Coprime Factorizability and Stabilizability of Plants Extended by Zeros and Parallelled Some Plants

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Abstract—**This paper is concerned with the factorization approach to control systems. It is known all models do not have both right- and left-coprime factorizations. In this paper, we consider the models in which some plants admit and in which some plants do not admit coprime factorization. In the case where the plants admits only one-side coprime factorization, it is known that the plant with additional zeros admits both side coprime factorizations. However we show that in the case where the plant do not admit coprime factorization, there exists a case where the plant with any finite additional zeros cannot admit coprime factorization. We will also consider the parallel plants, in which one of them is plant to be stabilized. Any stabilizable plants has some parallel plants that admit both side coprime factorizations. We will show that this fact can applied to stabilizable plants only, so that unstabilizable plants is not able to construct such parallel plants.**

Index Terms—**Linear systems, Feedback stabilization, One-side coprime factorization, Coprime factorization over commutative rings**

I. Introduction

In the factorization approach[1], [2], [3], [4], a transfer function is given as the ratio of *two stable causal transfer functions* and the set of stable causal transfer functions forms a *commutative ring*.

Since stabilizing controllers are not unique in general, the choice of stabilizing controllers is important for the resulting closed loop. In the classical case such as continuous-time LTI systems and discrete-time LTI systems, the stabilizing controllers can be parameterized by the method called "Youla-parameterization"[1], [2], [4], [5], [6], [7] (also called Youla-Kučera-parameterization). However, there exist models in which some stabilizable transfer matrices do not have their right- and left-coprime factorizations in general[8], [9]. In such models, we cannot employ the Youla-parameterization in general.

We consider the models in which some plants admit or do not admit coprime factorization. In the case where the plants admits only one-side coprime factorization, it is known that the plant with additional zeros admits both side coprime factorizations. However we show that in the case where the plant do not admit coprime factorization, there exists a case where the plant with any additional zeros cannot admit coprime factorization (Theorem 5). We will also consider the parallel plants, in which one of them is plant to be stabilized. Any stabilizable plants has some parallel plants that admit both side coprime factorizations. However, we will show that unstabilizable plants has not able to construct such parallel plants.

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II. Preliminaries

In the following we begin by introducing notations used in this paper. Then we give the formulation of the feedback stabilization problem.

A. Notations

a) Commutative Rings: We will consider that *the set of all stable causal transfer functions* is a commutative ring, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \text{ is a nonzerodivisor}\}$. This will be considered to be *the set of all possible transfer functions*. If the commutative ring $\mathcal{A}$ is an integral domain, $\mathcal{F}$ becomes a field of fractions of $\mathcal{A}$. However, if $\mathcal{A}$ is not an integral domain, then $\mathcal{F}$ is not a field, because any nonzero zerodivisor of $\mathcal{F}$ is not a unit.

b) Matrices: Suppose that x and y denote sizes of matrices.

The set of matrices over $\mathcal{A}$ of size $x \times y$ is denoted by $\mathcal{A}^{x\times y}$. In particular, the set of square matrices over $\mathcal{A}$ of size x is denoted by $(\mathcal{A})_x$. A square matrix is called *singular* over $\mathcal{A}$ if its determinant is a zerodivisor of $\mathcal{A}$, and *nonsingular* otherwise. The identity and the zero matrices are denoted by I_x and $O_{x\times y}$, respectively, if the sizes are required, otherwise they are denoted simply by I and O.

Matrices A and B over $\mathcal{A}$ are *right-coprime over* $\mathcal{A}$ if there exist matrices $\widetilde{X}$ and $\widetilde{Y}$ over $\mathcal{A}$ such that $\widetilde{X}A + \widetilde{Y}B = I$. Analogously, matrices $\widetilde{A}$ and $\widetilde{B}$ over $\mathcal{A}$ are *left-coprime over* $\mathcal{A}$ if there exist matrices X and Y over $\mathcal{A}$ such that $\widetilde{A}X + \widetilde{B}Y = I$. Further, pair (N, D) of matrices N and D is said to be a *right-coprime factorization of P over $\mathcal{A}$* if (i) the matrix D is nonsingular over $\mathcal{A}$, (ii) $P = ND^{-1}$ over $\mathcal{F}$, and (iii) N and D are right-coprime over $\mathcal{A}$. Also, pair $(\widetilde{N}, \widetilde{D})$ of matrices $\widetilde{N}$ and $\widetilde{D}$ is said to be a *left-coprime factorization of P over $\mathcal{A}$* if (i) $\widetilde{D}$ is nonsingular over $\mathcal{A}$, (ii) $P = \widetilde{D}^{-1}\widetilde{N}$ over $\mathcal{F}$, and (iii) $\widetilde{N}$ and $\widetilde{D}$ are left-coprime over $\mathcal{A}$. As we have seen, in the case where a matrix is potentially used to express *left* fractional form and/or *left* coprimeness, we usually attach a tilde '$\widetilde{\ }$' to a symbol; for example $\widetilde{N}$, $\widetilde{D}$ for $P = \widetilde{D}^{-1}\widetilde{N}$ and $\widetilde{Y}$, $\widetilde{X}$ for $\widetilde{Y}N + \widetilde{X}D = I$.

B. Feedback Stabilization Problem

The stabilization problem considered in this paper follows that of Sule in [10] and Mori and Abe in [11] who consider the feedback system Σ [3, Ch.5, Figure 5.1] as in Figure 1.

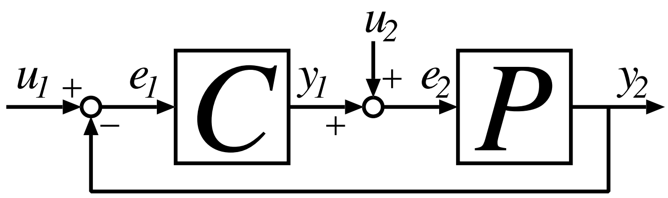

Fig. 1. Feedback system Σ.

For further details the reader is referred to [3], [11]. Throughout this paper, the plant we consider has m inputs and n outputs, and its transfer matrix, which itself is also called simply a *plant*, is denoted by P and belongs to $\mathcal{F}^{n\times m}$.

Definition 1: Define $\widehat{F}_{\rm ad}$ by

$$\begin{aligned}\widehat{F}_{\rm ad} = \ & \{(X,Y) \in \mathcal{F}^{x\times y} \times \mathcal{F}^{y\times x} \mid \\ & \quad \det(I_x + XY) \text{ is a unit of } \mathcal{F}, \\ & \quad x \text{ and } y \text{ are positive integers}\}.\end{aligned}$$

For $P \in \mathcal{F}^{n\times m}$ and $C \in \mathcal{F}^{m\times n}$, the matrix $H(P,C) \in (\mathcal{F})_{m+n}$ is defined by

$$H(P,C) = \begin{bmatrix} (I_n + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_n+PC)^{-1} & (I_m+CP)^{-1}\end{bmatrix} \quad (1)$$

provided $(P,C) \in \widehat{F}_{\rm ad}$. This $H(P,C)$ is the transfer matrix from $[\ u_1^t\ u_2^t\ ]^t$ to $[\ e_1^t\ e_2^t\ ]^t$ of the feedback system Σ. If (i) $(P,C) \in \widehat{F}_{\rm ad}$ and (ii) $H(P,C) \in (\mathcal{A})_{m+n}$, then we say that the plant P is *stabilizable*, P is *stabilized* by C, and C is a *stabilizing controller* of P. ▪

It is known that $W(P,C)$ defined below is over $\mathcal{A}$ if and only if $H(P,C)$ is over $\mathcal{A}$:

$$W(P,C) := \begin{bmatrix} C(I_n+PC)^{-1} & -CP(I_m+CP)^{-1} \\ PC(I_n+PC)^{-1} & P(I_m+CP)^{-1}\end{bmatrix}. \quad (2)$$

This $W(P,C)$ is the transfer matrix from u_1 and u_2 to y_1 and y_2. Then, we have

$$H(P,C) = I_{m+n} - FW(P,C),$$

where

$$F = \begin{bmatrix} O & I_n \\ -I_m & O\end{bmatrix}.$$

The matrix F is unimodular; in fact,

$$F^{-1} = \begin{bmatrix} O & -I_m \\ I_n & O\end{bmatrix},$$

which is over $\mathcal{A}$. Thus, $W(P,C)$ can be expressed in terms of F and $H(P,C)$:

$$W(P,C) = F^{-1}(I_{m+n} - H(P,C)).$$

Here we define the causality of transfer functions, which is an important physical constraint, used in this paper. We employ the definition of causality from Vidyasagar *et al.*[4, Definition 3.1] and Mori and Abe[11].

Definition 2: Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$, with $\mathcal{Z} \neq \mathcal{A}$, including all zerodivisors. Define the subsets $\mathcal{P}$ and $\mathcal{P}_{\rm s}$ of $\mathcal{F}$ as follows:

$$\begin{aligned}\mathcal{P} &= \{n/d \in \mathcal{F} \,|\, n \in \mathcal{A},\ d \in \mathcal{A}\backslash\mathcal{Z}\}, \\ \mathcal{P}_{\rm s} &= \{n/d \in \mathcal{F} \,|\, n \in \mathcal{Z},\ d \in \mathcal{A}\backslash\mathcal{Z}\}.\end{aligned}$$

A transfer function in $\mathcal{P}$ ($\mathcal{P}_{\rm s}$) is called *causal* (*strictly causal*). Similarly, if every entry of a transfer matrix over $\mathcal{F}$ is in $\mathcal{P}$ ($\mathcal{P}_{\rm s}$), the transfer matrix is called *causal* (*strictly causal*). ▪

It should be noted that when using "a stabilizing controller," we do not guarantee the causality. However, in the classical case of the factorization approach, once we restrict ourselves to strictly proper plants, it is known that any stabilizing controller of strictly causal plant is causal (cf. Corollary 5.2.20 of [3], Theorem 4.1 of [4], and Proposition 6.2 of [11]). One can see, in fact, that many practical systems are strictly causal. On the other hand, including noncausal stabilizing controllers seems to make the theory easy and simple in the mathematical viewpoint. From these observations, we have accepted the possibility of the noncausality of stabilizing controllers in the parametrization.

III. Previous Results

The first results below are for only one of right- and left-coprime factorizations, which were considered in [12].

Theorem 1 (Theorem 1 of [12], [13]): If there exists a right-(left-)coprime factorization of the plant $P \in \mathcal{P}^{n\times m}$, then the plant $[\,P^t \quad O^{m\times m}]^t \in \mathcal{P}^{(m+n)\times m}$ (the plant $[\,P \quad O^{n\times n}] \in \mathcal{P}^{n\times(m+n)}$ has both right- and left-coprime factorizations.

Theorem 2 (Theorem 2 of [12], [13]): Let $\mathcal{S}(P)$ and $\mathcal{S}(\mathrm{Diag}(P,O^{y\times x}))$ denote the sets of stabilizing controllers of the plants P and $\mathrm{Diag}(P,O^{y\times x})$, respectively. Then the following equation holds:

$$\mathcal{S}(P) = \{[\,I_m\ \ O^{m\times x}\,]C[\,\begin{smallmatrix} I_n \\ O^{y\times n}\end{smallmatrix}\,] \mid C \in \mathcal{S}(\mathrm{Diag}(P,O^{y\times x}))\}.$$

By using Theorems 1 and 2, we have the following theorem and Corollary.

Theorem 3 (Theorem 3 of [12], [13]): Suppose that there exists a right-coprime factorization (N,D) over $\mathcal{A}$ of the plant $P \in \mathcal{P}^{n\times m}$ with $\widetilde{Y}N + \widetilde{X}D = I_m$. Let $(\widetilde{N}', \widetilde{D}')$ be a left-coprime factorization over $\mathcal{A}$ of the plant $[\,P^t \quad O^{m\times m}\,]^t \in \mathcal{P}^{(m+n)\times m}$ with $\widetilde{N}'Y' + \widetilde{D}'X' = I_{m+n}$.

Then all of the stabilizing controllers of the plant P are of the form

$$(\widetilde{X} - R\widetilde{N}')^{-1}([\,\widetilde{Y} \quad O^{m\times m}\,] + R\widetilde{D}')][\,I_n \quad O^{n\times m}\,]^t \quad (3)$$

with $\widetilde{X} - R\widetilde{N}'$ nonsingular, where R is a parameter matrix of $\mathcal{A}^{m\times(m+n)}$.

Corollary 1 (Corollary 1 of [12]): Suppose that there exists a left-coprime factorization $(\widetilde{N},\widetilde{D})$ over $\mathcal{A}$ of the plant $P \in \mathcal{P}^{n\times m}$ with $\widetilde{N}Y + \widetilde{D}X = I_n$. Let (N', D') be a right-coprime factorization over $\mathcal{A}$ of the plant $[\,P \quad O^{n\times n}\,] \in \mathcal{P}^{n\times(m+n)}$ with $\widetilde{Y}'N' + \widetilde{X}'D' = I_{m+n}$.

Then all of the stabilizing controllers of the plant P are of the form

$$[\,I_m \quad O^{m\times n}\,]([\,Y^t \quad O^{n\times n}\,]^t + D'R)(X - N'R)^{-1} \quad (4)$$

with $X - N'R$ nonsingular, where R is a parameter matrix of $\mathcal{A}^{(m+n)\times n}$.

In [14], we considered the parallel plants, in which one of them is the plant P to be stabilized. Here we do not consider whether or not P admits coprime factorization. Even under this situation, we have the following result.

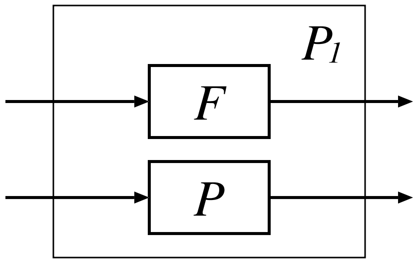

Fig. 2. New plant $\mathrm{Diag}(F, P)$.

Theorem 4 (Theorem 2 of [14]): Let P be a causal plant of $\mathcal{P}^{n\times m}$ and let F be a transfer matrix of $\mathcal{F}^{m'\times n'}$ ($m', n' \geq 0$). Assume that $\mathrm{Diag}(F, P)$ admits a doubly coprime factorization(cf. Figure 2). Let (N_1, D_1) and $(\widetilde{D}_1, \widetilde{N}_1)$ be a right- and a left-coprime factorizations of $\mathrm{Diag}(F, P)$ such that

$$\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1 = I_{m+n'}, \quad \widetilde{N}_1 Y_1 + \widetilde{D}_1 X_1 = I_{m'+n}, \quad (5)$$

where $Y_1, \widetilde{Y}_1 \in \mathcal{A}^{(m+n')\times(m'+n)}$, $X_1 \in \mathcal{A}^{(m'+n)\times(m'+n)}$, $\widetilde{X}_1 \in \mathcal{A}^{(m+n')\times(m+n')}$. Then, we have

$$\begin{aligned}\mathcal{H}(P) &= \{H \in \widehat{\mathcal{H}}(P) \mid H \text{ is nonsingular}\} \quad (6)\\ &= \Bigg\{ H := V \begin{bmatrix} I_{m'+n} - N_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & -N_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \\ D_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & D_1(\widetilde{X}_1 - R_1\widetilde{N}_1)\end{bmatrix} V^t \\ &\qquad \Bigg| \ R_1 \in \mathcal{A}^{(m+n')\times(m'+n)},\ H \text{ is nonsingular} \Bigg\} \quad (7)\end{aligned}$$

where

$$V = \begin{bmatrix} O_{n\times m'} & I_n & O_{n\times n'} & O_{n\times m} \\ O_{m\times m'} & O_{m\times n} & O_{m\times n'} & I_m \end{bmatrix}.$$

∎

Conversely, once we have $\mathcal{H}(P)$, it is also easy to obtain the set $\mathcal{S}(P)$ as

$$\mathcal{S}(P) = \left\{ H_{22}^{-1}H_{21} \in \mathcal{F}^{m\times n} \ \middle|\ \begin{array}{c|cc} & n & m \\ \hline n & H_{11} & H_{12} \\ m & H_{21} & H_{22} \end{array} \in \mathcal{H}(P) \right\} \quad (8)$$

(Lemma 2 of [15]). This implies that obtaining $\mathcal{S}(P)$ and obtaining $\mathcal{H}(P)$ are equivalent to each other.

IV. Example and Counterexamples

A. Example and Counterexamples of Theorem 1

Let us consider the model in [3, 8.1.73]. Then the set $\mathcal{A}$ of stable causal transfer functions is defined as $\mathcal{A} = C(S^2)$, that is, the set of all continuous real-valued functions on S^2, where S^2 denotes the unit sphere in $\mathbb{R}^3$.

Then let $A = [n_1\ n_2\ d] \in \mathcal{A}^{1\times 3}$, where we set $A(\cdot)$ equal to the unit outward normal vector at x, for each $x \in S^2$. Then we have $n_1^2 + d_1^2 + d^2 = 1$ but A cannot be complemented.

We now let P be

$$P = \begin{bmatrix} \frac{n_1}{d} & \frac{n_2}{d} \end{bmatrix}. \quad (9)$$

Then P have the left coprime factorization $(\widetilde{N}, \widetilde{D})$:

$$\widetilde{N}Y + \widetilde{D}X = I_1,$$

where $\widetilde{N} = [n_1 \quad n_2]$, $\widetilde{D} = [d]$, $Y[n_1 \quad n_2]^t$, and $X = [d]$. However P does not admit the right coprime factorization.

We consider P of (9). Recall that it admits the left coprime factorization but does not admit the right coprime factorization.

We now apply Theorem 1 to the plant P. Then new plant $P' = [P \quad 0]$ admits both side coprime factorization as follows:

$$\widetilde{N}'Y' + \widetilde{D}'X' = I_1, \quad \widetilde{Y}'N' + \widetilde{X}'D' = I_3, \quad (10)$$

where

$$\begin{aligned}\widetilde{N}' &= \begin{bmatrix} n_1 & n_2 & 0\end{bmatrix},\\ \widetilde{D}' &= \begin{bmatrix} d \end{bmatrix},\\ Y' &= \begin{bmatrix} n_1 & n_2 & 0\end{bmatrix}^t,\\ X' &= \begin{bmatrix} d \end{bmatrix},\\ N' &= \begin{bmatrix} -dn_1 & -dn_2 & n_1^2 + n_2^2 \end{bmatrix},\\ D' &= \begin{bmatrix} -1+n_1^2 & n_1n_2 & dn_1 \\ n_1n_2 & -1+n_2^2 & ,dn_2 \\ n_1 & n_2 & 2d - d^3 - dn_1^2 - dn_2^2\end{bmatrix},\\ \widetilde{Y}' &= \begin{bmatrix} -dn_1 & -dn_2 & n_1^2 + n_2^2 \end{bmatrix}^t,\\ \widetilde{X}' &= \begin{bmatrix} -1+n_1^2 & n_1n_2 & n_1 \\ n_1n_2 & -1+n_2^2 & n_2 \\ dn_1 & dn_2 & d\end{bmatrix}.\end{aligned}$$

We can apply Corollary 1 to obtain the parametrization of stabilizing controllers of P. By the straightforward calculation, the set of all stabilizing controllers of P based on Corollary 1 is as follows:

$$\frac{1}{d + dn_1r_1 + dn_2r_2 - r_3 + d^2r_3} \times \begin{bmatrix} n_1 - r_1 + n_1^2 r_1 + n_1n_2r_2 + dn_1r_3 \\ n_2 + n_1n_2r_1 - d^2r_2 - n_1^2r_2 + dn_2r_3 \end{bmatrix},$$

where $R = [r_1\ r_2\ r_3]^t \in \mathcal{A}^{3\times 1}$ is the parameter matrix.

From here, we consider another example, Anantharam's example. Anantharam [8] considered the case $\mathcal{A} = \mathbb{Z}[\sqrt{-5}] = \{u + v\sqrt{-5} \mid u, v \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers (This ring [16, pp.134–135] is isomorphic to $\mathbb{Z}[x]/(x^2+5)$ and is an integral domain but not a unique factorization domain. In fact, $6 \in \mathbb{Z}[\sqrt{-5}]$ has two factorizations, $2\cdot 3$ and $(1+\sqrt{-5})\cdot(1-\sqrt{-5})$). He showed that a single-input single-output plant $p = (1+\sqrt{-5})/2$ does not admit a coprime factorization but is stabilizable and $c = (1-\sqrt{-5})/(-2)$ is a stabilizing controller.

Let $P' = [p\ O^{1\times x}]$. This is adding a zero matrix from the right side of the plant. We will show that this cannot admit coprime factorization yet with any x.

Let $n = 1 + \sqrt{-5}$ and $d = 2$ with $p = n/d$. Then $F = [n\ O^{1\times x}\ d]$. The module generated by this F is not free. Thus P' cannot admit right-coprime factorization.

Next, we consider

$$G = \begin{bmatrix} n & & & O^{1\times n} \\ d & & & O^{1\times(n-1)} \\ 0 & d & & O^{1\times(n-2)} \\ 0 & 0 & d & O^{1\times(n-3)} \\ & & \cdots & \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The minors of this G consist of 0, $1+\sqrt{-5}$ and 2. Thus, the module generated by G is not free. Thus P' cannot admit left-coprime factorization.

Analogously, considering $P'' = [p\ O^{1\times x}]^t$, we have that P'' cannot admit right- and left-coprime factorizations.

By combining these results, we have the following:

Theorem 5: Let $\mathcal{A} = \mathbb{Z}[\sqrt{-5}]$ and $p = (1+\sqrt{-5})/2$. Then for any non-negative number n, the matrix

$$\begin{bmatrix} p & O^{1\times n} \\ O^{n\times 1} & O^{n\times n} \end{bmatrix}$$

cannot admit right- and left-coprime factorizations.

By these examples, we can say that if the plant has one side coprime factorization, then by finitely adding zeros, the added plant admits both right- and left-coprime factorizations. However, if the plant does not have coprime factorization, then even the plant added zeros does not have coprime factorization in general.

B. Example and Counterexamples of Theorem 4

V. EXAMPLE

Let us consider Example 3.4 of [11]. In this example, $\mathcal{A}$ is equal to $\mathbb{R}[d^2, d^3]$, where d denotes a unit delay operator. The impulse response of a transfer function being stable is finitely terminated and does not have the unit delay.

Here, we consider the following plant:

$$P = \begin{bmatrix} 2+d^2 & \dfrac{1-d^2}{1-d^3} \\ 1+d^2 & \dfrac{1-d^2}{1-d^3} \end{bmatrix}. \tag{11}$$

This plant P admits neither right- nor left-coprime factorization.

Now, we consider the following transfer function:

$$F = \frac{1+d^3}{1-d^2}. \tag{12}$$

Then, $\mathrm{Diag}(F,P)$ admits a right- and a left-coprime factorizations (N_1, D_1) and $(\widetilde{N}_1, \widetilde{D}_1)$ with $\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1 = I_3$ and $\widetilde{N}_1 Y_1 + \widetilde{D}_1 X_1 = I_3$, where

$$N_1 = \frac{1}{2}\begin{bmatrix} 0 & 1+d^3 & 1+d^2+d^4 \\ 4+2d^2 & 1-d^2 & -1-d^3 \\ 2+2d^2 & 1-d^2 & -1-d^3 \end{bmatrix},$$

$$D_1 = \frac{1}{2}\begin{bmatrix} 0 & 1-d^2 & 1-d^3 \\ 2 & 0 & 0 \\ 0 & 1-d^3 & -1-d^2-d^4 \end{bmatrix},$$

$$\widetilde{Y}_1 = \frac{1}{2}\begin{bmatrix} -1+d^2 & 3-d^2+d^3+d^5 & -2+d^2-2d^3-d^5 \\ 3-d^3 & -1-2d^2-2d^4-d^6 & 2+3d^2+3d^4+d^6 \\ 1-d^2 & 3+3d^2-d^3-d^5 & -6-3d^2+2d^3+d^5 \end{bmatrix},$$

$$\widetilde{D}_1 = \frac{1}{2}\begin{bmatrix} 1+d^3 & -2 & -1+d^2 \\ -1-d^2-d^4 & 0 & 1-d^3 \\ 1-d^3 & 0 & 1-d^2 \end{bmatrix},$$

$$\widetilde{N}_1 = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ 1+d^3 & 0 & -1+d^2 \\ 1+d^2+d^4 & 0 & 1+d^3 \end{bmatrix},$$

$$\widetilde{D}_1 = \frac{1}{2}\begin{bmatrix} 0 & 2 & -2 \\ 1-d^2 & 1+d^2-d^3-d^5 & -2-d^2+2d^3+d^5 \\ 1-d^3 & -1-2d^2-2d^4-d^6 & 2+3d^2+3d^4+d^6 \end{bmatrix},$$

$$Y_1 = \frac{1}{2}\begin{bmatrix} 0 & 3-d^3 & 1-d^2 \\ 4 & -2 & 0 \\ 0 & -1-d^2-d^4 & 3-d^3 \end{bmatrix},$$

$$X_1 = \frac{1}{2}\begin{bmatrix} 0 & -1-d^2-d^4 & 1-d^3 \\ -4-2d^2 & 3+2d^2+d^3 & -1+d^2 \\ -2-2d^2 & 1+2d^2+d^3 & -1+d^2 \end{bmatrix}.$$

Finally, since $m' = n' = 2$ and $m = n = 3$, the number of parameters is 25. On the other hand, the method of [15] (also [17]) requires 36 parameters.

The set $\mathcal{H}(P)$ of P can be given as in (7). As an example, letting R be as follows:

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 7-4d^2 & 0 \\ -3+d^4 & 0 & 0 \end{bmatrix},$$

we obtain a stabilizing controller C of P, based on (8), as follows:

$$C = \begin{bmatrix} -4 & 0 \\ d_{21} & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} 8 & -8 \\ c_{21} & c_{22} \end{bmatrix},$$

where

$$\begin{aligned} d_{21} &= -6-6d^2-4d^4+2d^6+2d^8, \\ d_{22} &= 7-11d^2-9d^3+4d^4+11d^5+2d^6-4d^7, \\ c_{21} &= 9+d^2-12d^3-8d^4-2d^5+d^6+12d^7 \\ &\qquad +d^8+2d^9-4d^{10}, \\ c_{22} &= -12+7d^2+24d^3+12d^4-8d^5-8d^6 \\ &\qquad -14d^7+3d^8-2d^9+4d^{10}. \end{aligned}$$

As we shown that by considering the parallel plants, we can obtain the extended plants that admits doubly coprime factorization. However, we need to say that Theorem 4 assumes that $\mathrm{Diag}(F,P)$ admits a doubly coprime factorization. This means that F is, in fact, stabilizable. If the original plant P is unstable, there never exist F such that $\mathrm{Diag}(F,P)$ admits a doubly coprime factorization. This is stated as following theorem:

Theorem 6: Let P be a causal plant of $\mathcal{P}^{n\times m}$. For any transfer matrix F of $\mathcal{F}^{m'\times n'}$ $(m', n' \geq 0)$, $\mathrm{Diag}(F,P)$ does not admit doubly coprime factorization if and only if P is unstabilizable.

VI. CONCLUSION AND FURTHER WORKS

In this paper, we have given some examples and counterexamples of previous results. We have also shown that in the case where the plant do not admit coprime factorization, there exists a case where the plant with any additional zeros cannot admit coprime factorization.

We will investigate the explicit criteria for the stabilizability, the stability, and the minimal number of parameters for the parametrization of stabilizing controllers.

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