t) \in \partial \Omega \times I
\end{cases}
$$

(1)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, with boundary $\partial \Omega$. In this paper, we consider the problem with periodic boundary.

$$
d(x) = (d_1(x), d_2(x))^T, \quad I = [0, T], \quad T > 0 \text{ is some fixed final time.} \quad f(u, x, t) \text{ is uniformly Lipschitz continuous with respect to } u.
$$

On the other hand, the coefficients of (1) satisfy

$$
\begin{align*}
(a) \quad &0 < a_1 \leq a(x) \leq a_2, \quad 0 < b_1 \leq b(x) \leq b_2, \quad |d| = \sqrt{d_1^2 + d_2^2} \leq d^* < c(x) \leq c(x), \\
(b) \quad &\left| \frac{d(x)}{c(x)} \right| + \left| \frac{\partial f}{\partial u} \right| \left( \frac{d(x)}{c(x)} \right) \leq K_1, \\
(c) \quad &\left| \frac{\partial f}{\partial x_1} \right| + \left| \frac{\partial^2 f}{\partial u \partial x_1^2} \right| \leq K_2, \quad i = 1, 2; \\
(d) \quad &u \in L^\infty(0, T; W^{q-2}_0(\Omega) \cap W^{q,p}(\Omega)); \\
(e) \quad &\frac{\partial u}{\partial t} \in L^2(0, T; W^{q-2}_0(\Omega)) \cap L^\infty(0, T; W^{q-2}_0(\Omega)); \\
(f) \quad &\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega) \cap H^1(\Omega)).
\end{align*}
$$

where $a_1, a_2, b_1, b_2, d^*, K_1$ and $K_2$ are positive constants and $p > 1, q \geq 2$. Problems of the form (1) commonly arise in the flow of fluids through fissured rocks [1], thermodynamics [2], the migration of moisture in soil [3], and other applications. For a discussion of existence and uniqueness results, see [4]-[8]. Various numerical treatment of this problem can be found in [9]-[19] and their references.

The characteristic method was first introduced by Douglas and Russell in [20]. And then extended by Russell [21] to nonlinear coupled systems in two and three spatial dimensions. The main concept of characteristic method is to combine the time derivative and the convection term as a directional derivative along the characteristics, leading to a characteristic time-stepping procedure. Then, the standard method can be applied to the problem whose form is similar to heat equation. Comparing with standard methods, it can use larger time steps in numerical simulation, and can eliminate the excessive numerical diffusion and nonphysical oscillation.

Finite volume element (FVE) method, as a type of important numerical tool for solving differential equations, was widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. This method is also known as a box method [22], [23] or generalized difference method [24], [25] in China. Perhaps the most important property of FVE method is that it can preserve the conservation laws (mass, momentum and heat flux) on each computational element. This important property, combined with adequate accuracy and ease of implementation, has attracted many researchers to do research [26]-[32]. The theoretical framework and the basic tools for the analysis of FVE method have been developed in the last two decades [26]-[32].

Two-grid method was first introduced by Xu [33]-[35] as a discretization method for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial equations. The basic concept of this method is to solve a complicated problem (nonlinear, etc.) on a coarse grid (with mesh size $H$) and then solve an easier problem (linear, etc.) on a fine grid (with mesh size $h$ and $h \ll H$) as correction. Later on, the two-grid method was further investigated by many authors, for instance, Dawson and Wheeler [36], [37] have applied two-grid mixed finite element method and two-grid finite difference method to a class of parabolic equations, respectively. Wu and Allen [38] have used the two-grid mixed...
finite element method to approximate the reaction-diffusion equations. Utnes [39] have applied this method to Navier-Stokes equations. Bi and Ginting [40] have studied the two-grid finite volume element method for linear and nonlinear elliptic equations. Chen [41], [42] and Chen, Bi [43] have applied the two-grid finite volume element method to a kind of nonlinear parabolic equations and convection diffusion equations, respectively.

In this paper, based on two linear conforming finite element spaces $U_H$ and $U_h$, on one coarse grid (with grid size $H$) and one fine grid (with grid size $h \ll H$), we use the two-grid characteristic finite volume element methods to approximate the semi-linear Sobolev equations (1). We first solve a nonsymmetric and nonlinear problem on the coarse grid, then we use the known coarse grid solution and a Taylor expansion to get the solution of a symmetric and linear system on the fine grid. As shown in [33], the approach can use coarser mesh on the coarse grid without loss of accuracy. The outline of the paper is as follows. In Section 2, preliminaries and notations are introduced. In Section 3, the characteristic FVE method and two-grid characteristic FVE method are presented, respectively. The error estimates in the $H^1$- and $L^2$-norm of characteristic FVE method are demonstrated in Section 4. In Section 5, the error estimates in the $H^2$-norm for the two-grid characteristic FVE method are presented.

Throughout this paper, the letter C denotes a generic positive constant which independent of the mesh parameter and may be different at its different occurrences.

II. PRELIMINARIES AND NOTATION

For a convex polygonal domain $\Omega \subset \mathbb{R}^2$, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ and supply it with a norm $\| \cdot \|_{m,p}$

$$
\|u\|_{m,p} = \left\{ \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^{p}_{p} \right)^{1/p}, \quad 1 \leq p < \infty, \\
\|\cdot\|_{m,p} = \|\cdot\|_{\infty} \right\} \quad \text{for} \quad p = \infty.
$$

Define $W^{m,p}(\Omega)$ as the closure of $C^\infty_c(\Omega)$ with respect to the norm $\| \cdot \|_{m,p}$. In particular when $p = 2$ we write $W^{m,2}(\Omega)$ and $W^{m,2}_0(\Omega)$ as $H^m(\Omega)$ and $H^m_0(\Omega)$, respectively. Note that $H^0(\Omega) = L^2(\Omega)$ and $H^1(\Omega) = \{v \in H^2(\Omega) : v = 0 \text{ on } \partial\Omega\}$.

Let $\Omega$ is a polygonal region with boundary $\partial\Omega$. Divide $\Omega$ into a sum of finite number of small triangles, each triangle is called an element and the vertices of the triangle are called nodes. All the elements $K$ constitute a triangulation of $\Omega$, denoted by $T_h$, where $h$ is the maximum length of all the sides.

Now we construct a dual decomposition $T_h^*$ related to $T_h$. Let $P_0$ be a node of a triangle, $P_1$ $(i = 1, 2, \cdots, 6)$ the adjacent nodes of $P_0$, and $M_i$ the midpoint of $P_0P_i$ (Figure 1). Choose a point $Q_i$ in an element $P_0P_iP_{i+1}$ $(P_7 = P_1)$ and connect successively $M_1, Q_1, \cdots, M_6, Q_6, M_1$ to form a polygonal region $K_{P_0}$, called a dual element. The modification of the definition is obvious when $P_0$ is on the boundary. All the dual elements constitute a new decomposition, called a dual decomposition. $Q_i$ is called a node of the dual decomposition, $Q_i$ is usually chosen as barycenter or circumcenter of the element $K \in T_h$.

In the sequel we denote by $\Omega_h$ the set of the nodes of the decomposition $T_h$, $\Omega_h^* = \Omega_h \setminus \partial\Omega$ the set of the interior nodes, and $\Omega_h^1$ the set of the nodes of the dual decomposition $T_h^*$. For $Q \in \Omega_h^*$, $K_Q$ denotes the triangular element containing $Q$. Let $S_Q$ and $S_P$, be the areas of the triangular element $K_Q$ and the dual element $K_{P_0}$, respectively. We call the mesh $T_h$ and $T_h^*$ are quasi-uniform if there exists constant $c_1, c_2, c_3 > 0$ independent of $h$ such that

$$
c_1h^2 \leq S_Q \leq h^2, Q \in \Omega_h^*,
$$

$$
c_2h^2 \leq S_{P_0} \leq c_3h^2, P_0 \in \Omega_h^1.
$$

The trial function space $U_h$ is chosen as the linear element space related to $T_h$,

$$
U_h = \{ u_h \in C(\overline{\Omega}) : u_h|_K \in P_1, \forall K \in T_h; u_h|_{\partial\Omega} = 0 \},
$$

and the test space $V_h$ is chosen as the piecewise constant function space with respect to $T_h^*$, spanned by the following basis functions: For any point $P_0 \in \Omega_h^*$

$$
\Psi_{P_0}(P) = \begin{cases} 
1, & P \in K_{P_0}^*, \\
0, & \text{elsewhere}. 
\end{cases}
$$

For any $v_h \in V_h$

$$
v_h = \sum_{P_0 \in \Omega_h^*} v_{P_0}(P_0) \Psi_{P_0}.
$$

Then we obtain $U_h = \text{span}\{ \phi_i(x) : P_i \in \Omega_h^1 \}$ and $V_h = \text{span}\{ \psi_i(x) : P_i \in \Omega_h^* \}$ where $\phi_i(x)$ is the nodal basis function associated with the node $P_i$, and $\psi_i(x)$ is the characteristic function of $K_{P_i}$.

For any $u \in H^1(\Omega) \cap H^2(\Omega)$, we define an interpolation operator $\Pi_h : C(\overline{\Omega}) \rightarrow U_h$, such that

$$
\Pi_h u = \sum_{P_0 \in \Omega_h^1} u(P_0) \phi_i(x).
$$

For any $u_h \in U_h$, we define another interpolation operator $\Pi_h^* : U_h \rightarrow V_h$, such that

$$
\Pi^*_h u_h = \sum_{P_0 \in \Omega_h^*} u_h(P_0) \psi_i(x).
$$

By the interpolation theory we have

$$
\|u_h - \Pi^*_h u_h\| \leq C h \|u_h\|_1,
$$

and in [31] that

$$
\|\Pi_h^* u_h\| \leq C \|u_h\|.
$$

Fig. 1. The dual element of $P_0$. (Advance online publication: 10 July 2015)
III. THE CHARACTERISTIC FVE METHOD AND TWO-GRID FVE METHOD

A. The characteristic FVE method

In the characteristic method, the time derivative and the convection term of (1) are combined as a directional derivative along the characteristics direction $\tau = \tau(x)$:

$$c(x) \frac{\partial u}{\partial t} + d(x) \cdot \nabla u = \sqrt{c(x)^2 + |d(x)|^2} \frac{\partial u}{\partial \tau},$$

where

$$\frac{\partial}{\partial \tau} = \frac{1}{\psi(x)} \left( c(x) \frac{\partial}{\partial t} + d(x) \cdot \nabla \right),$$

$$\psi(x) = \sqrt{c(x)^2 + |d(x)|^2}.$$

Then, (1) can be written as

$$\begin{cases}
\psi(x) u_{\tau} - \nabla \cdot (a(x) \nabla u_{\tau}) - \nabla \cdot (b(x) \nabla u) = f(u, x, t), & (x, t) \in \Omega \times I, \\
u(x, 0) = u_0(x), & x \in \Omega, \\
u(x, t) = 0, & (x, t) \in \partial \Omega \times I.
\end{cases}$$

We define a partition of the time interval $[0, T]$ by $t_n = n\Delta t, n = 0, 1, 2, \cdots, N$, with $\Delta t = T/N, \Delta x_n = u_h(t_n)$. In the standard characteristic method [20], the directional derivative along the characteristics is approximated by

$$\psi(x) \frac{\partial u^n}{\partial \tau} \approx \psi(x) u^n(x, t^n) - u^n(x, t^{n-1}) = c(x) \frac{u^n(x, t^n) - u^n(x, t^{n-1})}{\Delta t},$$

where $\tau = \frac{d(x)}{\sqrt{c(x)^2 + |d(x)|^2}} \Delta t$.

The variational problem related to (10) is: Find $u = u(\cdot, t) \in U$ such that

$$\begin{cases}
\psi(x) u_{\tau} + a_h(u, v) + b_h(u, v) = (f(u), v), & \forall v \in V, \\
u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$, and $a_h(u, v) = (a(x) \nabla u, \nabla v), b_h(u, v) = (b(x) \nabla u, \nabla v)$. (12)

Though the trial function space $U_h$ satisfies $U_h \subset U$ like finite element methods, the test space $V_h \not\subset U_h$. As in the case of nonconforming finite element methods, this is due to the loss of continuity of the functions in $V_h$ on the boundary of two neighboring elements. So the bilinear forms $a_h(u, v)$ and $b_h(u, v)$ must be revised accordingly. For nonconforming finite element methods, the idea is to write the integral on the whole region as a sum of the integrals on every element $K$, so $a_h(u, v)$ and $b_h(u, v)$ are rewritten as

$$a_h(u, v) = \sum_{K \in T_h} \int_K (a(x) \nabla u \nabla v) dx, \quad b_h(u, v) = \sum_{K \in T_h} \int_K (b(x) \nabla u \nabla v) dx. \quad \text{(14)}$$

Now $a_h(u, v)$ and $b_h(u, v)$ are well-defined on $U_h \times V_h$. For the FVE methods, i.e. generalized difference methods, we place a dual grid and interpret (14) in the sense of generalized function, i.e. $\delta$ functions on the boundary of neighboring dual elements. Or equivalently, we take $a_h(u, v)$ and $b_h(u, v)$ as the bilinear form resulting from the piecewise integration in parts on the dual elements $K_h^\ast$:

$$\int_{\Omega} v \Delta u dx = - \sum_{K_h \in T_h} \int_{K_h^\ast} \nabla u \nabla v dx + \sum_{K_h \in T_h} \int_{\partial K_h^\ast} (\nabla u) \cdot \mathbf{n} v ds. \quad \text{(16)}$$

where $\int_{\partial K_h^\ast}$ denotes the line integrals, in the counterclockwise direction, on the boundary $\partial K_h^\ast$ of the dual element. So, we have

$$a_h(u, v) = \sum_{K_h \in T_h} \int_{K_h^\ast} (a(x) \nabla u) \nabla v dx,$$

$$b_h(u, v) = \sum_{K_h \in T_h} \int_{\partial K_h^\ast} (b(x) \nabla u) \nabla v dx. \quad \text{(17)}$$

Since the test space $V_h$ is chosen as the piecewise constant function space, so we have

$$a_h(u, v) = - \sum_{K_h^\ast \in T_h} \int_{\partial K_h^\ast} (a(x) \nabla u) \nabla v ds, \quad \text{(19)}$$

$$b_h(u, v) = - \sum_{K_h^\ast \in T_h} \int_{\partial K_h^\ast} (b(x) \nabla u) \nabla v ds. \quad \text{(20)}$$

Then, the semi-discrete FVE formulation of (1) is: Find $u_h = u_h(\cdot, t) \in U_h (0 \leq t \leq T)$ such that

$$\begin{cases}
(u_h, r_{\Pi_h^1} v_h) + a_h(u_h, t_{\Pi^1} v_h) + b_h(u_h, t_{\Pi^1} v_h) = (f(u_h), v_h), & \forall v_h \in U_h, \quad t > 0, \\
u_h(x, 0) = u_{0\Omega}(x). \quad x \in \Omega,
\end{cases}$$

(21)

where $a_h(\cdot, \Pi^1_h \cdot)$ and $b_h(\cdot, \Pi_h^1 \cdot)$ are defined by, for any $u_h, v_h \in U_h$, (19) and (20), respectively, $u_{0\Omega}$ is a certain approximation to $u_0$ on $U_h$. At time $t = t_n$, we use the backward difference quotient

$$\frac{u^n_h - u_0}{\Delta t} \quad \text{(22)}$$

to approximate $u_{h,t}$, then we get the fully-discrete scheme of (1): Find $u^n_h \in U_h (n = 1, 2, \cdots, N)$ such that

$$\begin{cases}
(c(x) \frac{u^n_h - u_0}{\Delta t}, t_{\Pi^1} v_h) + a_h \left( \frac{u^n_h - u_0}{\Delta t}, \Pi_h^1 v_h \right) \\
+ b_h(u^n_h, t_{\Pi^1} v_h) = (f(u^n_h), \Pi_h^1 v_h), \forall v_h \in U_h, \quad t > 0, \\
u^n_h = u_{0\Omega}. \quad x \in \Omega,
\end{cases}$$

(23)

On the other hand, from Lemma 2, we know that there exists a unique local solution for (12) and (23)(see e.g. [24]).

(Advance online publication: 10 July 2015)
B. The two-grid characteristic FVE method

In order to present two-grid FVE method for the semilinear Sobolev equation (1), we introduce two quasi-uniform triangulations of \( \Omega, T_H \) and \( T_h \), with two different mesh sizes \( H \) and \( h(H \gg h) \). We introduce the corresponding finite element spaces \( U_H \) and \( U_h \) which satisfy \( U_H \subset U_h \). They will be called the coarse grid and fine grid spaces, respectively.

The basic idea of two-grid method is to use a coarse grid space to produce a rough approximation of the solution, and use the corresponding finite element spaces \( U_H \) and \( U_h \) which satisfy \( U_H \subset U_h \). We present the two-grid characteristic FVE method as two steps[40], [43]:

**Algorithm 1.**

Step 1: On the coarse grid \( T_H \), find \( u^n_H \in U_H(n = 1, 2, \cdots) \), such that

\[
\begin{aligned}
(c(x) u^n_H - \nabla \cdot \mathbf{v}_h) + a_H (u^n_H - u^{n-1}_H, v_H) \\
+ b_H (u^n_H, v_H) = (f, v_H),
\end{aligned}
\]

\( \forall v_H \in U_H, t > 0, \)

\( u^n_H = u_{0H}. \quad x \in \Omega, \quad (24) \)

Step 2: On the fine grid \( T_h \), find \( u^n_h \in U_h(n = 1, 2, \cdots) \), such that

\[
\begin{aligned}
(c(x) u^n_h - \nabla \cdot \mathbf{v}_h) + a_h (u^n_h - u^{n-1}_h, v_h) \\
+ b_h (u^n_h, v_h) = (f_h, v_h),
\end{aligned}
\]

\( \forall v_h \in U_h, t > 0, \)

\( u^n_h = u_{0h}. \quad x \in \Omega, \quad (25) \)

We note that the second step of Algorithm 1 is a linear problem but still nonsymmetric. In order to get a symmetric system, we introduce the following bilinear forms

\[
a_c (u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx, \forall u_h, v_h \in U_h, \quad (26)
\]

\[
b_c (u_h, v_h) = \int_{\Omega} \bar{b} \nabla u_h \cdot \nabla v_h \, dx, \forall u_h, v_h \in U_h, \quad (27)
\]

\[
a_{h,c} (u_h, \Pi_h v_h) = - \sum_{K_h \in T_h} \int_{\partial K_h} (\nabla \cdot \mathbf{u}_h) \cdot n \Pi_h v_h \, ds, \quad (28)
\]

\[
b_{h,c} (u_h, \Pi_h v_h) = - \sum_{K_h \in T_h} \int_{\partial K_h} (\nabla \cdot \mathbf{u}_h) \cdot n \Pi_h v_h \, ds, \quad (29)
\]

where \( \bar{b} = \text{area}(K) = K_h \), \( b_c (u_h, v_h) = b_h, \)

\[
a_k = \frac{1}{\text{area}(K)} \int_{K} a(x) \, dx \\
b_k = \frac{1}{\text{area}(K)} \int_{K} b(x) \, dx, \forall K \in T_h,
\]

Then from [29], [44], we have the following lemma.

**Lemma 1:** For any \( u_h, v_h \in U_h \), we have

\[
a_{h,c} (u_h, \Pi_h v_h) = a_c (u_h, v_h)
\]

From this lemma we can see that \( a_{h,c}(u_h, \Pi_h v_h) \) and \( a_{h,c}(u_h, \Pi_h v_h) \) are symmetric. Then we obtain the second algorithm.

**Algorithm 2.**

Step 1: On the coarse grid \( T_H \), find \( u^n_H \in U_H(n = 1, 2, \cdots) \), such that

\[
\begin{aligned}
(c(x) u^n_H - \nabla \cdot \mathbf{v}_h) + a_H (u^n_H - u^{n-1}_H, v_H) \\
+ b_H (u^n_H, v_H) = (f, v_H),
\end{aligned}
\]

\( \forall v_H \in U_H, t > 0, \)

\( u^n_H = u_{0H}. \quad x \in \Omega, \quad (30) \)

Step 2: On the fine grid \( T_h \), find \( u^n_h \in U_h(n = 1, 2, \cdots) \), such that

\[
\begin{aligned}
(c(x) u^n_h - \nabla \cdot \mathbf{v}_h) + a_h (u^n_h - u^{n-1}_h, v_h) \\
+ b_h (u^n_h, v_h) = (f_h, v_h),
\end{aligned}
\]

\( \forall v_h \in U_h, t > 0, \)

\( u^n_h = u_{0h}. \quad x \in \Omega, \quad (31) \)

We note that the coefficient matrixes of the system in Algorithm 2 are symmetric. So the system is easier to solve (e.g. conjugate-gradient-like methods can be applied effectively). We call these algorithms as two-grid characteristic FVE methods.

**IV. ERROR ANALYSIS FOR CHARACTERISTIC FVE METHOD**

A. Some lemmas

**Lemma 2:** ([29], [24]) There exist positive constants \( h_0, \alpha \) and \( M \) such that when \( 0 < h \leq h_0 \), the coercive property

\[
a_h (u_h, \Pi_h v_h) \geq \alpha \| u_h \|_1^2, \forall u_h \in U_h
\]

and the boundness property

\[
|a_h (u_h, \Pi_h v_h) \leq M \| u_h \|_1 \| v_h \|_1
\]

hold true.

**Lemma 3:** ([24]) Set \( \| u_h \|_{0} = (u_h, \Pi_h u_h)^{1/2} \) and \( \| u_h \|_{1} = (b(u_h, \Pi_h u_h))^{1/2} \). Then \( \| u_h \|_{0} \) and \( \| u_h \|_{1} \) are equivalent to \( \| u \|_{0} \) and \( \| u \|_{1} \) on \( U_h \), respectively, that is, there exist positive constants \( c_1, c_2, c_3, c_4 \) such that

\[
c_1 \| u_h \|_{1} \leq \| u_h \|_{0} \leq c_2 \| u_h \|_{1}, \forall u_h \in U_h.
\]

\[
c_3 \| u_h \|_{0} \leq \| u_h \|_{1} \leq c_4 \| u_h \|_{0}, \forall u_h \in U_h.
\]

**Lemma 4:** ([24], [44]) For any \( u_h, v_h \in U_h \), there are

\[
(a_h (u_h, \Pi_h v_h) - a_h (v_h, \Pi_h u_h)) \leq C \| u_h \|_1 \| v_h \|_1
\]

and

\[
(a_h (u_h, \Pi_h v_h) = (v_h, \Pi_h u_h))
\]

**Lemma 5:** ([24], [44]) Define an elliptic operator \( P_h : C(\Omega) \rightarrow U_h \), such that

\[
a_h (u - P_h u, \Pi_h v_h) = 0, \forall v_h \in U_h.
\]
Then we have
\[ \| u - P_h u \| \leq Ch \| u \|_2, \]
\[ \| u - P_h u \| \leq Ch^2 \| u \|_{3,p}, p > 1, \]
\[ \| u - P_h u \|_{0,\infty} \leq C h^2 \| h^n \| \left( \| u \|_3 + \| u \|_{2,\infty} \right). \]

**Lemma 6:** (Gronwall lemma) Let \( a_k, b_k, \alpha_k, d_k, \) are non-negative sequence, for \( k \geq 0, \) and satisfy
\[ a_J + \sum_{k=0}^{J} b_k \Delta t \leq \sum_{k=0}^{J} a_k d_k \Delta t + \sum_{k=0}^{J} \alpha_k \Delta t, \]
then
\[ a_J + \sum_{k=0}^{J} b_k \Delta t \leq \exp \left[ 2 \sum_{k=0}^{J} d_k \Delta t \right] \sum_{k=0}^{J} \alpha_k \Delta t. \]

**Lemma 7:** (43) Let \( u \in L^\infty(0,T; H^1(\Omega)) \) and \( \pi^{-1} = u (\mathcal{T}, T^{-1} \mathcal{T}), \) where \( \mathcal{T} \) is defined by (11), then we have
\[ \| u_n - \pi^{-1} \| \leq C \| u - \pi^{-1} \|_1 \Delta t. \]

**B. Error analysis for characteristic FVE method**

Now we consider the error estimate for the characteristic finite volume element method of (1). The error estimates in the \( H^1 \) and \( L^2 \) norm will be given in the following Theorem 4.1 and Theorem 4.2.

**Theorem 4.1:** Let \( u \) and \( u_h \) be the solutions of (12) and (23), respectively. Under assumption (a)-(f), for \( \Delta t \) small enough, if \( u_n^h = P_h u_0 \) with \( P_h \) defined by Lemma 5, we have, for \( t_n \leq T, \)
\[ \max_{1 \leq n \leq N} \| u^n - u_n^h \|_1 \leq C (\Delta t + h). \] \[ (32) \]
where \( C = C \left( \| u \|_{L^\infty(H^1(\Omega))}, \| u \|_{H^1(\Omega)} \right) \) and \( \| u \|_{H^1(\Omega)} \) is independent of \( h \) and \( \Delta t. \)

**Proof:** For convenience, let \( u_n^n = u_n^h - P_h u^n \) and \( u^n - P_h u^n = e^n - \rho^n. \) Then from (12) and (23), we get the following error equations at \( t_n \)
\[ (c(x)\frac{e^n - \pi^{-1}}{\Delta t}, \Pi_h v_h) + b_h (\partial_t e^n, \Pi_h v_h) + a_h (u^n_t, \Pi_h v_h) + b_h (u^n_t, \Pi_h v_h) + a_h (u^n_{\partial_t} v_h, \Pi_h v_h) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h v_h) + (f(u^n) - f(u^n), \Pi_h v_h) \] \[ = (\pi^{-1} - P_h u^n, \Pi_h v_h) + \psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h v_h) + a_h (u^n_t, \Pi_h v_h) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h v_h) + (f(u^n) - f(u^n), \Pi_h v_h). \] \[ (33) \]
Letting \( \partial_t e^n \equiv \frac{e^n - e^{n-1}}{\Delta t} \) and choosing \( v_h = \partial_t e^n \), we obtain
\[ (c(x)\frac{e^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + b_h (\partial_t e^n, \Pi_h \partial_t e^n) + \psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + a_h (u^n_t, \Pi_h \partial_t e^n) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n). \]

Rewritten (34) as
\[ (c(x)\partial_t e^n, \Pi_h \partial_t e^n) + a_h (\partial_t e^n, \Pi_h \partial_t e^n) + b_h (e^n, \Pi_h \partial_t e^n) + (\psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + a_h (u^n_t, \Pi_h \partial_t e^n) + (f(u^n) - f(u^n), \Pi_h \partial_t e^n) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + (\psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + a_h (u^n_t, \Pi_h \partial_t e^n) + (f(u^n) - f(u^n), \Pi_h \partial_t e^n). \] \[ (34) \]
Now we estimate (34). First
\[ b_h (e^n, \Pi_h \partial_t e^n) = \frac{1}{2 \Delta t} \left[ b_h (e^n + e^{n-1}, \Pi_h (e^n - e^{n-1})) + b_h (e^n - e^{n-1}, \Pi_h (e^n - e^{n-1})) \right] \]
\[ \leq \frac{1}{2 \Delta t} \left[ \left( \psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + a_h (u^n_t, \Pi_h \partial_t e^n) + (f(u^n) - f(u^n), \Pi_h \partial_t e^n) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \right) \right. \]
\[ + a_h (u^n_t, \Pi_h \partial_t e^n) + (f(u^n) - f(u^n), \Pi_h \partial_t e^n) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \] \[ + \frac{1}{2} \left[ b_h (\partial_t e^n, \Pi_h \partial_t e^n) - b_h (e^n, \Pi_h \partial_t e^n) \right]. \] \[ (35) \]

By (34) and (35), we have
\[ (c(x)\partial_t e^n, \Pi_h \partial_t e^n) + a_h (\partial_t e^n, \Pi_h \partial_t e^n) + \frac{1}{2 \Delta t} \left[ b_h (e^n + e^{n-1}, \Pi_h (e^n - e^{n-1})) + b_h (e^n - e^{n-1}, \Pi_h (e^n - e^{n-1})) \right] \]
\[ \leq \frac{1}{2 \Delta t} \left[ \left( \psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) + a_h (u^n_t, \Pi_h \partial_t e^n) + (f(u^n) - f(u^n), \Pi_h \partial_t e^n) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \right) \right. \]
\[ + a_h (u^n_t, \Pi_h \partial_t e^n) + (f(u^n) - f(u^n), \Pi_h \partial_t e^n) + (c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \] \[ + \frac{1}{2} \left[ b_h (\partial_t e^n, \Pi_h \partial_t e^n) - b_h (e^n, \Pi_h \partial_t e^n) \right]. \] \[ (36) \]

Multiplying by \( \Delta t \) and summing over \( l \) from 1 to \( l(1 \leq n \leq N) \) at both sides of (36), by Lemma 3 and Lemma 4, since \( e^0 = 0 \) we have
\[ \frac{1}{2} \left( \| e^n \|^2 + C \sum_{l=1}^{n} \| \partial_t e^n \|^2 \Delta t + C \sum_{l=1}^{n} \| \partial_t e^n \|_1^2 \Delta t \right) \]
\[ \leq \sum_{l=1}^{n} \left( \psi(x) \frac{\partial u^n}{\partial t} - c(x) \frac{u^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \right) \Delta t \]
\[ + \sum_{l=1}^{n} a_h (u^n_t, \Pi_h \partial_t e^n) \Delta t \]
\[ + \sum_{l=1}^{n} \left( c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \right) \Delta t \]
\[ + \sum_{l=1}^{n} \left( c(x)\frac{\rho^n - \pi^{-1}}{\Delta t}, \Pi_h \partial_t e^n) \right) \Delta t. \]
\[\begin{align*}
&+ \sum_{l=1}^{n} \left( c(x) \frac{v_l - v_l^1}{\Delta t}, \Pi_k^l \partial_t e^l \right) \Delta t \\
&+ \frac{1}{2} \sum_{l=1}^{n} \left[ b_h \left( \partial_t e^l, \Pi_k^l e^l \right) - b_h \left( e^l, \Pi_k^l \partial_t e^l \right) \right] \Delta t \\
&+ \sum_{l=1}^{n} \left( f(u_h^l) - f(u^l), \Pi_k^l \partial_t e^l \right) \Delta t \equiv \sum_{i=1}^{7} T_i \quad (37)
\end{align*}\]

We now estimate the right-hand terms of (37) For \( T_1 \), from the results given in [21], that

\[\begin{align*}
\left\| \psi \frac{\partial u^l}{\partial t} - c(x) \frac{u^l - u^{l-1}}{\Delta t}, \Pi_k^l \partial_t e^l \right\| &\leq C \Delta t \left\| \frac{\partial^2 u}{\partial t^2} \right\| \left\| \Pi_k^l \partial_t e^l \right\| \\
&\leq C \Delta t \left\| \frac{\partial^2 u}{\partial t^2} \right\| \left\| \Pi_k^l \partial_t e^l \right\| L^2(I _{t_{n-1} - t_0}, L^2(\Omega)).
\end{align*}\]

by (9) and \( \varepsilon \)-inequality

\[\begin{align*}
|T_1| &\leq \sum_{l=1}^{n} \left| \left( \psi(x) \frac{\partial u^l}{\partial t} - c(x) \frac{u^l - u^{l-1}}{\Delta t} \right), \Pi_k^l \partial_t e^l \right| \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left| \left( \psi(x) \frac{\partial u^l}{\partial t} - c(x) \frac{u^l - u^{l-1}}{\Delta t} \right) \right| \Delta t \\
&+ \varepsilon \sum_{l=1}^{n} \left\| \Pi_k^l \partial_t e^l \right\| \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_2^2 (\Delta t)^2 \\
&+ \varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_2^2 \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_2^2 (\Delta t)^2 \\
&+ \varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_2^2 \Delta t
\end{align*}\]

For \( T_2 \), from the results given in [21], we get

\[\begin{align*}
|T_2| &\leq \sum_{l=1}^{n} \left| a_h \left( \partial_t u^l - u^l, \Pi_k^l \partial_t e^l \right) \right| \Delta t \\
&\leq M \sum_{l=1}^{n} \left\| \partial_t u^l - u^l \right\|_1 \left\| \Pi_k^l \partial_t e^l \right\|_1 \Delta t \\
&\leq MC(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t u^l - u^l \right\|_1^2 \Delta t + \varepsilon \sum_{l=1}^{n} \left\| \Pi_k^l \partial_t e^l \right\|_1^2 \Delta t \\
&\leq MC(\varepsilon) \sum_{l=1}^{n} \left( \int_{t_{l-1}}^{t_l} \left\| u_{tt}^l \right\| dt \right)^2 \Delta t \\
&+ \varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\
&\leq MC(\varepsilon) \left( \int_{0}^{t} \left\| u_{tt}^l \right\| dt \right) (\Delta t)^2 \\
&+ \varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t
\end{align*}\]

For \( T_3 \), from Lemma 5, we get

\[\begin{align*}
|T_3| &\leq \sum_{l=1}^{n} \left| \left( c(x) \frac{v_l^1 - v_l}{\Delta t}, \Pi_k^l \partial_t e^l \right) \right| \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \frac{v_l^1 - v_l}{\Delta t} \right\|_1 \left\| \Pi_k^l \partial_t e^l \right\| \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\
&+ C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t
\end{align*}\]

By Lemma 5 and Lemma 7, for \( T_4 \), we obtain

\[\begin{align*}
|T_4| &\leq \sum_{l=1}^{n} \left| \left( c(x) \frac{v_l - v_l^1}{\Delta t}, \Pi_k^l \partial_t e^l \right) \right| \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \frac{v_l - v_l^1}{\Delta t} \right\| \left\| \Pi_k^l \partial_t e^l \right\| \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| \Pi_k^l \partial_t e^l \right\|_1 \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| \Pi_k^l \partial_t e^l \right\|_1 \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t \\
&+ C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t
\end{align*}\]

For \( T_5 \), we have the similar result,

\[\begin{align*}
|T_5| &\leq \sum_{l=1}^{n} \left| \left( c(x) \frac{v_l^1 - v_l}{\Delta t}, \Pi_k^l \partial_t e^l \right) \right| \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \frac{v_l^1 - v_l}{\Delta t} \right\| \left\| \Pi_k^l \partial_t e^l \right\| \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| \Pi_k^l \partial_t e^l \right\|_1 \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| \Pi_k^l \partial_t e^l \right\|_1 \Delta t \\
&\leq C(\varepsilon) \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t + C\varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1^2 \Delta t
\end{align*}\]

For \( T_6 \), by Lemma 4 and the inverse estimate, we have

\[\begin{align*}
|T_6| &\leq \frac{1}{2} \sum_{l=1}^{n} \left| b_h \left( \partial_t e^l, \Pi_k^l e^l \right) - b_h \left( e^l, \Pi_k^l \partial_t e^l \right) \right| \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| e^l \right\|_1 \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| e^l \right\|_1 \Delta t \\
&\leq C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| e^l \right\|_1 \Delta t
\end{align*}\]

(Advance online publication: 10 July 2015)
For $T_7$, at any point $x \in \Omega$, by the Taylor expansion, we have
\[
\frac{f(u_h^n) - f(u^n)}{\Delta t} = f'(\tilde{u}^n)(u_h^n - u^n) = f'(\tilde{u}^n)(\epsilon^n - \rho^n),
\]
for some value $\tilde{u}^n$ between $u_h^n$ and $u^n$. By assumption (c) and Lemma 5, we have
\[
|T_7| \leq \sum_{i=1}^{n} \frac{|f(u_h^i) - f(u^i)|}{\Delta t} \leq \sum_{i=1}^{n} \frac{\|f(u_h^i) - f(u^i)\|}{\Delta t} \leq C(\epsilon \sum_{i=1}^{n} \|f(u_h^i) - f(u^i)\|^2 \Delta t)
\]
and kicking the two terms into the left side of (49), we obtain
\[
(c(x)\partial_t e^n, \Pi_h^* e^n) + a_h (\partial_t e^n, \Pi_h^* e^n) + b_h (e^n, \Pi_h^* e^n)
\]
\[
\leq \frac{1}{2\Delta t} \left[ a_h (e^n - e^{n-1}, \Pi_h^* (e^n + e^{n-1})) + b_h (e^n, \Pi_h^* e^n) \right]
\]
(50)
For the first term of the left-hand side of (50), we have
\[
(c(x)\partial_t e^n, \Pi_h^* e^n) = \frac{1}{2\Delta t} \left[ a_h (e^n - e^{n-1}, \Pi_h^* (e^n + e^{n-1})) + b_h (e^n, \Pi_h^* e^n) \right]
\]
(51)
For the second term of the left-hand side of (50), we have
\[
a_h (\partial_t e^n, \Pi_h^* e^n) = \frac{1}{2\Delta t} \left[ a_h (e^n - e^{n-1}, \Pi_h^* (e^n + e^{n-1})) + b_h (e^n, \Pi_h^* e^n) \right]
\]
(52)
By (50)-(52), we have
\[
\frac{1}{2\Delta t} \frac{1}{2} \frac{c(x)(\|e^n\|_2^2 - \|e^{n-1}\|_2^2)}{\Delta t} + \frac{1}{2} \frac{c(x)(\|e^n\|_2^2 - \|e^{n-1}\|_2^2)}{\Delta t} + b_h (e^n, \Pi_h^* e^n)
\]
(53)
Together Lemma 5, we have the estimate (32).

Theorem 4.2: Let $u$ and $u_h$ be the solutions of (12) and (23), respectively. Under assumption (a)-(f), for $\Delta t$ small enough, if $u_0^h = P_h u_0$ with $P_h$ defined by Lemma 5, we have, for $t_n \leq T$,
\[
\frac{\max_{1 \leq n \leq N} \|u^n - u_h^n\|}{\Delta t} \leq C(\Delta t + h^2).
\]
(49)
where $C = C([\|u\|_{L^\infty(H^2(\Omega))}, \|u_r\|_{L^2(H^1(\Omega))}, \|u_{tt}\|_{L^2(H^1(\Omega))}, \|u_t\|_{L^\infty(W^{3,p}(\Omega))}, \|u_t\|_{L^\infty(W^{3,p}(\Omega))})$ is independent of $h$ and $\Delta t$.

Proof: In (33), choosing $v_n = e^n$ we obtain
\[
(c(x)\partial_t e^n, \Pi_h^* e^n) + a_h (\partial_t e^n, \Pi_h^* e^n) + b_h (e^n, \Pi_h^* e^n)
\]
(54)
\[
\leq \frac{1}{2\Delta t} \left[ a_h (e^n - e^{n-1}, \Pi_h^* (e^n + e^{n-1})) + b_h (e^n, \Pi_h^* e^n) \right]
\]
(55)
Then we have
\[
\|e^n\|_1 \leq C(h + \Delta t),
\]
(48)
By Lemma 2 and (50)-(52) multiplying $\Delta t$ and summing over $l$ from 1 to $n(1 \leq n \leq N)$ at both sides of (53), since $e^0 = 0$ we have
\[
C \|e^0\|^2 + \|e^n\|^2 + C \sum_{l=1}^{n} \|e^l\|^2 \Delta t \leq \sum_{l=1}^{n} \left( \psi(x) \frac{\partial u^l}{\partial x^l} - c(x) \frac{u^l - \Pi^l_{\epsilon} e^l}{\Delta t}, \Pi^l_{\epsilon} e^l \right) \Delta t
\]
\[+ \sum_{l=1}^{n} a_h \left( u^l_t - \partial_t u^l \Pi^l_{\epsilon} e^l \right) \Delta t + \frac{1}{2} \sum_{l=1}^{n} \left[ a_h \left( e^l \Pi^l_{\epsilon} \partial_t e^l \right) - a_h \left( \partial_t e^l \Pi^l_{\epsilon} e^l \right) \right] \Delta t
\]
\[+ \sum_{l=1}^{n} \left( f(u^l_t) - f(u^l) \Pi^l_{\epsilon} e^l \right) \Delta t \equiv Q_{41} + Q_{42}.
\]
For $Q_{11}$, similar as the estimate of $T_1$, we have
\[
|Q_{11}| \leq C \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2((0,T);L^2(\Omega))] \Delta t)^2 + C \varepsilon \sum_{l=1}^{n} \|e^l\|^2 \Delta t.
\]
\[\text{Similarly, for } Q_{2}, \text{ we have}
\]
\[
|Q_{2}| \leq MC(\varepsilon) \left\| \int_0^t \|u_{tt}\|^2 dt \right\| (\Delta t)^2 + C \varepsilon \sum_{l=1}^{n} \|e^l\|^2 \Delta t.
\]
For $Q_{3}$, we have
\[
|Q_{3}| \leq C(\varepsilon)h^2 \left\| \frac{\partial u}{\partial t} \right\|_{L^2((0,T);L^2(\Omega))] \Delta t} + C \varepsilon \sum_{l=1}^{n} \|e^l\|^2 \Delta t.
\]
For $Q_{4}$, we have
\[
|Q_{4}| = \sum_{l=1}^{n} \left( \psi(x) \frac{u^l - \Pi^l_{\epsilon} e^l}{\Delta t}, \Pi^l_{\epsilon} e^l \right) \Delta t
\]
\[\leq \sum_{l=1}^{n} \left( c(x) \frac{u^l - \Pi^l_{\epsilon} e^l}{\Delta t}, \Pi^l_{\epsilon} e^l - e^l \right) \Delta t
\]
\[+ \sum_{l=1}^{n} \left( c(x) \frac{u^l - \Pi^l_{\epsilon} e^l}{\Delta t}, e^l \right) \Delta t
\]
\[\equiv Q_{41} + Q_{42} + Q_{43}.
\]
For $Q_{41}$, by (8), Lemma 5 and Lemma 7, we obtain
\[
Q_{41} \leq \sum_{l=1}^{n} \left( \psi(x) \frac{u^l - \Pi^l_{\epsilon} e^l}{\Delta t}, \Pi^l_{\epsilon} e^l - e^l \right) \Delta t 
\]
\[\leq C \sum_{l=1}^{n} \left\| \frac{u^l - \Pi^l_{\epsilon} e^l}{\Delta t} \right\| \left\| \Pi^l_{\epsilon} e^l - e^l \right\| \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| \frac{\rho^{l-1} - \rho^{l-1}}{\Delta t} \right\| \left\| \Pi^l_{\epsilon} e^l - e^l \right\| \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| \rho^{l-1} \right\|_1 \left\| e^l \right\|_1 \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| \rho^{l-1} \right\|_1 ^2 \Delta t + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t
\]
\[\leq C h^2 \sum_{l=1}^{n} \left\| u^l \right\|_{L^2((0,T);H^2(\Omega))} + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t
\]
In [20] Douglas and Russell have proved that
\[
\left( e(x) \frac{\rho^{l-1} - \rho^{l-1}}{\Delta t}, e^l \right) \leq C \left\| \frac{\rho^{l-1} - \rho^{l-1}}{\Delta t} \right\|_1 \left\| e^l \right\|_1 ^2,
\]
using this inequality and Lemma 5, we have
\[
Q_{42} \leq C \sum_{l=1}^{n} \left\| \rho^{l-1} \right\|_1 ^2 \Delta t + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t
\]
\[\leq C h^4 \sum_{l=1}^{n} \left\| u^l \right\|_{L^2((0,T);H^2(\Omega))} + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t.
\]
Then, there is
\[
|Q_4| \leq C h^4 \sum_{l=1}^{n} \left\| u^l \right\|_{L^2((0,T);H^2(\Omega))} + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t
\]
\[+ C h^4 \sum_{l=1}^{n} \left\| u^l \right\|_{L^2((0,T);H^2(\Omega))}.
\]
For $Q_5$, by Lemma 7 and (9), we have
\[
|Q_5| \leq \sum_{l=1}^{n} \left( e(x) \frac{\rho^{l-1} - \rho^{l-1}}{\Delta t}, \Pi^l_{\epsilon} e^l \right) \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| \frac{\rho^{l-1} - \rho^{l-1}}{\Delta t} \right\| \left\| \Pi^l_{\epsilon} e^l \right\| \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| e^l \right\|_1 \left\| \Pi^l_{\epsilon} e^l \right\| \Delta t
\]
\[\leq C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t.
\]
For $Q_6$, we have
\[
|Q_6| \leq \sum_{l=1}^{n} \left| a_h \left( e^l, \Pi^l_{\epsilon} \partial_t e^l \right) - a_h \left( \partial_t e^l, \Pi^l_{\epsilon} e^l \right) \right| \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 \left\| e^l \right\|_1 \Delta t \leq \sum_{l=1}^{n} \left\| \partial_t e^l \right\| \left\| e^l \right\|_1 ^2 \Delta t
\]
\[\leq C \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t + C \varepsilon \sum_{l=1}^{n} \left\| \partial_t e^l \right\|_1 ^2 \Delta t.
\]
For $Q_7$, we have
\[
|Q_7| \leq \sum_{l=1}^{n} \left( f(u^l) - f(u^l), \Pi^l_{\epsilon} e^l \right) \Delta t
\]
\[\leq \sum_{l=1}^{n} \left\| f(u^l) - f(u^l) \right\| \left\| \Pi^l_{\epsilon} e^l \right\| \Delta t
\]
\[\leq C(\varepsilon) \sum_{l=1}^{n} \left\| f(u^l) - f(u^l) \right\| \left\| \Pi^l_{\epsilon} e^l \right\| \Delta t
\]
\[\leq C(\varepsilon) \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t + C \varepsilon \sum_{l=1}^{n} \left\| e^l \right\|_1 ^2 \Delta t.
\]

(Advance online publication: 10 July 2015)
Combining the error estimates of $Q_i (1 \leq i \leq 7)$ with (54), we have
\[
C \| e^n \|^2 + \| e^n \|^2 + C \sum_{l=1}^{n} \| e^n \|^2 \Delta t.
\]

Then we get, for all $t < \frac{T}{2}$ and kicking the two terms into the left hand side of (65), and applying discrete Gronwall Lemma 6, we get
\[
\| e^n \|^2 + \| e^n \|^2 \Delta t.
\]

Choosing proper $\varepsilon$ and kicking the two terms into the left hand side of (65), and applying discrete Gronwall Lemma 6, we get
\[
\| e^n \|^2 + \| e^n \|^2 + C \sum_{l=1}^{n} \| e^n \|^2 \Delta t.
\]

Then we get, for all
\[
\| e^n \| \leq C (\Delta t + h^2),
\]
Together Lemma 5, we have the estimate (49).

**V. ERROR ANALYSIS FOR TWO-GRID CHARACTERISTIC FVE METHOD**

In this section, we consider the error estimates in the $H^1$-norm for the two-grid characteristic FVE method. For the two-grid characteristic FVE method Algorithm 1, we have:

**Theorem 5.1:** Let $u$ and $u_h$ are the solutions of (12) and the two-grid FVE method Algorithm 1, respectively. Under assumption(a)-(l) and the coarse grid size $H$ and the time step $\Delta t$ satisfies $H^{-1} \Delta t < C$. For $\Delta t$ small enough, if $u_0^h = P_h u_0$ with $P_h$ defined by Lemma 5, we have, for $t_n \leq t$, the following estimate
\[
\max_{1 \leq n \leq N} \| u^n - u^n_h \|_1 \leq C (\Delta t + h + H^3).
\]

**Proof:** Once again, let $u^n_h - u^n = (u^n_h - P_h u^n) - (u^n - \rho^n)$. Then from (12) and (30), we get the following error equation at $t_n$:
\[
\begin{align*}
\left( c(x) \frac{e^n - \rho^n}{\Delta t}, \Pi_h v_h \right) &+ \rho_h \left( e^n, \Pi_h v_h \right) = \left( \psi \frac{\partial u}{\partial t} - c(x) \frac{u^n - \rho^n}{\Delta t}, \Pi_h v_h \right) \\
&+ a_h \left( u^n_t - \delta_t u^n_h, \Pi_h v_h \right) + \left( c(x) \frac{\rho^n - \rho^n}{\Delta t}, \Pi_h v_h \right) \\
&- (f(u^n) - f(u^n_H)) - f'(u^n_H)(u^n_h - u^n_H), \Pi_h v_h),
\end{align*}
\]

Choosing $v_h = \delta_t e^n$ and (69) can be written as
\[
\begin{align*}
\left( c(x) \delta_t e^n, \Pi_h \delta_t e^n \right) &+ a_h \left( \delta_t e^n, \Pi_h \delta_t e^n \right) \\
&+ b_h (e^n, \Pi_h \delta_t e^n) = \left( \psi \frac{\partial u}{\partial t} - c(x) \frac{u^n - \rho^n}{\Delta t}, \Pi_h \delta_t e^n \right) \\
&+ a_h \left( u^n_t - \delta_t u^n_h, \Pi_h \delta_t e^n \right) + \left( c(x) \frac{\rho^n - \rho^n}{\Delta t}, \Pi_h \delta_t e^n \right) \\
&- (f(u^n) - f(u^n_H)) - f'(u^n_H)(u^n_h - u^n_H), \Pi_h \delta_t e^n).
\end{align*}
\]

By Lemma 2 and (37), we have
\[
\begin{align*}
\frac{\alpha}{2} \| e^n \|^2 + C \sum_{l=1}^{n} \| \delta_t e^n \|^2 \Delta t + C \sum_{l=1}^{n} \| \delta_t e^n \|^2 \Delta t \\
&\leq \sum_{l=1}^{n} \left( \psi \frac{\partial u}{\partial t} - c(x) \frac{u^n - \rho^n}{\Delta t}, \Pi_h \delta_t e^n \right) + \sum_{l=1}^{n} \left( c(x) \frac{\rho^n - \rho^n}{\Delta t}, \Pi_h \delta_t e^n \right) \\
&+ \sum_{l=1}^{n} \left( f(u^n) - f(u^n_H) - f'(u^n_H)(u^n_h - u^n_H), \Pi_h \delta_t e^n \right) \\
&\equiv \sum_{l=1}^{n} T_i + T_i'.
\end{align*}
\]

For $T_i + T_i'$, we can estimate them similarly as in Theorem 4.1. For the last term of the right-hand side of (71), a Taylor
expansion about \( u_H^n \) yields

\[
f(u^n) = f(u_H^n) + f'(u_H^n)(u^n - u_H^n) + \frac{1}{2} f''(\tilde{u})(u^n - u_H^n)^2,
\]

for some function \( \tilde{u} \). Then

\[
f(u^n) - f(u_H^n) = f'(u_H^n)(u^n - u_H^n) + \frac{1}{2} f''(\tilde{u})(u^n - u_H^n)^2
\]

\[
= f'(u_H^n)(u^n - u_H^n) + \frac{1}{2} f''(\tilde{u})(u^n - u_H^n)^2
\]

So by assumption (c) and Lemma 5, we have

\[
|T^n_T| \leq \sum_{l=1}^n \left| (f(u^n) - f(u_H^n)) - f'(u_H^n)(u^n - u_H^n) \right| \|\Pi^n_\epsilon \partial_t e^n\| \Delta t
\]

\[
\leq \sum_{l=1}^n \left| (f(u^n) - f(u_H^n)) - f'(u_H^n)(u^n - u_H^n) \right| \|\Pi^n_\epsilon \partial_t e^n\| \Delta t
\]

\[
\leq C(\epsilon) \sum_{l=1}^n \left| (f(u^n) - f(u_H^n)) - f'(u_H^n)(u^n - u_H^n) \right| \|\partial_t e^n\| \Delta t
\]

\[
+ C(\epsilon) \sum_{l=1}^n \left| (u^l - u_H^l)^2 \right| \Delta t + C(\epsilon) \sum_{l=1}^n \left| (u^l - u_H^l)^2 \right| \Delta t
\]

\[
\leq C(\epsilon) \sum_{l=1}^n \left| (u^l - u_H^l)^2 \right| \Delta t + C(\epsilon) \sum_{l=1}^n \left| (u^l - u_H^l)^2 \right| \Delta t(72)
\]

So we have

\[
\|e^n\|^2 + \sum_{l=1}^n \|\partial_t e^n\|^2 \Delta t + \sum_{l=1}^n \|\partial_t e^n\|^2 \Delta t
\]

\[
\leq C \left( \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(0,T;L^2(\Omega))} + \int_0^T \|u_H^n\|^2 \, dt \right) (\Delta t)^2
\]

\[
+ C(\epsilon) \left( \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(0,T;H^2(\Omega))} + \|u\|_{L^\infty(0,T;H^2(\Omega))} \right) h^2
\]

\[
+ C(\epsilon) \sum_{l=1}^n \left| (u^l - u_H^l)^2 \right| \Delta t. \quad (73)
\]

For the last term of (73), we have

\[
\|u^n - u_H^n\|^{2} \leq \|u^n - u_H^n\|_{0,\infty}^{2} \|u^n - u_H^n\|^{2}
\]

\[
\leq \left( \|u^n - P_H u^n\|_{0,\infty} + \|P_H u^n - u_H^n\|_{0,\infty} \right)^2
\]

\[
\|u^n - u_H^n\|^{2}, \quad (74)
\]

where \( P_H \) is defined in the same way as \( P_h \) is defined by Lemma 5. By Theorem 4.2, and the inverse estimate, we get

\[
\leq C \left( (u^n - u_H^n)^2 \right)^2 \leq C \left( (H^2 \|u^n\| + H^{-1} \|P_H u^n - u_H^n\|^2 \right)^2 \Delta t + H^2 \Delta t + H^2 \|u_H^n\|^2 + H^{-1} (\Delta t)^2 + 2 \|H^2 \Delta t + H^2 \|^2, \quad (75)
\]

We can choose \( H \) and \( \Delta t \) such that \( H^{-1} \Delta t < C \), then we have

\[
\|u^n - u_H^n\|^{2} \leq C \left( \Delta t + H^3 \right)^2, \quad (76)
\]

with (73), we get

\[
\|e^n\| \leq C \left( \Delta t + H + H^3 \right). \quad (77)
\]

For the two-grid characteristic FVE method Algorithm 2, we can have a similar result.

Theorem 5.2: Let \( u \) and \( u_h \) be the solutions of (12) and the two-grid FVE method Algorithm 2, respectively. Under assumption (a)-(f), and the coarse grid size \( H \) and the step \( \Delta t \) that satisfy \( H^{-1} \Delta t < C \). For \( \Delta t \) small enough, if \( u_h = P_h u_0 \) with \( P_h \) defined by Lemma 5, we have, for \( t_n \leq T \),

\[
\max_{1 \leq n \leq N} \|u^n - u_H^n\| \leq C \left( \Delta t + H + H^3 \right). \quad (78)
\]

VI. Conclusions

In this paper, we have presented the error estimates for the two-grid FVE method and the two-grid characteristic FVE method for a semi-linear Sobolev equation. The theorems above demonstrate a remarkable fact about two-grid characteristic FVE method: we can iterate on a very coarse grid \( T_H \) and still get good approximations by taking one iteration on the fine grid \( T_h \). It is proved that the coarse grid can be much coarser than the fine grid (\( h \ll H \)). We can achieve optimal approximation in \( H^1 \)-norm error estimate as long as the mesh sizes satisfy \( h = O(H^3) \).

REFERENCES


