Forwarding Indices of 4-regular Circulants *

Shuting Liu,Jixiang Meng

Abstract—**An all-to-all routing R (or a routing for short) of a connected graph G of order n is a collection of $n(n-1)$ elementary paths connecting every ordered pair of vertices of G. For a given routing R of G, the vertex-forwarding index $\xi(G;R)$ of G is the maximum number of paths in R passing through any vertex of G. The vertex-forwarding index $\xi(G)$ of G is defined as the minimum $\xi(G;R)$ over all routings R of G. Similarly, given a routing R of G, we define the edge-forwarding index $\pi(G;R)$ of G a routing R as the maximum number of paths in R passing through any edge of G. The edge-forwarding index $\pi(G)$ of G is defined as the minimum $\pi(G;R)$ over all routings R of G. The forwarding index corresponds to the maximum load of the graph. Thus, it is important, for given graphs, to find routings minimizing these indices.**

In this paper, we construct shortest paths whose expressions are specifically given between any two distinct vertices and obtain the exact values of vertex-forwarding indices of 4-regular circulant graphs with order $n (n \geq 6)$. Furthermore, based on the relations, known so far, between vertex-forwarding indices and edge forwarding indices, some bounds of edge forwarding indices for this kind of graphs can be presented immediately.

Keywords: forwarding index, Circulant graph, routing, distance, disjoint paths

1 Introduction

An interconnection network is often modeled by a connected graph $G = (V(G), E(G))$, where the vertex set $V(G)$ corresponds to node set in a network represent communication centers or processors, and the edge set $E(G)$ represents link set with which to communicate data or messages between different vertices. For notation not defined here, see [1] for references.

Designers of interconnection network specify a set of routes for every pair (x, y) of vertices, indicating a fixed route which carries the data transmitted from the message source x to the destination y. The load of any vertex is limited by the capacity of the vertex, for otherwise it would reduce the efficiency of transmission and even cause the fault of the network. Whether or not the network capacity could be fully used will depend on the choice of these routes.

Let G be a connected and simple graph of order n. A routing R of G is a set of $n(n-1)$ elementary paths $R(x, y)$ specified for all ordered pairs (x, y) of vertices of G. if $R(x, y) = R(y, x)$ specified by R, that is to say the path $R(y, x)$ is the reverse of the path $R(x, y)$ for all x, y , then we say that the routing is *symmetric*. If each of the paths specified by R is shortest, the routing R is said to be *minimal*, denoted by R_m. For any two vertices x and y, and a vertex z belonging to the path $R(x; y)$ specified by R, the path $R(x; y)$ is the concatenation of the paths $R(x, z)$ and $R(z, y)$.

In order to measure the load of a vertex, Chung et al introduced in [2] the notion of forwarding index, which has received considerable attention due to its importance in networks.(see nice survey [10], and references therein). The forwarding index is one of the fault tolerance parameters of a network. Some other interesting studies about a network can be seen in [5, 6] and so on.

Let the sets of routings and minimum routings in a graph G be denoted by $\mathcal{R}(G)$ and $\mathcal{R}_m(G)$ respectively. For a given $R \in \mathcal{R}(G)$ and $x \in V(G)$, the *load of a vertex x* in a given routing R of a graph G , denoted by $\xi_x(G, R)$, is defined as the number of paths specified by R passing through x and admitting x as an inner vertex. The *forwarding index of G with respect to R* is the maximum number of paths of R going through any vertex x in G and is denoted by

$$\xi(G, R) = max\{\xi_x(G, R) : x \in V(G)\}.$$

The minimum forwarding index over all possible routings of a graph G, denoted by

$$\xi(G) = min\{\xi(G, R) : R \in \mathcal{R}(G)\}$$

is called the forwarding index of G.

In [3], Similar concepts for the edge version of a graph was introduced by Heydemann et al. The *load of an edge e with respect to R*, denoted by $\pi_e(G, R)$, is defined as the number of the paths specified by R going though it . The *edge forwarding index of G with respect to R* is the maximum number of paths specified by R going through

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any edge of G and is denoted by

$$\pi(G,R) = max\{\pi_x(G,R) : x \in E(G)\}.$$

We call

$$\pi(G) = min\{\pi(G,R) : R \in \mathcal{R}_m(G)\}.$$

the *edge forwarding index of G* .

For routings of shortest paths, define

$$\xi_m(G) = min\{\xi(G,R_m) : R_m \in \mathcal{R}_m(G)\}$$

and

$$\pi_m(G) = min\{\pi(G,R_m) : R_m \in \mathcal{R}_m(G)\}.$$

Clearly, $\xi(G) \leq \xi_m(G)$ and $\pi(G) \leq \pi_m(G)$. The equality, however, does not always hold and some examples were given in [3].

The forwarding index problem is NP-complete even if the diameter of the graph is 2, seen in [9]. In [4], Solé proved that the vertex forwarding indices of graphs in a class of quasi-Cayley graphs, a new class of vertex-transitive graphs, which contains Cayley graphs, achieve the minimum. However, in general it is difficult to find the exact value or a good estimate of the forwarding index of a graph, even for some special classes of graphs such as circulant graphs. In [13], it established tight upper and lower bounds of forwarding indices for circulant graphs. But these bounds are difficult to compute generally. Moreover, a uniform routing of shortest paths may not exist for circulant graphs, just as the case for Cayley graphs [7]. The circulant graph G is denoted by $C_n(d_1,\ d_2,\ \cdots,\ d_k)$ or briefly $C_n(d_i)$, where $0 < d_1 < d_2 < \cdots < d_k < (n+1)/2$, The sequence (d_i) is called a jump sequence and d_i is called a jump. In [13], for the circulant digraph $G(d^n;S)$ with $S = \{1, d, \cdots, d^{n-1}\}$, $d \geq 2$ and $n \geq 2$, Xu et al obtained

$$\xi(G(d^n;S)) = \frac{1}{2}(d-1)d^n n - (d^n - 1)$$

and

$$\pi(G(d^n;S) = \frac{1}{2}(d-1)d^n.$$

In [12], for the circulant graph $G = C_n(1, 3d+1, 3d^2-1)$, where $n = 3d^2 + 3d + 1$, Thomson and Zhou determined

$$\pi(G) = \frac{1}{3}d(d+1)(2d+1), \ \texttt{for}\ d \geq 2.$$

Generally, computing the forwarding index of a graph is very difficult. The purpose of this paper is to study the forwarding indices of circulant graphs with degree 4.

2 Preliminaries

Before proceeding, we collect some known results which will be useful in the proofs of our main results.

Theorem 2.1. *[11] If* gcd $(\mathrm{n}, \mathrm{a}) = 1$ *or* gcd $(\mathrm{n}, \mathrm{b}) = 1$*, then there exists an integer k satisfying $C_n(1,k) \cong C_n(a,b)$.*

Theorem 2.2. *[8] $C_n(d_1, d_2, \cdots, d_k)$ is connected if and only if gcd $(d_1, d_2, \cdots, d_k, n) = 1$.*

conjecture 2.3. *(Heydemann et al. [3], 1989) In any vertex-transitive graph $G = (V,E)$, there exists a routing of shortest paths in which the load of every vertex, and therefore the vertex-forwarding index is equal to $\sum_{y \in v} d_G(x,y) - (n-1)$ for any vertex u of G.*

The conjecture is not true for symmetric routings of shortest paths. But the conjecture is true for Cayley graphs as stated in the following theorem.

Theorem 2.4. *(Heydemann et al. [3], 1989). If $G = (V,E)$ is a Cayley graph with order n, then, for any vertex x in V,*

$$\xi(G) = \xi_m(G) = \sum_{y \in V} d_G(x,y) - (n-1). \tag{2.1}$$

Heydemann et al. found that the equality 2.1 is not valid for $\pi(G)$, and proposed conjectures in [3]. So far, There are no results on these conjectures and some relationships between vertex and edge forwarding indices are given as follows.

Theorem 2.5. *(Heydemann et al. [3], 1989). For any connected graph G of order n, maximum degree Δ,and minimum degree δ,*

(a) $2\xi(G) + 2(n-1) \leq \Delta\pi(G)$;

(b) $\pi(G) \leq \xi(G) + 2(n-1)$;

(c) $\pi_m(G) \leq \xi_m(G) + 2(n-\delta)$.

All these inequalities are also valid for symmetric routings and the inequality in (a) is also valid for minimal routings.

Remark 2.6. *(Heydemann et al. [3], 1989) In (a) the equality holds for $C_n, W_n, K_{1,n}$ the n-cube, the Petersen graph and its line graph with the given values. In (b) the equality holds for the complete graph.*

Lemma 2.7. *For a circulant graph $G = C_n(1,d)$,*

$$\frac{1}{2}(\xi(G) + (n-1)) \leq \pi(G) \leq \xi(G) + 2(n-4)$$

or

$$\frac{1}{2}\sum_{y \in V} d_G(x,y) \leq \pi(G) \leq \sum_{y \in V} d_G(x,y) + n - 7.$$

Proof. By theorems 2.1, 2.2, 2.5 and 2.4, clearly. □

3 Main results

In the following, we assume that the points of a circulant graph are labeled clockwise by $0,\ 1,\ 2,\ \cdots,\ n-1$, (corresponding to $0,\quad -n+1,\ \cdots,\ -2,\ -1$ respectively) and we refer to point i instead of saying the point labeled by i. In general, we can show that a circulant graph is connected by identifying the existence of a path from 0 to t for each vertex t. That is, we need a combination of elements of S that sum to $t: \sum_{j=1}^{\eta}(\alpha_j)(d_j) \equiv t (\text{mod n})$, where (α_m) and (d_m) respectively denote the step number and the step factor of the 0t-path. For example, $t = (k)(d) + (i)(1)$ corresponds to a path $(0, d, 2d, \cdots, kd, kd+1, kd+2, \cdots, kd+i-1, t)$, where $t = kd+i$ is one vertex of a circulant graph G with order n.

Let $G = C_n(1, d)$ $(n \geq 6)$ be a circulant graph, then all the shortest paths from 0 to t for each vertex t can be constructed and $\sum_{t\in v} d_G(0, t)$ can be calculated, from which the forwarding indices can be determined. We firstly give two notations which will be used in the subsequent lemmas and theorems :

(1)Let (n, d) be a fixed pair of positive integers satisfying $n = Kd + n_0$, *where* $1 < d \leq \lfloor \frac{n}{2} \rfloor$, $d \neq \frac{n}{2}$, $0 \leq n_0 < d$, and $K > 2$.

(2) For all $0 < t \leq \lfloor n/2 \rfloor$, let $t = kd+i$, where $0 \leq i < d$, then $0 \leq k \leq \lfloor \frac{K}{2} \rfloor$, and $k + i \geq 1$.

3.1 $n = Kd$

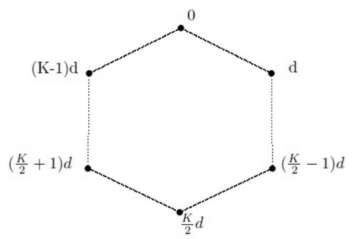

Fig 1. K is even

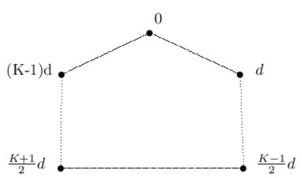

Fig 2. K is odd

Lemma 3.1. *Let $G = C_n(1, d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd$ and $K \geq 4$ is even. then*

$$\sum_{t\in V} d_G(0, t) = \begin{cases} H - \frac{K}{4}, & where\ d\ is\ even; \\ H, & where\ d\ is\ odd. \end{cases}$$

where $H = \frac{K}{2}(\frac{K}{2}d + \frac{d^2}{2} - \frac{1}{2})$.

Proof. On the conditions that K is even and $n = Kd$, we divide G into K equal parts of order d. See Fig.1. By the vertex transitivity of G, to construct shortest paths between any two vertices, we only need to consider the shortest paths between vertices 0 and t. For any vertex t of G, let $t = kd + i$ and $0 \leq k \leq \frac{K}{2} - 1$.

Case 1. $kd + 1 \leq t \leq kd + \lfloor \frac{d}{2} \rfloor$.

We construct a 0t-path according the equality $t = (k)(d) + (t - kd)(1)$. obviously, the path given is a shortest path between vertices 0 and t, therefore $d(0, t) = k + t - kd$.

Case 2. $kd + \lfloor \frac{d}{2} \rfloor + 1 \leq t \leq (k + 1)d$.

Similar to case 1, the shortest path can be constructed by the equality $t = (k+1)(d) + ((k+1)d - t)(-1)$, so the distance $d(0, t) = k + 1 + (k + 1)d - t$.

We then have the following

$$\begin{aligned}
&\sum_{t\in V} d_G(0,t) \\
=\ & 2 * \Big(\sum_{k=0}^{\frac{K}{2}-1}\Big(\sum_{t=kd+1}^{kd+\lfloor\frac{d}{2}\rfloor}(k+t-kd) \\
&+ \sum_{t=kd+\lfloor\frac{d}{2}\rfloor+1}^{(k+1)d}(k+1+(k+1)d-t)\Big)\Big) - \frac{K}{2} \\
=\ & 2 * \Big(\sum_{k=0}^{\frac{K}{2}-1}\Big(\sum_{j=1}^{\lfloor\frac{d}{2}\rfloor}(k+j) \\
&+ \sum_{j=1}^{\lceil\frac{d}{2}\rceil}(k+1+\lceil\tfrac{d}{2}\rceil-j)\Big)\Big) - \frac{K}{2} \\
=\ & d\frac{K}{2}(\frac{K}{2}-1) + \big(\lfloor\tfrac{d}{2}\rfloor(1+\lfloor\tfrac{d}{2}\rfloor) \\
&+ \lceil\tfrac{d}{2}\rceil(1+\lceil\tfrac{d}{2}\rceil)\big)\frac{K}{2} - \frac{K}{2} \\
=\ & d\frac{K}{2}(\frac{K}{2}-1) + (\lfloor\tfrac{d}{2}\rfloor^2 + \lceil\tfrac{d}{2}\rceil^2 + d)\frac{K}{2} - \frac{K}{2} \\
=\ & \frac{K}{2}(\frac{K}{2}d + \lfloor\tfrac{d}{2}\rfloor^2 + \lceil\tfrac{d}{2}\rceil^2 - 1) \\
=\ & \begin{cases} \frac{K^2d}{4} + \frac{Kd^2}{4} - \frac{K}{2}, & \text{where } d \text{ is even;} \\ \frac{K^2d}{4} + \frac{Kd^2}{4} - \frac{K}{4}, & \text{where } d \text{ is odd.} \end{cases}
\end{aligned}$$

□

Theorem 3.2. *Let $G = C_n(1, d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd$ and $K \geq 4$ is even. Then*

$$\xi(G) = \begin{cases} L + \frac{d-K}{4}, & where\ d\ is\ even; \\ L + \frac{d}{4}, & where\ d\ is\ odd. \end{cases}$$

where $L = \frac{1}{4}(K + d - 4)(n - 1)$.

Proof. By theorem 2.4,

$$\begin{aligned}
&\xi(G) = \xi_m(G) \\
=\ & \frac{K}{2}(\frac{K}{2}d + \lfloor\tfrac{d}{2}\rfloor^2 + \lceil\tfrac{d}{2}\rceil^2 - 1) - (n-1) \\
=\ & \frac{K}{2}(\frac{K}{2}d + \lfloor\tfrac{d}{2}\rfloor^2 + \lceil\tfrac{d}{2}\rceil^2 - 2d - 1) + 1 \\
=\ & \begin{cases} \frac{n}{4}(K+d-4) - \frac{K}{2} + 1, & where\ d\ is\ even; \\ \frac{n}{4}(K+d-4) - \frac{K}{4} + 1, & where\ d\ is\ odd. \end{cases} \\
=\ & \begin{cases} L + \frac{d-K}{4}, & where\ d\ is\ even; \\ L + \frac{d}{4}, & where\ d\ is\ odd. \end{cases}
\end{aligned}$$

□

Lemma 3.3. *Let $G = C_n(1, d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd$ and $K \geq 3$ is odd. Then*

$$\sum_{t\in V} d_G(0,t) = \begin{cases} M - \frac{K-1}{4}, & \text{where } d \text{ is even;} \\ M - \frac{1}{4}, & \text{where } d \text{ is odd.} \end{cases}$$

where $M = \frac{K-1}{2}(\frac{K-1}{2}d + \frac{d^2}{2} + d - \frac{1}{2}) + \frac{d^2}{4}$.

Proof. We consider two cases. Let $t = kd + i$ be any one vertex of G.

Case 1. $0 \leq k \leq \frac{K-1}{2} - 1$.

Case 1.1. $kd + 1 \leq t \leq kd + \lfloor \frac{d}{2} \rfloor$.

The equality $t = kd + (t - kd)(1)$ determines a shortest 0t-path and the distance $d(0, t) = k + t - kd$.

Case 1.2. $kd + \lfloor \frac{d}{2} \rfloor + 1 \leq t \leq (k + 1)d$.

The equality $t = (k+1)d + ((k+1)d - t)(-1)$ determines a shortest 0t-path and the distance $d(0, t) = k + 1 + (k + 1)d - t$.

Case 2. $k = \frac{K-1}{2}$.

Case 2.1. d is odd.(see Fig.3)

$$\begin{aligned} & \textstyle\sum_{y\in v} d_G(x,y) \\ = \; & 2 * \Big(\textstyle\sum_{k=0}^{\frac{K-1}{2}-1} \Big(\sum_{t=kd+1}^{kd+\lfloor \frac{d}{2}\rfloor}(k + t - kd) \\ & + \textstyle\sum_{t=kd+\lfloor \frac{d}{2}\rfloor+1}^{(k+1)d}(k + 1 + (k + 1)d) - t\Big)\Big) \\ & +2 * \Big(\textstyle\sum_{t=(\frac{K-1}{2})d+1}^{(\frac{K-1}{2})d+\lfloor \frac{d}{2}\rfloor}(\frac{K-1}{2} + t - \frac{K-1}{2}d)\Big) \\ = \; & d\frac{K-1}{2}(\frac{K-1}{2} - 1) + (\lfloor \tfrac{d}{2}\rfloor^2 + \lceil \tfrac{d}{2}\rceil^2 \\ & +d)\frac{K-1}{2} + (K - 1)\lfloor \tfrac{d}{2}\rfloor + \lfloor \tfrac{d}{2}\rfloor(1 + \lfloor \tfrac{d}{2}\rfloor) \\ = \; & \frac{K-1}{2}(\frac{K-1}{2}d + \lfloor \tfrac{d}{2}\rfloor^2 + \lceil \tfrac{d}{2}\rceil^2 \\ & +2\lfloor \tfrac{d}{2}\rfloor) + \lfloor \tfrac{d}{2}\rfloor(1 + \lfloor \tfrac{d}{2}\rfloor) \\ = \; & \frac{K-1}{2}(\frac{K-1}{2}d + \frac{d^2}{2} + d - \frac{1}{2}) + \frac{d^2}{4} - \frac{1}{4}. \end{aligned} \tag{3.1}$$

Case 2.2. d is even. (see Fig.4)

Based on the discussion in case 2.1, we only need to minus $d_G(0, \frac{n}{2})$, where $d_G(0, \frac{n}{2}) = \frac{K-1}{2} + \frac{d}{2}$, from $\sum_{t\in V} d_G(0,t)$ in equality 3.1. So

$$\begin{aligned} & \textstyle\sum_{y\in v} d_G(x,y) \\ = \; & 2 * \Big(\textstyle\sum_{k=0}^{\frac{K-1}{2}-1} \Big(\sum_{t=kd+1}^{kd+\frac{d}{2}}(k + t - kd) \\ & + \textstyle\sum_{t=kd+\frac{d}{2}+1}^{(k+1)d}(k + 1 + (k + 1)d) - t\Big)\Big) \\ & +2 * \Big(\textstyle\sum_{t=(\frac{K-1}{2})d+1}^{(\frac{K-1}{2})d+\frac{d}{2}}(\frac{K-1}{2} + t - \frac{K-1}{2}d)\Big) \\ & -(\frac{K-1}{2} + \frac{d}{2}) \\ = \; & \frac{K-1}{2}(\frac{K-1}{2}d + \lfloor \tfrac{d}{2}\rfloor^2 + \lceil \tfrac{d}{2}\rceil^2 + 2\lfloor \tfrac{d}{2}\rfloor) \\ & +\lfloor \tfrac{d}{2}\rfloor(1 + \lfloor \tfrac{d}{2}\rfloor) - (\frac{K-1}{2} + \frac{d}{2}) \\ = \; & \frac{K-1}{2}(\frac{K-1}{2}d + \lfloor \tfrac{d}{2}\rfloor^2 + \lceil \tfrac{d}{2}\rceil^2 + d - 1) + \frac{d^2}{4} \\ = \; & \frac{K-1}{2}(\frac{K-1}{2}d + \frac{d^2}{2} + d - 1) + \frac{d^2}{4} \\ = \; & \frac{K-1}{2}(\frac{K-1}{2}d + \frac{d^2}{2} + d - \frac{1}{2}) + \frac{d^2}{4} - \frac{K-1}{4}. \end{aligned} \tag{3.2}$$

This completes the proof. □

Theorem 3.4. *Let $G = C_n(1, d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd$ and $K \geq 3$ is odd. Then*

$$\xi(G) = \begin{cases} L - \frac{K-2}{4}, & \text{where } d \text{ is even;} \\ L, & \text{where } d \text{ is odd.} \end{cases}$$

Where $L = \frac{1}{4}(K + d - 4)(n - 1)$.

Proof. If d is odd, by theorem 2.4 we have

$$\begin{aligned} & \xi(G) \\ = \; & \tfrac{1}{4}[(K - 1)^2 d + (K - 1)d^2 + 2(K - 1)d \\ & -(K - 1)] + \tfrac{d^2}{4} - \tfrac{1}{4} - n + 1 \\ = \; & \tfrac{1}{4}[(K - 1)(n - d) + (n - d)d + 2(n - d) \\ & -(K - 1)] + \tfrac{d^2}{4} - \tfrac{1}{4} - n + 1 \\ = \; & \tfrac{1}{4}[Kn - n - n + d + nd - d^2 + 2(n - d) \\ & -K + 1] + \tfrac{d^2}{4} - \tfrac{1}{4} - n + 1 \\ = \; & \tfrac{1}{4}[Kn + nd - d - K] - n + 1 \\ = \; & \tfrac{1}{4}[Kn + nd - 4n] - \tfrac{1}{4}[K + d - 4] \\ = \; & \tfrac{1}{4}(n - 1)(K + d - 4). \end{aligned} \tag{3.3}$$

If d is even, compare equalities 3.2, 3.1 and 3.3, we obtain

$$\begin{aligned} \xi(G) &= \tfrac{1}{4}(n - 1)(K + d - 4) - \tfrac{K-1}{4} + \tfrac{1}{4} \\ &= \tfrac{1}{4}(n - 1)(K + d - 4) - \tfrac{K-2}{4}. \end{aligned}$$

□

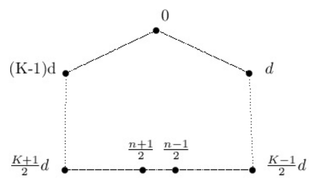

Fig 3 K is odd and d is odd

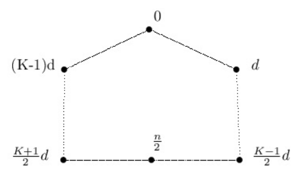

Fig 4 K is odd and d is even

3.2 $n = Kd + n_0,\ where\ 0 < n_0 < d.$

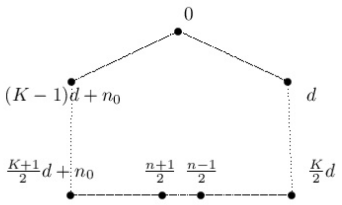

Fig 5 K is odd and d is odd

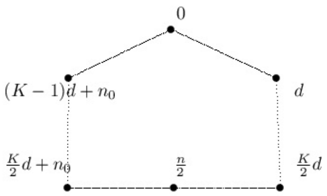

Fig 6 K is odd and d is even

Lemma 3.5. *Let $G = C_n(1,d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$ and $K \geq 4$ is even, then*

(1) If n_0 is odd,

$$\sum_{t\in V} d_G(0,t)) = \begin{cases} N - \frac{1}{4}, & where\ d\ is\ even;\\ N + \frac{K-1}{4}, & where\ d\ is\ odd.\end{cases}$$

(2) If n_0 is even,

$$\sum_{t\in V} d_G(0,t)) = \begin{cases} N, & where\ d\ is\ even;\\ N + \frac{K}{4}, & where\ d\ is\ odd.\end{cases}$$

where $N = \frac{K}{2}(\frac{K}{2}d + \frac{d^2}{2} + n_0 - 1) + \frac{n_0^2}{4}$.

Proof. Let $t = kd+i$ be any a vertex of G. When $0 \leq k \leq \frac{K}{2} - 1$, the method of discussion is similar to that of in lemma 3.1. In this proof, we particularly consider the $\frac{K}{2}$-th part from the vertex $\frac{K}{2}d+1$ to the vertex $\frac{Kd}{2} + \lfloor\frac{n_0}{2}\rfloor = \lfloor\frac{n}{2}\rfloor$.

When $\frac{Kd}{2} + 1 \leq t \leq \frac{Kd}{2} + \lfloor\frac{n_0}{2}\rfloor = \lfloor\frac{n}{2}\rfloor$a shortest 0t-path is determined by $t = (\frac{K}{2})(d) + (t - \frac{K}{2}d)(1)$ and so the distance $d(0,t) = \frac{K}{2} + (t - \frac{K}{2}d)$.

Case 1. n_0 is odd.(see Fig.5)

$$\begin{aligned}
&\sum_{t\in V} d_G(0,t)\\
=\ & 2*\Big(\sum_{k=0}^{\frac{K}{2}-1}\Big(\sum_{t=kd+1}^{kd+\lfloor\frac{d}{2}\rfloor}(k+t-kd)\\
&+\sum_{t=kd+\lfloor\frac{d}{2}\rfloor+1}^{(k+1)d}(k+1+(k+1)d)-t\Big)\Big)\\
&+2*\sum_{t=\frac{K}{2}d+1}^{\frac{K}{2}d+\lfloor\frac{n_0}{2}\rfloor}(\frac{K}{2}+(t-\frac{K}{2}d))\\
=\ & d\frac{K}{2}(\frac{K}{2}-1)+(\lfloor\frac{d}{2}\rfloor^2+\lceil\frac{d}{2}\rceil^2+d)\frac{K}{2}\\
&+K\lfloor\frac{n_0}{2}\rfloor+\lfloor\frac{n_0}{2}\rfloor(1+\lfloor\frac{n_0}{2}\rfloor)\\
=\ & \frac{K}{2}(\frac{K}{2}d+\lfloor\frac{d}{2}\rfloor^2+\lceil\frac{d}{2}\rceil^2+2\lfloor\frac{n_0}{2}\rfloor)+\lfloor\frac{n_0}{2}\rfloor(1+\lfloor\frac{n_0}{2}\rfloor)\\
=\ & \begin{cases}\frac{K}{2}(\frac{K}{2}d+\frac{d^2}{2}+n_0-1)+\frac{n_0^2-1}{4}, & \text{d is even;}\\ \frac{K}{2}(\frac{K}{2}d+\frac{d^2}{2}+n_0-1)+\frac{n_0^2-1}{4}+\frac{K}{4}, & \text{d is odd.}\end{cases}\\
=\ & \begin{cases}\frac{K^2d}{4}+\frac{Kd^2}{4}+K\frac{n_0}{2}+\frac{n_0^2}{4}-\frac{K}{2}-\frac{1}{4}, & \text{d is even;}\\ \frac{K^2d}{4}+\frac{Kd^2}{4}+K\frac{n_0}{2}+\frac{n_0^2}{4}-\frac{K}{4}-\frac{1}{4}, & \text{d is odd.}\end{cases}
\end{aligned}$$

Case 2. n_0 is even.(see Fig.6)

Based on the discussion in case 1, we only need to minus $d_G(0,\frac{n}{2})$, where $d_G(0,\frac{n}{2}) = \frac{K}{2} + \frac{n_0}{2}$, from $\sum_{t\in V} d_G(0,t)$ above. So

$$\begin{aligned}
&\sum_{t\in V} d_G(0,t)\\
=\ & 2*\Big(\sum_{k=0}^{\frac{K}{2}-1}\Big(\sum_{t=kd+1}^{kd+\lfloor\frac{d}{2}\rfloor}(k+t-kd)\\
&+\sum_{t=kd+\lfloor\frac{d}{2}\rfloor+1}^{(k+1)d}(k+1+(k+1)d)-t\Big)\Big)\\
&+2*\sum_{t=\frac{K}{2}d+1}^{\frac{K}{2}d+\frac{n_0}{2}}(\frac{K}{2}+(t-\frac{K}{2}d))-(\frac{K}{2}+\frac{n_0}{2})\\
=\ & d\frac{K}{2}(\frac{K}{2}-1)+(\lfloor\frac{d}{2}\rfloor^2+\lceil\frac{d}{2}\rceil^2+d)\frac{K}{2}\\
&+K\lfloor\frac{n_0}{2}\rfloor+\lfloor\frac{n_0}{2}\rfloor(1+\lfloor\frac{n_0}{2}\rfloor)-(\frac{K}{2}+\frac{n_0}{2})\\
=\ & \frac{K}{2}(\frac{K}{2}d+\lfloor\frac{d}{2}\rfloor^2+\lceil\frac{d}{2}\rceil^2\\
&+2\lfloor\frac{n_0}{2}\rfloor)+\lfloor\frac{n_0}{2}\rfloor(1+\lfloor\frac{n_0}{2}\rfloor)-(\frac{K}{2}+\frac{n_0}{2})\\
=\ & \frac{K}{2}(\frac{K}{2}d+\lfloor\frac{d}{2}\rfloor^2+\lceil\frac{d}{2}\rceil^2+n_0-1)+\frac{n_0^2}{4}\\
=\ & \begin{cases}\frac{K}{2}(\frac{K}{2}d+\frac{d^2}{2}+n_0-1)+\frac{n_0^2}{4}, & \text{d is even;}\\ \frac{K}{2}(\frac{K}{2}d+\frac{d^2}{2}+n_0-1)+\frac{n_0^2}{4}+\frac{K}{4}, & \text{d is odd.}\end{cases}\\
=\ & \begin{cases}\frac{K^2d}{4}+\frac{Kd^2}{4}+K\frac{n_0}{2}+\frac{n_0^2}{4}-\frac{K}{2}, & \text{d is even;}\\ \frac{K^2d}{4}+\frac{Kd^2}{4}+K\frac{n_0}{2}+\frac{n_0^2}{4}-\frac{K}{4}, & \text{d is odd.}\end{cases}
\end{aligned}$$

□

Theorem 3.6. *Let $G = C_n(1,d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$ and $K \geq 4$ is even, then*

(1) If n_0 is odd,

$$\xi(G) = \begin{cases} T, & where\ d\ is\ even;\\ T + \frac{K}{4}, & where\ d\ is\ odd.\end{cases}$$

(2) If n_0 is even,

$$\xi(G) = \begin{cases} T + \frac{1}{4}, & where\ d\ is\ even;\\ T + \frac{K+1}{4}, & where\ d\ is\ odd.\end{cases}$$

where $T = \frac{1}{4}(d+K-4)(n-1) + \frac{1}{4}(K-d+n_0+1)(n_0-1)$.

To explain the proofs of the following lemmas and theorems, we give four figures.

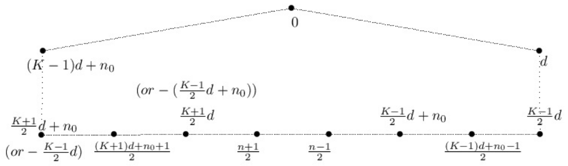

Fig 7. K is odd, n_0 is odd and d is even

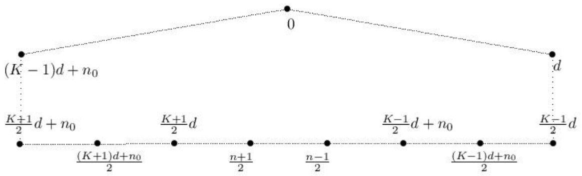

Fig 8. K is odd, n_0 is even and d is odd

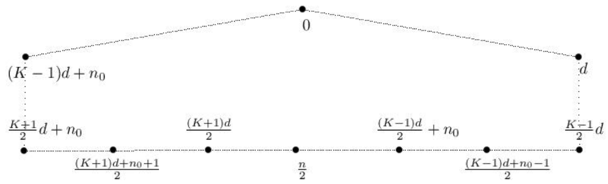

Fig 9. K is odd, n_0 is odd and d is odd

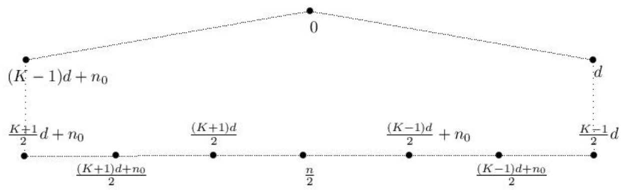

Fig 10. K is odd n_0 is even and d is even

Lemma 3.7. *Let $G = C_n(1,d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$ and $K \geq 3$ is odd, then*

(1) If $d+n_0$ is odd,

$$\sum_{t\in V} d_G(0,t) = \begin{cases} S - \frac{K+4}{4}, \text{where } d \text{ is odd} \\ S - \frac{2K+1}{4}, \text{where } d \text{ is even.} \end{cases}$$

(2) If $d+n_0$ is even,

$$\sum_{t\in V} d_G(0,t) = \begin{cases} S - \frac{K+1}{2}, \text{where } d \text{ is even.} \\ S - \frac{K+1}{4}, \text{where } d \text{ is odd.} \end{cases}$$

where $S = \frac{K-1}{2}(\frac{K+1}{2}d + \frac{d^2}{2} + n_0) + \frac{{n_0}^2}{2} + (\frac{d-n_0}{2})^2 + d$

Proof. Let $t = kd + i$ be a vertex of G. When $0 \leq k \leq \frac{K-1}{2} - 1$, the method we use is similar to those in lemma 3.1 and 3.3. In the proof, we particularly consider the $\frac{K-1}{2}$-th part from vertex $\frac{K-1}{2}d+1$ to vertex $\lfloor \frac{n}{2} \rfloor = \frac{K-1}{2}d + \lfloor \frac{d+n_0}{2} \rfloor$.

Case 1. $d+n_0$ is odd. we consider three subcases:

Subcase 1.1. $\frac{K-1}{2}d + 1 \leq t \leq \frac{K-1}{2}d + \lfloor \frac{n_0}{2} \rfloor$.

A shortest 0t path is determined by the equality $t = \frac{K-1}{2}d + (t - \frac{K-1}{2}d)(1)$, and so $d(0,t) = \frac{K-1}{2} + t - \frac{K-1}{2}d$.

Subcase 1.2. $\frac{K-1}{2}d + \lfloor \frac{n_0}{2} \rfloor + 1 \leq t \leq \frac{K-1}{2}d + n_0$.

In this case, we construct a shortest 0t-path according to the following equality $t \equiv t - n(mod n) = (\frac{K-1}{2}+1)(-d) + (n - t - (\frac{K-1}{2}+1)d)(-1)$ and so $d(0,t) = \frac{K-1}{2} + 1 + n - t - (\frac{K-1}{2}+1)d$.

Subcase 1.3. $\frac{K-1}{2}d + n_0 + 1 \leq t \leq \lfloor \frac{n}{2} \rfloor = \frac{K-1}{2}d + \lfloor \frac{d+n_0}{2} \rfloor$.

A shortest 0t-path can be constructed by the equality $t \equiv t - n(mod n) : (\frac{K-1}{2}+1)(-d) + ((\frac{K-1}{2}+1)d - (n-t))(1)$ and so $d(0,t) = \frac{K-1}{2} + 1 + (\frac{K-1}{2}+1)d - (n-t)$.

It is easy to check that the paths constructed are the shortest ones.

$$\begin{aligned}
&\sum_{t\in V} d_G(0,t) \\
= \; & 2 * \Big(\sum_{k=0}^{\frac{K-1}{2}-1} \Big(\sum_{t=kd+1}^{kd+\lfloor \frac{d}{2} \rfloor}(k + t - kd) \\
& + \sum_{t=kd+\lfloor \frac{d}{2} \rfloor+1}^{(k+1)d}(k+1+(k+1)d) - t\Big)\Big) \\
& +2 * \sum_{t=\frac{K-1}{2}d+1}^{\frac{K-1}{2}d+\lfloor \frac{n_0}{2} \rfloor}(\frac{K-1}{2} + (t - \frac{K-1}{2}d)) \\
& +2 * \sum_{t=\frac{K-1}{2}d+\lfloor \frac{n_0}{2} \rfloor+1}^{\frac{K-1}{2}d+n_0}(\frac{K-1}{2} + 1 \\
& +n - t - (\frac{K-1}{2}+1)d) \\
& +2 * \sum_{t=\frac{K-1}{2}d+n_0+1}^{\frac{n-1}{2}}(\frac{K-1}{2} + 1 \\
& +(\frac{K-1}{2}+1)d - (n-t)) \\
= \; & d\frac{K-1}{2}(\frac{K-1}{2} - 1) + (\lfloor \frac{d}{2} \rfloor^2 + \lceil \frac{d}{2} \rceil^2 + d)\frac{K-1}{2} \\
& +2\sum_{j=1}^{\lfloor \frac{n_0}{2} \rfloor}(\frac{K-1}{2} + j) \\
& +2\sum_{j=1}^{\lceil \frac{n_0}{2} \rceil}(\frac{K-1}{2} + 1 + \lceil \frac{n_0}{2} \rceil - j) \\
& +2\sum_{j=1}^{\lfloor \frac{d+n_0}{2} \rfloor - n_0}(\frac{K-1}{2} + 1 + j) \\
= \; & \frac{K-1}{2}(\frac{K-1}{2}d + \lfloor \frac{d}{2} \rfloor^2 + \lceil \frac{d}{2} \rceil^2 + 2\lfloor \frac{d+n_0}{2} \rfloor) \\
& +\lfloor \frac{n_0}{2} \rfloor^2 + \lceil \frac{n_0}{2} \rceil^2 + n_0 \\
& +(\lfloor \frac{d+n_0}{2} \rfloor - n_0)(\lfloor \frac{d+n_0}{2} \rfloor - n_0 + 3) \\
= \; & \frac{K-1}{2}(\frac{K-1}{2}d + \lfloor \frac{d}{2} \rfloor^2 + \lceil \frac{d}{2} \rceil^2 + d + n_0 - 1) \\
& +\lfloor \frac{n_0}{2} \rfloor^2 + \lceil \frac{n_0}{2} \rceil^2 + n_0 + (\frac{d-n_0-1}{2})(\frac{d-n_0+5}{2}) \\
= \; & \begin{cases} S - \frac{K+4}{4}, \text{ where } d \text{ is odd;} \\ S - \frac{2K+1}{4}, \text{ where } d \text{ is even.} \end{cases}
\end{aligned}$$

Case 2. $d+n_0$ is even.

Based on the discussion in case 1, we only need to minus $d_G(0, \frac{n}{2})$, where $d_G(0, \frac{n}{2}) = \frac{K+1}{2} + \frac{d-n_0}{2}$, from $\sum_{t\in V} d_G(0,t)$ above. So

$$
\begin{aligned}
& \sum_{t\in V} d_G(0,t) \\
= \; & 2*\Big(\sum_{k=0}^{\frac{K-1}{2}-1}\Big(\sum_{t=kd+1}^{kd+\lfloor\frac{d}{2}\rfloor}(k+t-kd) \\
& +\sum_{t=kd+\lfloor\frac{d}{2}\rfloor+1}^{(k+1)d}(k+1+(k+1)d)-t\Big)\Big) \\
& +2*\sum_{t=\frac{K-1}{2}d+1}^{\frac{K-1}{2}d+\lfloor\frac{n_0}{2}\rfloor}(\frac{K-1}{2}+(t-\frac{K-1}{2}d)) \\
& +2*\sum_{t=\frac{K-1}{2}d+\lfloor\frac{n_0}{2}\rfloor+1}^{\frac{K-1}{2}d+n_0}(\frac{K-1}{2}+1 \\
& +n-t-(\frac{K-1}{2}+1)d) \\
& +2*\sum_{t=\frac{K-1}{2}d+n_0+1}^{\frac{n}{2}}(\frac{K-1}{2}+1+(\frac{K-1}{2}+1)d \\
& -(n-t))-(\frac{K+1}{2}+\frac{d-n_0}{2}) \\
= \; & \frac{K-1}{2}(\frac{K-1}{2}d+\lfloor\frac{d}{2}\rfloor^2+\lceil\frac{d}{2}\rceil^2 \\
& +2\lfloor\frac{d+n_0}{2}\rfloor)+\lfloor\frac{n_0}{2}\rfloor^2+\lceil\frac{n_0}{2}\rceil^2+n_0 \\
& +(\lfloor\frac{d+n_0}{2}\rfloor-n_0)(\lfloor\frac{d+n_0}{2}\rfloor-n_0+3) \\
& -(\frac{K+1}{2}+\frac{d-n_0}{2}) \\
= \; & \begin{cases} S-\frac{K+1}{2}, & \text{where } d \text{ is even;} \\ S-\frac{K+1}{4}, & \text{where } d \text{ is odd.} \end{cases}
\end{aligned}
$$

□

Theorem 3.8. *Let $G = C_n(1,d)$ $(d \geq 2)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$ and $K \geq 3$ is odd, then*

(1) If $d + n_0$ is odd,

$$\xi(G) = \begin{cases} S-n-\frac{K}{4}, & \textit{where } d \textit{ is odd;} \\ S-n-\frac{2K-3}{4}, & \textit{where } d \textit{ is even.} \end{cases}$$

(2) If $d + n_0$ is even,

$$\xi(G) = \begin{cases} S-n-\frac{K-1}{2}, & \textit{where } d \textit{ is even;} \\ S-n-\frac{K-3}{4}, & \textit{where } d \textit{ is odd.} \end{cases}$$

where $S = \frac{K-1}{2}(\frac{K+1}{2}d + \frac{d^2}{2} + n_0) + \frac{{n_0}^2}{2} + (\frac{d-n_0}{2})^2 + d.$

4 Conclusions

In this paper, the exact values of vertex-forwarding indices of 4-regular circulant graphs with order $n(n \geq 6)$ are obtained. Particularly, expressions of the shortest paths are specifically given between any two distinct vertices. Moreover, some bounds of edge forwarding indices for this kind of graphs are also presented. However, the exact values of the edge-forwarding indices of these graphs remains unknown.

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