Total Restrained Domination in Trees

Xing Chen, Dongyang Xie, Wei Xiong and Jixiang Meng

***Abstract*—Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a *total restrained dominating set* of G, if every vertex in V has at least one neighbor in D and every vertex in $V - D$ has at least one neighbor in $V - D$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of all total restrained dominating sets of G. In this paper, we give some results on total restrained domination number of trees. And then, we characterize the trees which satisfies $\gamma_{tr}(T) = n - 3$ or $n - 4$, where n is the order of T.**

***Index Terms*—path, tree, diameter, total restrained dominating set**

I. Introduction

It is well-known that an interconnection network can be modeled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore various problems in networks can be studied by graph theoretical methods. Now dominations have become one of the major areas in graph theory after more than 20 years' development. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems, such as facility location problems. Let $G = (V, E)$ be a graph. For any vertex $v \in V$, the *open neighborhood* of v, denoted by $N(v)$, is defined by $\{u \in V | uv \in E\}$ and the *closed neighborhood* of v denoted by $N[v]$, is defined by $N(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, is the cardinality of $N(v)$. Similarly, the *open neighborhood* of a subset $S \subseteq V$, denoted by $N(S)$, is defined by $\cup_{v \in S} N(v)$ and the *closed neighborhood* of S denoted by $N[S]$, is defined by $N(S) \cup S$. A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and $E = \{v_0v_1, v_1v_2, \cdots, v_{n-2}v_{n-1}\}$, where v_i are all distinct. The vertices v_0 and v_{n-1} are linked by P and are called its *ends*, the vertices $v_1, v_2, \cdots, v_{n-2}$ are the *inner* vertices of P. The number of vertices/ edges in a path is its *order/ length* and the path of order n is denoted by P_n. If $v_0 = v_{n-1}$, we call it a *cycle* and denoted by C_n. Recall that an *acyclic graph* is one that contains no cycles. A connected acyclic graph is called a *tree*. Acyclic graphs are usually called *forests*. Let G be a connected graph, then the distance between two vertices u and v is defined as the length of a shortest path from u to v and the *diameter* of G is the number $\mathrm{diam}(G) = \max\{d(u, v) : u, v \in V(G)\}$. Let $V' \subseteq V$, the subgraph of G whose vertex set is V' and edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and denoted by $G[V']$. A set $D \subseteq V$ is a *dominating set* of G if every vertex not in D is adjacent to a vertex of D. The *domination number* of G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations is now well studied in graph theory. The book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [10], [11].

In this paper, we continue the study of a variation of the domination, namely total restrained domination. A set $D \subseteq V$ is a *total restrained dominating set* (denoted by $TRDS$) of G if every vertex is adjacent to a vertex in D and every vertex in $V - D$ is also adjacent to a vertex in $V - D$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS. A TRDS of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set. Note that total restrained domination is defined only for graphs without isolated vertices and each graph without isolated vertices has a TRDS, since $D = V$ is such a set. The concept of total restrained domination in graphs was introduced by Telle and Proskurowski [13], albeit indirectly, as a vertex partition problem has been studied.

Recently, the total restrained domination and the domination of tree have been studied by many authors, for example, in ([3], [4]-[18], [20]-[23]). Henning and Maritz [6] investigated upper bounds on total restrained domination number of a graph. Hattingh et al.[4] gave some lower bounds of $\gamma_{tr}(G)$ of a tree and characterized the extremal trees achieving these lower bounds. Further, Hattingh et al.[5] gave an upper bound for graphs which is not one of several forbidden graphs and $\delta \geq 3$. In [19], The distance number of symmetric Lobseer-like tree was studied. In [23], the bounds on the vertex-edge dominating number of trees has been studied. J. Raczek and J. Cyman [14] have characterized the trees with equal total and total restrained dominating numbers and gave a lower bound on the total restrained dominating number of a tree in terms of its order and the number of leaves. Moreover, in [2], the authors determined the total restrained domination number for certain classes of graphs, and characterized those graphs achieving these bounds. Here, we characterize the trees which satisfies that $\gamma_{tr}(T) = n - 3$ or $n - 4$. Notation and definitions not given here can be found in [1].

A. *the Main Results*

Let S_n denote the star of order n. The empty graph is the graph without edges. In a tree, a *leaf* is a vertex of degree one. It is clear that $\gamma_{tr}(G) \leq n$ for any connected graph G of order n. In [2], the authors described the connected graph of order n which satisfies that $\gamma_{tr}(G) = n$ and $\gamma_{tr}(G) = n-2$. Following we continue this work and characterize the trees which satisfies that $\gamma_{tr}(T) = n - 3$ or $n - 4$. Let $S = \{v \in V(G) | d_G(v) = 1 \text{ or } \exists\ u \in N(v) \text{ and } d_G(u) = 1\}$.

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Xing Chen is with Xinjiang Institute of Engineering, Urumqi, Xinjiang, P.C. 830091 China e-mail: chenxingxjnu@163.com.

Dongyang Xie is with State Grid Changji Electric Power Supply Company, Changji, Xinjiang, P.C. 831100 China.

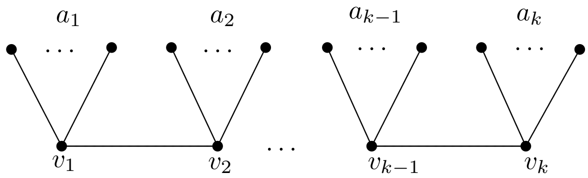
Wei Xiong and Jixiang Meng are with same University, they are College of Mathematics and Systems Sciences, Xinjiang University, Urumqi, Xinjiang, P.C. 830046 China e-mail: mjxxju@163.com. (Jixiang Meng)

Proposition 1.1: [4] Let $n \geq 4$ be a positive integer, then $\gamma_{tr}(P_n) = n - 2\lfloor\frac{n-2}{4}\rfloor$.

Lemma 1.2: [2] Let G be a connected graph of order $n \geq 4$. Then $\gamma_{tr}(G) = n$ if and only if $G - S$ is an empty graph.

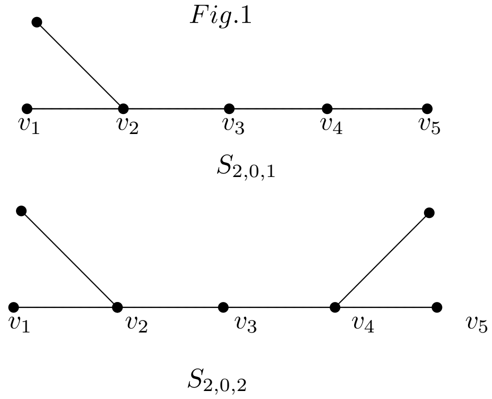
Lemma 1.3: [2] Let G be a connected graph of order $n \geq 4$ and $G - S$ be a forest, then $\gamma_{tr}(G) = n - 2$ if and only if $G - S$ has only one nontrivial component and it is a path on at most five vertices and only the ends of path are adjacent to the vertices of S.

Let $P_k = v_1v_2\cdots v_k$, we denote $S_{a_1,a_2,\ldots,a_k}$ be a graph which obtained from P_k by attach a_1 vertices to v_1, a_2 vertices to $v_2, \cdots$, and a_k vertices to v_k, respectively (see Fig.1). Clearly, $S_{a,b}$ is a double star and $S_{1,0,0,\cdots,0,1}$ is P_{k+2}.



$S_{a_1,a_2,\ldots,a_k}$

$Fig.1$



$Fig.2$

Theorem 1.4: Let T be a tree of order $n \geq 5$. Then $\gamma_{tr}(T) = n - 3$ if and only if $T - S$ has only one nontrivial component B and it is satisfies one of the following:

(a) : $B \cong S_3$ and all the vertices of S_3 have neighbors in S;

(b) : $B \cong S_4$ or $S_{2,1}$ or $S_{2,0,2}$ (see $Fig.2$) and only the leaves of B have neighbors in S;

(c) : $B \cong P_4$ and exact one inner vertex and all ends of B have neighbors in S;

(d) : $B \cong P_5 = v_1v_2v_3v_4v_5$ and there is at least one vertex of $\{v_2, v_4\}$ and all the ends of B have neighbors in S;

(e) : $B \cong S_{2,0,1}$ (see $Fig.2$) and at most v_4 and all the leaves of B have neighbors in S.

Proof. It is easy to verify that if $T - S$ has only one nontrivial component B and it satisfies one of the above conditions, then $\gamma_{tr}(T) = n - 3$. Conversely, let T be a tree of order $n \geq 5$ and $\gamma_{tr}(T) = n - 3$. By Lemma 2.2, $T - S$ contains at least one nontrivial component.

Claim 1. $T - S$ contains only one nontrivial component, say B.

Otherwise, suppose that $T - S$ contains at least two nontrivial components, say B_1 and B_2. Let $e_i = u_iv_i \subseteq B_i$ $(i = 1, 2)$. Note that any vertex in $T - S$ is not adjacent to any leaf of T and $N_{B_i}(u_i) \cap N_{B_i}(v_i) = \emptyset$ $(i = 1, 2)$, we have that $V(T) - \{u_1, v_1, u_2, v_2\}$ is a TRDS of T and $\gamma_(tr) \leq n - 4$, a contradiction and we complete the proof of claim 1.

Claim 2. $2 \leq diam(B) \leq 4$.

If diam$(B) = 1$, then $B \cong P_2$. By Lemma 2.3, $\gamma_{tr}(T) = n - 2$, a contradiction. If diam$(B) \geq 5$, let P be one of the longest paths in B of order at least 6, say $P = v_1v_2\cdots v_p$ $(6 \leq p \leq |V(B)|)$. First, we give the following two observations which will be used repeatedly in the following proof.

Observations: (1) both v_1 and v_p are leaves of B since P is one of the longest pathes in B,

(2) every leaf of B is dominated by S since every vertex v in B satisfies that $d_T(v) \geq 2$.

By the observations, both v_1 and v_p are dominated by S, we have that $V(T) - \{v_1, v_2, v_5, v_6\}$ is a TRDS of T and $\gamma_(tr) \leq n - 4$, a contradiction and we complete the proof of claim 2. We consider the following three cases.

Case 1: diam$(B) = 2$.

Let P be one of the longest paths in B, say $P = v_1v_2v_3$. Clearly, $B \cong S_{|V(B)|}$ $(|V(B)| \geq 3)$, where $d_B(v_2) = |V(B)| - 1$. If $B \cong S_3 = P$ and only the ends of P have neighbors in S, by Lemma 2.3, $\gamma_{tr}(T) = n - 2$, a contradiction. If $|V(B)| \geq 5$ or $B \cong S_4$ and the center vertex v_2 has neighbors in S, then $V(T) - \{v_1, v_2, u, v_3\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of T and $\gamma_(tr) \leq n - 4$, also a contradiction. Thus, $B \cong S_3$ and all the vertices of S_3 have neighbors in S or $B \cong S_4$ and only the leaves of S_4 have neighbors in S.

Case 2: diam$(B) = 3$.

Let P be one of the longest paths in B, say $P = v_1v_2\cdots v_4$. Since diam$(B) = 3$, we have that $B \cong S_{a,b}$ $(a \geq 1, b \geq 1)$ where $d_B(v_2) = a + 1$ and $d_B(v_3) = b + 1$. If $B \cong S_{1,1} = P$. By Lemma 2.3, at least one inner vertices of P has neighbors in S. Furthermore, if all the inner vertices of P have neighbors in S, then $V(T) - \{v_1, v_2, v_3, v_4\}$ is a TRDS of T and $\gamma_(tr) \leq n - 4$, a contradiction. Thus, there is exact one inner vertex and all the ends of P have neighbors in S. Following, we may assume that $a \geq 2$ or $b \geq 2$. If both $a \geq 2$ and $b \geq 2$, then $V(T) - \{v_1, v_2, v_3, v_4\}$ is a TRDS of T and $\gamma_(tr) \leq n - 4$, a contradiction. Thus, $B \cong S_{a,1}$ $(a \geq 2)$. Let $\{u_2, u_3, \cdots, u_a\}$ be neighbors of v_2 in $B - P$. If $a \geq 3$, then $V(T) - \{v_2, u_2, u_3, \cdots, u_a, v_3\}$ is a TRDS of size at least $n - 4$, a contradiction. Thus, $a = 2$ and $B \cong S_{2,1}$. But in this case, if v_2 or v_3 has neighbors in B, then $V(T) - \{v_1, v_2, u_2, v_3\}$ or $V(T) - \{v_1, v_2, v_3, v_4\}$ is a TRDS of size at least $n - 4$, also a contradiction. Thus, $B \cong S_{2,1}$ and only the leaves of $S_{2,1}$ have neighbors in S.

Case 3: diam$(B) = 4$.

The same to Case 2, let P be one of the longest paths in B, say $P = v_1v_2\cdots v_5$. We say $d_T(v_3) = 2$. Otherwise, $V(T) - \{v_1, v_2, v_4, v_5\}$ is a TRDS of T and $\gamma_(tr) \leq n - 4$, a contradiction. Thus, $B \cong S_{a,0,c}$ $(a \geq 1, c \geq 1)$ where $d_B(v_2) = a + 1$ and $d_B(v_4) = c + 1$. If $a = c = 1$, then $B \cong P$. By Lemma 2.3, there is at least one vertices of $\{v_2, v_4\}$ has neighbors in S. Following, we may assume that $a \geq 2$ or $c \geq 2$. If $a \geq 3$ or $c \geq 3$, say $a \geq 3$. Let $\{u_2, u_3, \cdots, u_a\}$ be neighbors of v_2 in $B - P$, then $V(T) - \{u_2, u_3, \cdots, u_a, v_2, v_3\}$ is a TRDS of size at least $n - 4$, a contradiction. Thus, $B \cong S_{2,0,2}$ or $S_{2,0,1}$. If $B \cong S_{2,0,2}$ and v_2 or v_4 have neighbors in S, say v_2. Then $V(T) - \{v_1, v_2, u_2, v_3\}$ is a TRDS of size $n - 4$, also a contradiction. Thus, $d_T(v_2) = d_T(v_4) = 3$ and only the leaves of $S_{2,0,2}$ have neighbors in S. If $B \cong S_{2,0,1}$, the same to $S_{2,0,2}$, we have that $d_T(v_2) = 3$. Thus, at most v_4 and all the leaves of $S_{2,0,1}$ have neighbors in S.

We discuss all the cases and complete the proof of the theorem.

Lemma 1.5: Let T be a tree of order $n \geq 6$. If $\gamma_{tr}(T) = n-4$, then $T-S$ contains at most two nontrivial components.

Proof. Let T be a tree of order $n \geq 6$ and $\gamma_{tr}(T) = n-4$. By Lemma 2.2, $T-S$ contains at least one nontrivial component. Suppose that $T-S$ contains at least three nontrivial components, say B_1, B_2 and B_3. Let $e_i = u_iv_i \subseteq B_i$ $(i = 1, 2, 3)$. Since any vertex in $T-S$ is not adjacent to any leaf of T and $N_{B_i}(u_i) \cap N_{B_i}(v_i) = \emptyset$, we have that $V(T) - \{u_1, v_1, u_2, v_2, u_3, v_3\}$ is a TRDS of size $n-6$, a contradiction. Thus, $T-S$ contains at most two nontrivial components and we complete the proof of the lemma.

Theorem 1.6: Let T be a tree of order $n \geq 6$ and $T-S$ contains two nontrivial components, then $\gamma_{tr}(T) = n-4$ if and only if the two nontrivial components of $T-S$ are paths on at most five vertices and only the ends of them are adjacent to the vertices of S.

Proof. Let T be a tree of order $n \geq 6$ and $\gamma_{tr}(T) = n-4$ and B_1, B_2 are two nontrivial components of $T-S$. We consider the subtrees $T[B_1 \cup S]$ and $T[B_2 \cup S]$.

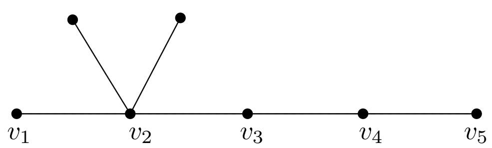
Claim: $\gamma_{tr}(T[B_i \cup S]) = |V(B_i)| + |S| - 2$ $(i = 1, 2)$.

Otherwise, let D_1 and D_2 be two $\gamma_{tr}-$sets of $B_1 \cup S$ and $B_2 \cup S$ respectively. It is clear that $S \subseteq D_1 \cap D_2$. Thus, $D_1 \cup D_2 \cup (V(T) - B_1 - B_2 - S)$ is a TRDS of T. If both $i = 1$ and 2 satisfies that $\gamma_{tr}(G[B_i \cup S]) < |V(B_i)| + |S| - 2$, then

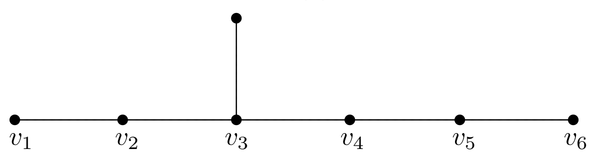
$$\begin{aligned}
\gamma_{tr}(T) &\leq |D_1 \cup D_2 \cup (V(T - B_1 - B_2 - S)| \\
&\leq |D_1 \cup D_2| + |V(T - B_1 - B_2 - S)| \\
&= |D_1| + |D_2| - |S| + |V(T)| - |V(B_1)| - \\
&\quad\ |V(B_2)| - |S| \\
&< |V(B_1)| + |S| - 2 + |V(B_2)| + |S| - 2 - \\
&\quad\ |S| + |V(T)| - |V(B_1)| - |V(B_2)| - |S| \\
&= n - 4.
\end{aligned}$$

a contradiction. Thus, there is at least one subtree of $T[B_1 \cup S]$ and $T[B_2 \cup S]$ satisfies that $\gamma_{tr}(T[B_i \cup S]) > |V(B_i)| + |S| - 2$. We may assume that $\gamma_{tr}(T[B_1 \cup S]) > |V(B_1)| + |S| - 2$. By the definition of TRDS, we have that $\gamma_{tr}(T[B_1 \cup S]) = |V(B_i)| + |S|$. But in this case, by Lemma 2.3, $T[B_1 \cup S] - S = B_1$ is an empty graph, a contradiction. Thus, we have that $\gamma_{tr}(T[B_i \cup S]) = |V(B_i)| + |S| - 2$ $(i = 1, 2)$.

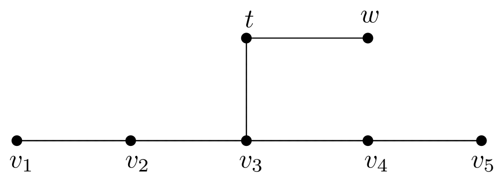
By Claim and Lemma 2.3, B_1 and B_2 are paths on at most five vertices and only the ends of them are adjacent to the vertices of S and we complete the proof of theorem.



$S_{3,0,1}$



$S_{1,1,0,1}$



$S_{1,P_3,1}$

$Fig.3$

Theorem 1.7: Let T be a tree of order $n \geq 6$ and $T-S$ contains only one nontrivial component, say B. Then $\gamma_{tr}(T) = n-4$ if and only if one of the following holds:
(a) : $B \cong S_4$ or P_4 and all the vertices of B have neighbors in S;
(b) : $B \in \{S_5, S_{2,2}, S_{3,1}, S_{3,0,3}, S_{1,2,1}, S_{1,2,0,1}, S_{1,1,1,0,1}, S_{1,0,2,0,1}\}$ and only the leaves of B have neighbors in S;
(c) : $B \cong S_{2,1}$ and exact one vertex with $d_B(v) = 3$ or $d_B(v) = 2$ and all the leaves of B have neighbors in S;
(d) : $B \cong S_{2,0,1}$ (see $Fig.2$) and $\{v_2\}$ or $\{v_2, v_4\}$ and all the leaves of B have neighbors in S;
(e) : $B \cong S_{2,0,2}$ (see $Fig.2$) and at least one of vertices with $d_B(v) = 3$ and all the leaves of B have neighbors in S;
(f) : $B \cong S_{3,0,2}$ and at most the vertex with $d_B(v) = 3$ and all the leaves of B have neighbors in S;
(g) : $B \cong S_{3,0,1}$ (see $Fig.3$) and at most v_4 and all the leaves of B have neighbors in S;
(h) : $B \cong S_{1,P_3,1}$ (see $Fig.3$) and at most one vertex of $\{v_2, v_4, t\}$ and all the leaves of B have neighbors in S;
(i) : $B \cong S_{1,1,1}$ and at most one vertex which is not a leave of B and all the leaves of B have neighbors in S;
(j) : $B \cong P_5 = v_1v_2v_3 \cdots v_5$ and $\{v_3\}$ or $\{v_2, v_3\}$ or $\{v_3, v_4\}$ and all leaves of B have neighbors in S;
(k) : $B \cong S_{1,1,0,1}$ (see $Fig.3$) and at most one vertex of $\{v_2, v_3, v_4\}$ and all the leaves of B have neighbors in S;
(l) : $B \cong P_6 = v_1v_2 \cdots v_6$ and $\{v_2, v_3\}$ or $\{v_2, v_5\}$ or $\{v_3, v_4\}$ or $\{v_4, v_5\}$ or exactly one vertex of $\{v_2, v_3, v_4, v_5\}$ and all the leaves of B have neighbors in S;
(m) : $B \cong S_{1,0,1,0,1}$ and at most the vertex with $d_B(v) = 3$ and all the leaves of B have neighbors in S.
(n) : $B \cong P_7 = v_1v_2 \cdots v_7$ and at most $\{v_3, v_4\}$ or $\{v_4, v_5\}$ or exact one vertex of $\{v_3, v_4, v_5\}$ and all the ends B have neighbors in S;
(o) : $B \cong P_8 = v_1v_2 \cdots v_8$ and at most v_4 or v_5 and all the ends of B have neighbors in S;
(p) : $B \cong P_9 = v_1v_2 \cdots v_9$ and at most v_5 and all the ends of B have neighbors in S;

Proof. It is easy to verify that if T satisfies one of the above conditions, then $\gamma_{tr}(T) = n-4$. Conversely, let T be a tree of order $n \geq 6$ with $\gamma_{tr}(T) = n-4$ and B be the nontrivial component of $T-S$.

Claim. $2 \leq$diam$(B) \leq 8$.

Otherwise, if diam$(B) = 1$, then $B \cong P_2$. By Lemma 2.3, $\gamma_{tr}(T) = n-2$, a contradiction. We may assume that diam$(B) \geq 9$. Let P be one of the longest paths in B of order at least 10, say $P = v_1v_2 \cdots v_p$ $(10 \leq p \leq |V(B)|)$. By the observations of Theorem 2.4, we have that $V(T) - \{v_1, v_2, v_5, v_6, v_{p-1}, v_p\}$ is a TRDS of size $n-6$, also a contradiction and we complete the proof of claim. Next, we consider the following seven cases.

Case 1 : diam$(B) = 2$.

Let P be one of the longest paths in B, say $P = v_1v_2v_3$. Clearly, $B \cong S_{|V(B)|}$ and $d_B(v_2) = |V(B)| - 1$. By Lemma 2.3 and Theorem 2.4, we have that $|V(B)| \geq 4$. If $|V(B)| \geq 6$ or $|V(B)| = 5$ and the center vertex v_2 of B has neighbors in S, then $V(T) - \{v_1, v_2, u, w, v_3\}$ where $\{u, w\} \subseteq N_{B-P}(v_2)$ is a TRDS of size $n-5$, a contradiction. Thus, $B \cong S_5$ and only the leaves of S_5 have neighbors in S. If $B \cong S_4$, by Theorem 2.4, we have that all vertices of S_4 have neighbors in S.

Case 2 :diam$(B) = 3$.

Let P be one of the longest paths in B, say $P = v_1v_2 \cdots v_4$. Since diam$(B) = 3$, we have that $B \cong S_{a,b}$ ($a \geq 1$, $b \geq 1$), where $d_B(v_2) = a+1$ and $d_B(v_3) = b+1$. If $B \cong P_4$, by Lemma 2.3 and the Case 2 of Theorem 2.4, we have that all the vertices of P_4 have neighbors in S. In the following, we may assume that $a \geq 2$ or $b \geq 2$. If $a \geq 4$ or $b \geq 4$, say $a \geq 4$. Then $V(T) - \{v_1, v_2, u, w, v_3\}$ where $\{u, w\} \subseteq N_{B-P}(v_2)$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. Thus, $a \leq 3$ and $b \leq 3$. Let $a = 3$. If $b \geq 2$, then $V(T) - \{v_1, v_2, u, v_3, v_4\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of size $n-5$, a contradiction. Thus, $b = 1$. Furthermore, if v_2 has neighbors in S, then $V(T) - \{v_1, v_2, u, w, v_3\}$ where $\{u, w\} \subseteq N_{B-P}(v_2)$ is a TRDS of size $n-5$, a contradiction. If v_3 has neighbors in S, then $V(T) - \{v_1, v_2, u, v_3, v_4\}$ is a TRDS of size $n-5$, also a contradiction. Thus, $B \cong S_{3,1}$ and only the leaves of B have neighbors in S. Following, we may assume that $a \leq 2$ and $b \leq 2$. If $B \cong S_{2,2}$ and v_2 or v_3 has neighbors in S, say v_2. Then $V(T) - \{v_1, v_2, u, v_3, v_4\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of size $n-5$, a contradiction. Thus, only the leaves of $S_{2,2}$ have neighbors in S. If $B \cong S_{2,1}$, by Theorem 2.4, at least one vertex of $\{v_2, v_3\}$ has neighbors in S. If both v_2 and v_3 have neighbors in S, then $V(T) - \{v_1, v_2, u, v_3, v_4\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of size $n-5$, a contradiction. Thus, there is exactly one vertex of $\{v_2, v_3\}$ and all the leaves of $S_{2,1}$ have neighbors in S.

Case 3 : diam$(B) = 4$.

Let P be one of the longest paths in B, say $P = v_1v_2 \cdots v_5$. We say that there are at most two vertices of $\{v_2, v_3, v_4\}$ have neighbors in $T - P_5$. Otherwise, $V(T) - \{v_1, v_2, v_3, v_4, v_5\}$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. We consider the following three subcases.

Subcase 3.1. There are two vertices of $\{v_2, v_3, v_4\}$ have neighbors in $B - P$.

If $\{v_2, v_3\}$ or $\{v_3, v_4\}$ has neighbors in $B - P$, say $\{v_2, v_3\}$. Then $V(T) - \{v_1, v_2, u, v_4, v_5\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of size $n-5$, a contradiction. We may assume that $\{v_2, v_4\}$ has neighbors in $B - P$, then $B \cong S_{a,0,c}$ ($a \geq 2$, $c \geq 2$) where $d_B(v_2) = a+1$ and $d_B(v_4) = c+1$. Furthermore, If $a \geq 4$ or $a = 3$ and v_2 has neighbors in S, then $V(T) - \{v_1, v_2, v_3, u, w\}$ where $\{u, w\} \subseteq N_{B-P}(v_2)$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. The same result to $c \geq 4$ or $c = 3$ and v_4 has neighbors in S. Thus, $a \leq 3$ and $c \leq 3$. Furthermore, if $B \cong S_{3,0,3}$, by the discussion of the above, only the leaves of $S_{3,0,3}$ have neighbors in S. If $B \cong S_{3,0,2}$, then at most the vertex v_4 and all the leaves of $S_{3,0,2}$ have neighbors in S. If $B \cong S_{2,0,2}$, by case 3 of Theorem 2.4, we have that there is at least one vertex with $d_B(v) = 3$ and all the leaves of $S_{2,0,2}$ have neighbors in S.

Subcase 3.2. There is only one vertex of $\{v_2, v_3, v_4\}$ has neighbors in $B - P$.

Let v_2 or v_4 has neighbors in $B - P$, say v_2. Then $B \cong S_{a,0,1}$ ($a \geq 2$) where $d_B(v_2) = a+1$ and $d_B(v_4) = 2$. If v_3 has neighbors in S, then $V(T) - \{v_1, v_2, u, v_4, v_5\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. Thus, $d_T(v_3) = 2$. Furthermore, If $a \geq 4$ or $a = 3$ and v_2 has neighbors in S, then $V(T) - \{v_1, v_2, u, w, v_3\}$ where $\{u, w\} \subseteq N_{B-P}(v_2)$ is a TRDS of size $n-5$, also a contradiction. Thus, $a < 3$ or $a = 3$ and $d_T(v_2) = 4$. Furthermore, if $B \cong S_{2,0,1}$, by Case 3 of Theorem 2.4, v_2 or $\{v_2, v_4\}$ and all leaves of $S_{2,0,1}$ have neighbors in S. Next, let v_3 has neighbors in $B - P$. Since $diam(B) = 4$, if there are at least two paths of order 3 attach to v_3 in $B - P$, say $v_3t_1w_1$ and $v_3t_2w_2$ (where it is possible that $t_1 = t_2$). We have that $V(T) - \{v_1, v_2, t_1, w_1, t_2, w_2\}$ is a TRDS of size at least $n-5$, a contradiction. Thus, there is at most one path of order 3 attach to v_3 in $B - P$ and we consider the following two subcases.

Subcase 3.2.1. There is only one path of order 3 attach to v_3 in $B - P$, say $P_3 = v_3tw$.

If there is one vertex in $S \cup V(B - P - P_3)$ adjacent to v_3, then $V(T) - \{v_1, v_2, t, w, v_4, v_5\}$ is a TRDS of T and $\gamma_(tr) \leq n-6$, a contradiction. Thus, $B \cong S_{1,P_3,1}$ and $d_T(v_3) = 3$. Furthermore, if there are at least two vertices of $\{v_2, t, v_4\}$ have neighbors in S, say v_2 and v_4. Then $V(T) - \{v_1, v_2, v_3, v_4, v_5\}$ is a TRDS of size $n-5$, a contradiction. Thus, there is at most one vertex of $\{v_2, t, v_4\}$ has neighbors in S.

Subcase 3.2.2. There is no path of order 3 attach to v_3 in $B - P$.

It is clear that $B \cong S_{1,b,1}(b \geq 1)$. If $b \geq 3$ or $b = 2$ and v_3 has neighbors in S, then $V(T) - \{v_2, v_3, u, w, v_4\}$ where $\{u, w\} \subseteq N_{B-P}(v_3)$ is a TRDS of T of and $\gamma_(tr) \leq n-5$, a contradiction. Thus, $b < 2$ or $b = 2$ and $d_T(v_3) = 4$. Furthermore, If $B \cong S_{1,2,1}$ and v_2 or v_4 has neighbors in S, say v_2, then $V(T) - \{v_1, v_2, v_3, u, v_4\}$ where $u \in N_{B-P}(v_3)$ is a TRDS of size $n-5$, also a contradiction. Thus, $B \cong S_{1,2,1}$ and only the leaves have neighbors in S. If $B \cong S_{1,1,1}$ and $\{v_2, v_3\}$ or $\{v_3, v_4\}$ or $\{v_2, v_4\}$ has neighbors in S, then $V(T) - \{v_1, v_2, v_3, u, v_4\}$ or $V(T) - \{v_2, v_3, u, v_4, v_5\}$ where $u \in N_{B-P}(v_2)$ or $V(T) - \{v_1, v_2, v_3, v_4, v_5\}$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. Thus, at most one vertex of $\{v_2, v_3, v_4\}$ and all leaves of $S_{1,1,1}$ have neighbors in S.

Subcase 3.3. $B \cong P_5$.

By Lemma 2.3, at least one vertex of $\{v_2, v_3, v_4\}$ has neighbors in S. If there is only one vertex of $\{v_2, v_4\}$ has neighbors in S, by Theorem 2.4, $\gamma_{tr}(T) = n-3$, a contradiction. If all the vertices of $\{v_2, v_3, v_4\}$ have neighbors in S, then $V(T) - \{v_1, v_2, v_3, v_4, v_5\}$ is a TRDS of T and $\gamma_(tr) \leq n-5$, also a contradiction. Thus, $\{v_3\}$ or $\{v_2, v_3\}$ or $\{v_3, v_4\}$ and all the ends of P_5 have neighbors in S.

Case 4 : diam$(B) = 5$.

Let P be one of the longest paths in B, say $P = v_1v_2v_3v_4v_5v_6$. If v_2 or v_5 has neighbors in $B - P$, say v_2. Then $V(T) - \{v_1, v_2, u, v_5, v_6\}$ where $u \in N_{B-P}(v_2)$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. If both v_3 and v_4 have neighbors in $B - P$, then $V(T) - \{v_2, w, v_3, v_5, v_6\}$ where $w \in N_{B-P}(v_3)$ is a TRDS of T and $\gamma_(tr) \leq n-5$, also a contradiction. Thus, there is at most one vertex of

$\{v_3, v_4\}$ has neighbors in $B-P$ and we consider the following two subcases.

Subcase 4.1. There is exactly one vertex of $\{v_3, v_4\}$ has neighbors in $B-P$.

We may assume that v_3 has neighbors in $B-P$. Since diam$(B) = 5$, if there is a path $P_3 = v_3uw$ attach to v_3 in B, then $V(T) - \{v_1, v_2, u, w, v_5, v_6\}$ is a TRDS of T and $\gamma_(tr) \leq n-6$, a contradiction. Thus, $B \cong S_{1,b,0,1}$ $(b \geq 1)$ where $d_B(v_3) = b+2$. Furthermore, if $b \geq 3$ or $b = 2$ and v_3 has neighbors in S, then $V(T) - \{v_2, v_3, u, w, v_4\}$ where $\{u, w\} \subseteq N_{B-P}(v_3)$ is a TRDS of size $n-5$, a contradiction. Thus, $B \cong S_{1,1,0,1}$ or $B \cong S_{1,2,0,1}$ and $d_T(v_3) = 4$. Let $B \cong S_{1,1,0,1}$, if $\{v_2, v_3\}$ or $\{v_2, v_4\}$ or $\{v_3, v_4\}$ has neighbors in S, then $V(T) - \{v_1, v_2, v_3, u, v_4\}$ or $V(T) - \{v_1, v_2, v_3, v_4, v_5\}$ or $V(T) - \{v_2, v_3, u, v_4, v_5\}$ where $u \in N_{B-P}(v_3)$ is a TRDS of size $n-5$, a contradiction. If v_5 has neighbors in S, then $V(T) - \{v_1, v_2, v_4, v_5, v_6\}$ is a TRDS of size $n-5$, a contradiction. Thus, at most one vertex of $\{v_2, v_3, v_4\}$ has neighbors in S. Let $B \cong S_{1,2,0,1}$ where $d_T(v_3) = 4$ and $\{u, w\} \subseteq N_{B-P}(v_3)$. By the same discussion to $S_{1,1,0,1}$, we have that there is at most one vertex of $\{v_2, v_4\}$ has neighbors in S. Furthermore, if v_2 or v_4 has neighbors in S, then $V(T) - \{v_1, v_2, v_3, u, w\}$ or $V(T) - \{u, w, v_3, v_4, v_5\}$ is a TRDS of size $n-5$, a contradiction. Thus, only the leaves of $S_{1,2,0,1}$ have neighbors in S.

Subcase 4.2. There is no vertex of $\{v_2, v_3, v_4, v_5\}$ has neighbors in $B-P$.

Clearly, $B \cong P_6 = v_1v_2\cdots v_6$. If $\{v_2, v_4\}$ or $\{v_3, v_5\}$ has neighbors in S, then $V(T) - \{v_1, v_2, v_3, v_5, v_6\}$ or $V(T) - \{v_1, v_2, v_4, v_5, v_6\}$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. Thus, there is at most $\{v_2, v_3\}$ or $\{v_2, v_5\}$ or $\{v_3, v_4\}$ or $\{v_4, v_5\}$ or exactly one vertex of $\{v_2, v_3, v_4, v_5\}$ and all the ends of P_6 have neighbors in S.

Case 5: diam$(B) = 6$.

Let P be one of the longest paths in B, say $P = v_1v_2v_3v_4v_5v_6v_7$. If v_2 or v_6 has neighbors in $T-P$, say v_2. Then $V(T) - \{v_1, v_2, v_3, v_6, v_7\}$ is a TRDS and $\gamma_(tr) \leq n-5$, a contradiction. Thus, if both v_3 and v_5 have neighbors in $T-P$, then $V(T) - \{v_2, v_3, v_4, v_6, v_7\}$ is a TRDS of size $n-5$, also a contradiction. We consider the following subcases.

Subcase 5.1. $\{v_3, v_4\}$ or $\{v_4, v_5\}$ has neighbors in $B-P$.

We may assume that both v_3 and v_4 have neighbors in $B-P$. If v_3 or v_4 has at least two neighbors in $B-P$, say v_3. Then $V(T) - \{v_2, v_3, u, v_4, v_5\}$ where $u \in N_{B-P}(v_3)$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. Thus, both v_3 and v_4 have exactly one neighbor in $B-P$. Since diam$(B) = 6$, if there is one path $P_3 = v_3uw$ attach to v_3 in B, then $V(T) - \{v_1, v_2, u, w, v_6, v_7\}$ is a TRDS of size $n-6$, also a contradiction. By the same discussion to v_4, we have that $B \cong S_{1,1,1,0,1}$. Furthermore, if one vertex of $\{v_3, v_4, v_5\}$ has neighbors in S, then $V(T) - \{v_2, v_3, u, v_4, v_5\}$ where $u \in N_{B-P}(v_3)$ or $V(T) - \{v_2, v_3, v_4, w, v_5\}$ where $w \in N_{B-P}(v_4)$ or $V(T) - \{v_2, v_3, v_4, v_5, v_6\}$ is a TRDS of size $n-5$, a contradiction. Thus, $B \cong S_{1,1,1,0,1}$ and only the leaves of $S_{1,1,1,0,1}$ have neighbors in S.

Subcase 5.2. There is only one vertex of $\{v_3, v_4, v_5\}$ has neighbors in $B-P$.

If v_3 or v_5 has neighbors in $B-P$, say v_3. Then $V(T) - \{v_2, v_3, u, v_6, v_7\}$ where $u \in N_{B-P}(v_3)$ is a TRDS of size $n-5$, a contradiction. Thus, at most v_4 has neighbors in $B-P$. Since diam$(B) = 6$, if there is a path P_{t+1} of length at least 2 attach to v_4, say $P_{t+1} = v_4u_1, \cdots, u_t$ $(3 \geq t \geq 2)$. Then $V(T) - \{v_1, v_2, u_{t-1}, u_t, v_6, v_7\}$ is a TRDS of T and $\gamma_(tr) \leq n-6$, a contradiction. Thus, $B \cong S_{1,0,c,0,1}$ $(c \geq 1)$ where $d_B(v_4) = c+2$. If v_2 or v_6 has neighbors in S, then $V(T) - \{v_1, v_2, v_3, v_5, v_6\}$ or $V(T) - \{v_2, v_3, v_5, v_6, v_7\}$ is a TRDS of size $n-5$, a contradiction. If v_3 or v_5 has neighbors in S, then $V(T) - \{v_1, v_2, v_4, u, v_5\}$ or $V(T) - \{v_3, v_4, u, v_6, v_7\}$ where $u \in N_{B-P}(v_4)$ is a TRDS of size $n-5$, also a contradiction. Thus, $d_T(v_2) = d_T(v_3) = d_T(v_5) = d_T(v_6) = 2$. Furthermore, If $c \geq 3$ or $c = 2$ and v_4 have neighbors in S, then $V(T) - \{v_3, v_4, u, w, v_5\}$ where $\{u, w\} \subseteq N_{B-P}(v_4)$ is a TRDS of size $n-5$, a contradiction. Thus, $B \cong S_{1,0,2,0,1}$ and only the leaves of $S_{1,0,2,0,1}$ have neighbors in S or $B \cong S_{1,0,1,0,1}$ and at most v_4 and all the leaves of $S_{1,0,1,0,1}$ have neighbors in S.

Subcase 5.3. $B \cong P_7 = v_1v_2\cdots v_7$.

By the discussion of the above, it is clear that at most $\{v_3, v_4\}$ or $\{v_4, v_5\}$ or exactly one vertex of $\{v_3, v_4, v_5\}$ and all the ends of P_7 have neighbors in S.

Case 6 : diam$(B) = 7$.

Let P be one of the longest paths in B of order 8, say $P = v_1v_2\cdots v_8$. If v_2 or v_7 has a neighbor in $T-P$, say v_2, then $V(T) - \{v_1, v_2, v_3, v_6, v_7\}$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. If v_3 or v_6 has neighbors in $T-P$, say v_3, then $V(T) - \{v_2, v_3, v_4, v_7, v_8\}$ is a TRDS of size $n-5$, also a contradiction. Thus, at most v_4 or v_5 has neighbors in $T-P$. If there is a vertex $u \in V(B-P)$ adjacent to v_4 or v_5, then $V(T) - \{v_3, v_4, u, v_7, v_8\}$ where $u \in N_{B-P}(v_4)$ or $V(T) - \{v_1, v_2, v_5, w, v_6\}$ where $w \in N_{B-P}(v_5)$ is a TRDS of size $n-5$, also a contradiction. Thus, $B \cong P_8$ and at most v_4 or v_5 and all the ends of B have neighbors in S.

Case 7 : diam$(B) = 8$.

Let P be one of the longest paths in B of order 9, say $P = v_1v_2\cdots v_9$. The same to Case 6, we have that v_2 or v_3 or v_4 has no neighbors in $T-P$. Otherwise, $V(T) - \{v_1, v_2, v_3, v_8, v_9\}$ or $V(T) - \{v_2, v_3, v_4, v_8, v_9\}$ or $V(T) - \{v_3, v_4, v_5, v_8, v_9\}$ is a TRDS of T and $\gamma_(tr) \leq n-5$, a contradiction. By the symmetric, v_6 or v_7 or v_8 also has no neighbors in $T-P$. If there is a vertex $u \in V(B-P)$ adjacent to v_5 in B, then $V(T) - \{v_1, v_2, v_5, u, v_8, v_9\}$ is a TRDS of size $n-6$, also a contradiction. Thus, $B \cong P_9$ and at most v_5 and all the ends of B have neighbors in S.

We discuss all the cases and complete the proof of the theorem.

By our result, we can give an algorithm to determine a tree T of order n with $\gamma_{tr}(T) = n-i$ where $i = 0, 2, 3, 4$.

Algorithm 1.8: Input: A tree T of order n with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$.
Output: A subset vertex set of T, denote by S and the cardinality of S, denoted by r.
1 : set $S := \emptyset$ and $r := 0$
2 : for $i = 1, 2, \cdots, n$ do
3 : if v_i is a leaf or adjacent to a leaf of T then
4 : add v_i to S and increment r by 1
5 : end if
6 : end for
7 : return (S, r).

By the algorithm, for any tree T, we can get a subgraph $T[V-S]$. If $T[V-S]$ has at most two nontrivial components, and satisfies Lemma 2.2 or Lemma 2.3 or Theorem 2.4 or Theorem 2.6 or Theorem 2.7, then $\gamma_{tr}(T) = n - i$ where $i = 0, 2, 3, 4$. Otherwise, $r \leq \gamma_{tr}(T) \leq n - 5$.

REFERENCES

[1] G. Chartrand and L. Lesniak, *Graphs and digraphs*, third ed., Chapman and Hall, London. 1996.

[2] X. Chen, J. Liu and J.X. Meng, "Total restrained domination in graphs", *Computers and Mathematics with Applications*, vol.8, pp.2892-2898, 2011.

[3] P. Dankelmann, D. Day, J.H. Hattingh, L.R. Markus and H.C. Swart, "On equality in an upper bound for the restrained and total domination number of a graphs", *Discrete Mathematics*, vol.307, pp.2845-2852, 2007.

[4] J.H. Hattingh, E. Jonck, E.J. Joubert and A.R. Plummer., "Total restrained domination in trees", *Discrete Mathematics*, vol.307, pp.1643-1650, 2007.

[5] J.H. Hattingh, E. Jonck and E.J. Joubert., "Bounds on total restrained domination number of a graph", *Graphs and Combinatorics*, vol.26, pp.77-93, 2010.

[6] M.A. Henning and J.E. Maritz., "Total restrained domination in graphs with minimum degree two", *Discrete Mathematics*, vol.308, pp.1909-1920, 2008.

[7] J.H. Hettingh, E. Jonck and E.J. Joubert, "An upper bound on the total restrained domination number of a tree", *J. Comb Optim.*, vol.20, N.3, pp.205-223, 2010.

[8] J.H. Hattingh, E. Jonck, E.J. Joubert and A.R. Plummer., "Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs", *Discrete Mathematics*, vol.308, pp.1080-1087, 2008.

[9] Cyman, J. and Raczek, J., "On the total restrained domination number of a graph". *Australas. J. Combin.*, vol.36, pp.91-100, 2006.

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of domination in graphs*, first Edition, Marcel Dekker, New York, 1997.

[11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs: Advanced Topics*, first Edition, Marcel Dekker, New York, 1997.

[12] H. Jiang, L. Kang and E. Shan, "Total restrained domination number in Cubic graphs", *J. Graphs and Cominatorics*, vol.25, pp.341-350, 2009.

[13] J.A. Telle and A. Proskurowski, "Algorithms for vertex partitioning problems on partial $k-$trees", *SIAM J. Discrete Math.*, vol.10, pp.529-550, 1997.

[14] J. Raczek and J. Cyman., "Total restrained domination number of trees", *Discrete Mathematics*, vol.308, pp.44-50, 2008.

[15] N.J. Rad., "Results on total restrained domination", *Int. J. Contemp. Math. Sciences*, vol.3, N.8, pp.383-387, 2008.

[16] J. Southey and M.A. Henning, "An Improved Upper Bound on the Total Restrained Domination Number in Cubic Graphs", *Graphs and Combinatorics*, vol.28, pp.547-554, 2012.

[17] Z. Tahmasbzadehbaee, D.S. Nandappa, H.A. Ahangar, D.A. Mojdeh and Y.C. Zhao, "Total restrained domination in graphs of diameter 2 or 3", *Mathematical Sciences*, vol.7, N.26, pp.346-351, 2013.

[18] E. Joubert, "Total restrained domination in claw-free graphs with minimum degree at least two", *Discrete Applied Mathematics*, vol.159, pp.2078-2097, 2011.

[19] L.C. Zuo, H.F. Guo and C.H. Shang, "The Multi-level Distance Number for Symmetric Lobster-like Trees about the Weight Center", *IAENG International Journal of Applied Mathematics*, vol.45, no.1, pp.41-50, 2015.

[20] B. Krishnakumaria, Y.B. Venkatakrishnana and M. Krzywkowski, "Bounds on the vertex$-$edge domination number of a tree", *C. R. Acad. Sci. Paris, Ser. I* , vol.7, pp.363-366, 2014.

[21] M.A. Henning and S.A. Marcon, "Domination versus disjunctive domination in trees", *Discrete Applied Mathematics*, vol.184, pp.171-177, 2015.

[22] M. Atapour, S.M. Sheikholeslami and L.Volkmann, "Global Roman Domination in Trees", *Graphs and Combinatorics*, vol.31, pp.813-825, 2015.

[23] W.J. Desormeaux, T.W. Haynes and M.A. Henning, "Improved bounds on the domination number of a tree", *Discrete Applied Mathematics*, vol.177, pp.88-94, 2014.