On a Special Class of Solutions of the Dirac Equation for Massive Particles With Electric Potential

A. Bucio R. *IAENG, Member*, V. Martinez-Garza-Garcia *IAENG, Member*, R. Castillo-Perez.

***Abstract*—We study a special class of solutions for the Dirac equation considering electric potentials depending upon one spacial variable, based upon the Taylor series in formal powers, in order to examine the probability functions that describe the dynamics of the quantum particles within a circular domain, emphasizing a particular behavior us such probability functions upcoming for all classes of electrical potential considered in this work.**

***Index Terms*—Biquaternions, Dirac equation, Vekua equation.**

I. Introduction

According to the previously published the works [2] and [8], the study of the massive Dirac equation with electric potentials is significant for different branches Sciences, as it were Theoretical and Experimental Physics, Engineering and Applied Mathematics. Thus this work compiles the results presented in [2] and [8], where the massive Dirac equation was analyzed with an arbitrary electric potential depending upon one spacial variable, and where it was first appointed a special behavior of the probability functions upcoming from the solutions of the Dirac equation.

This class of solutions is obtained once the Dirac equation is written down in biquaternionic form, and decoupled into a pair of partial differential equations: one that can be immediately solved, and another that it is indeed a biquaternionic Vekua equation.

The main objective of this work is to enhance a special behavior of the probability functions obtained from the corresponding solutions for the massive Dirac equation when, in general, this decoupling technique is employed.

We would like to enhance, as we did in [2], that the exact representations of these probability functions are in general, not possible to obtain, thus we review the numerical method that will allow the construction of the probability functions.

II. Preliminaries: Elements of biquaternionic analysis

As posed in [6], we will consider the set of biquaternionic functions $\mathbb{H}(\mathbb{C})$, where the elements $q \in \mathbb{H}(\mathbb{C})$ have the form:

$$q = \sum_{n=0}^{3} q_n \mathbf{e}_n, \tag{1}$$

A. Bucio R. is with the SEPI-UPIITA Instituto Politecnico Nacional, Av. IPN 2580, C.P. 07340, Mexico, ari.bucio@gmail.com

V. Martinez-Garza-Garcia is with the Engineering Faculty of Universidad La Salle, B. Franklin 47, C.P. 06140, Mexico, and with the Biomedical Engineering Department of the Hospital General de Mexico *Dr. Eduardo Liceaga*, Dr. Balmis 148, C.P. 06726, vanmtzg@hotmail.com

R. Castillo-Perez is with SEPI, ESIME Zacatenco, Instituto Politecnico Nacional, Av. IPN S/N, C.P. 07738, Mexico, rcastillo@ipn.mx

being q_n complex-valued functions: $q_n = \text{Re}\ q_n + i\text{Im}\ q_n$, i is the standard imaginary unit $i^2 = -1$, $e_0 = 1$, and $\{\mathbf{e}_n\}_{n=1}^{3}$ are the quaternionic units:

$$\begin{aligned} \mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3, \\ \mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1, \\ \mathbf{e}_3\mathbf{e}_1 &= -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2, \\ \mathbf{e}_1^2 &= \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1. \end{aligned} \tag{2}$$

The imaginary unit i, by definition, commutes with the quaternionic units: $i\mathbf{e}_n = \mathbf{e}_n i$. An auxiliary representation of the biquaternions $q \in \mathbb{H}(\mathbb{C})$ that will be used in these pages is:

$$q = \mathbf{Sc}\ q + \mathbf{Vec}\ q, \tag{3}$$

where

$$\mathbf{Sc}\ q = q_0,$$

and where

$$\mathbf{Vec}\ q = \sum_{n=1}^{3} q_n \mathbf{e}_n.$$

The relations (2) immediately indicate that the multiplication between two biquaternions $p, q \in \mathbb{H}(\mathbb{C})$ is not commutative in general. Then we introduce a notation for the multiplication by the right-hand side of q by p:

$$M^p q = q \cdot p.$$

Aside, we consider the partial differential operator D, the Moisil-Theodoresco operator, introduced in the form:

$$D = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3.$$

Here we have employed the notation $\partial_n = \frac{\partial}{\partial x_n}$, for $n = 1, 2, 3$. This operator is defined in the space of at least once-derivable biquaternions with respect to the spacial variables $x_1,\ x_2$, and x_3.

A. A special class of biquaternionic partial differential equations.

As it was appointed in [3] and [8], the classical Dirac equation for massive particles under the influence of an arbitrary electric potential, depending upon one spatial variable, is closely related with a partial differential biquaternionic equation of the form:

$$\left(D - M^{g\mathbf{e}_1 + m\mathbf{e}_2}\right) f = 0, \tag{4}$$

where m is an purely scalar real constant $m = \mathbf{Sc}\ \mathbf{Re}\ m$, and g is a purely scalar imaginary function depending upon

(Advance online publication: 18 May 2016)

the variable x_1: $g = i\mathbf{Sc}\ \mathbf{Im}\ g(x_1)$. Notice $f \in \mathbb{H}(\mathbb{C})$ is a complete biquaternionic function.

Several works have been dedicated to approximate exact solutions for this equation, utilizing elements of the modern pseudoanalytic functions theory (see [5], [7] and more recently [3] and [8]). Here we will deploy the in brief form the techniques presented in the cited works. Thus, as proposed first in [7], let us consider:

$$f = \alpha Q, \tag{5}$$

where α is a purely scalar function $\alpha = \mathbf{Sc}\ \alpha$, and $Q \in \mathbb{H}(\mathbb{C})$ is a biquaternion. Expanding the differential equation (4) we will have:

$$D\alpha \cdot Q + \alpha DQ - \alpha Q g \mathbf{e}_1 - \alpha Q m \mathbf{e}_2 = 0. \tag{6}$$

Hereafter the selection of the decoupling of the last equation will be fundamental for obtaining solutions of the Dirac equation. Yet we will appoint that at the very final step, when the probability distributions are exhibited after using any variation of such decoupling technique, the emerging probability functions seem to be, in some way, independent of the electrical potential employed in the calculations, which is indeed an unexpected result.

Let us start with the decoupling employed in [2], assuming that the following relations hold:

$$DQ - Qg\mathbf{e}_1 = 0, \tag{7}$$

$$D\alpha \cdot Q - \alpha Q m \mathbf{e}_2 = 0. \tag{8}$$

If we also assume that Q is not a *zero divisor* (see [6] for a detailed explanation), this is, that there exist a $Q^{-1} \in \mathbb{H}(\mathbb{C})$ such that: $Q \cdot Q^{-1} = 1$, and that:

$$Q = q_1\mathbf{e}_1 + q_3\mathbf{e}_3, \tag{9}$$

the equations (7) and (8) will reach a pair of decoupled partial differential equations:

$$DQ - Qg\mathbf{e}_1 = 0, \tag{10}$$

and

$$\partial_1\alpha + g\alpha = 0. \tag{11}$$

It is evident that:

$$\alpha = Ke^{-mx_2}, \tag{12}$$

where K is a real constant, is the general solution of (11), whereas the equation (10) is fully equivalent to a special kind of biquaternionic Vekua equation [5]:

$$\partial_{\overline{z}}W - \frac{\partial_{\overline{z}}p}{p}\overline{W} = 0, \tag{13}$$

where

$$\partial_{\overline{z}} = \partial_1 + \mathbf{e}_1\partial_3,$$

$$W = q_1 - q_3\mathbf{e}_1, \quad \overline{W} = q_1 + q_3\mathbf{e}_1 \tag{14}$$

and

$$p = e^{\int g dx_1}. \tag{15}$$

As appointed in [2], the extension of the pseudoanalytic function theory [1] that allows the construction of the general solution for the biquaternionic Vekua equation (13) [9], can be found in the work of V. Kravchenko [5], still the main contribution of [2] was the preliminary analysis of the probability distributions rising up from the solutions of the Dirac equation, since many of them are only accessible by virtue of the numerical analysis.

We now browse the propositions presented by in [5], adapted by the authors on behalf of the results that will allow the construction of the general solution of (13) in terms of Taylor series in formal powers [1].

The general solution of (13) can be written down as:

$$W = \sum_{n=0}^{\infty} Z_0^{(n)}(a_n, z_0; z) \tag{16}$$

where a_n are biquaternionic constants:

$$a_n = a'_n + a''_n\mathbf{e}_1,$$

being

$$a'_n = \mathbf{Re}\ a'_n + i\,\mathbf{Im}\ a'_n;\ a''_n = \mathbf{Re}\ a''_n + i\,\mathbf{Im}\ a''_n.$$

Also, z is a purely real quaternionic variable:

$$z = x_1 + \mathbf{e}_1 x_3;$$

and z_0 is a fixed point in the quaternionic plane. Specifically, as it was done in [2], we will consider z_0 at the origin $z_0 = 0$.

Following [1] and [5], each formal power $Z_0^{(n)}(a_n, z_0; z)$ is a solution of (13), and possesses the property:

$$Z_0^{(n)}(a_n, z_0; z) = a' Z_0^{(n)}(1, z_0; z) + a'' Z_0^{(n)}(\mathbf{e}_1, z_0; z).$$

This implies that we can center our attention into the construction of some elements of the set:

$$\left\{ Z_0^{(n)}(1, 0; z),\ Z_0^{(n)}(\mathbf{e}_1, 0; z) \right\}_{n=0}^{\infty} \tag{17}$$

for everyone of them can be related with a solution of the Dirac equation in classical form, and in consequence, to provide a probability distribution for a specific massive quantum particle governed by such equation.

The complete propositions to approach the formal powers (17) can be found in [5]. Here for the sake of briefness we present only abbreviate principles, and without providing the proofs. We will circumscribe our explanations to the construction of the elements:

$$\left\{ Z_0^{(n)}(1, 0; z) \right\}_{n=0}^{\infty}$$

because it is possible to assure that the procedures are identical to those corresponding to the rest of the elements of (17).

First, let us introduce the functions:

$$F_0 = p^{-1},\ G_0 = \mathbf{e}_1 p,\ F_1 = p,\ G_1 = \mathbf{e}_1 p^{-1}, \tag{18}$$

where p possesses the form indicated in (15). These must be grouped in two pairs: (F_0, G_0) and (F_1, G_1), and they will be called *generating pairs* [1][5], because they fulfill the condition:

$$\mathbf{Vec}\left(\overline{F}G\right) \neq 0,$$

where $\overline{F} = \mathbf{Sc}\ F - \mathbf{Vec}\ F$. We can now introduce complementary functions:

$$F_0^* = -\mathbf{e}_1 p^{-1},\ G_0^* = p,\ F_1^* = -\mathbf{e}_1 p,\ G_0^* = p^{-1}, \tag{19}$$

where (F_0^*, G_0^*) will be called the *adjoint* pair of (F_0, G_0), and (F_1^*, G_1^*) the adjoint of (F_1, G_1).

Moreover, the pairs (F_0, G_0) and (F_1, G_1) are *embedded* into a *periodic generating sequence* (see e.g. [5]), with period $c = 2$. Taking into account all previous material, the formal powers can be constructed as:

$$Z_0^{(0)}(1,0;z) = \lambda F_0,$$

where λ is a constant such that $\lambda F_0(0) = 1$. The subsequent formal powers will be determined by the formulas:

$$Z_{\mathbf{j}}^{(n+1)}(1,0;z) = n\ F_{\mathbf{j}}\ \mathbf{Sc}\int_\Gamma G_{\mathbf{j}}^* Z_{\mathbf{k}}^{(n)}(1,0;z)\,dz+$$
$$+n\ G_{\mathbf{j}}\ \mathbf{Sc}\int_\Gamma F_{\mathbf{j}}^* Z_{\mathbf{k}}^{(n)}(1,0;z)\,dz, \qquad (20)$$

where Γ is a rectifiable curve going from 0 till z, and where $\mathbf{k} = 0$ if $\mathbf{j} = 1$, as well as $\mathbf{k} = 1$ if $\mathbf{j} = 0$.

III. The Dirac equation for massive particles under the influence of an electric potential

A. Results of the first decoupling.

Consider the Dirac equation in classical form:

$$\left[\gamma_0\partial_t - \sum_{n=1}^{3}\gamma_n\partial_n + im + \gamma_0 u(x_1)\right]\Phi(t,x) = 0, \qquad (21)$$

where m is the mass of the particle, u is an arbitrary electric potential depending on x_1, $\partial_t = \frac{\partial}{\partial t}$, t is the time variable, and γ_n, $n = 0,1,2,3$; are the Pauli-Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us consider the time-harmonic representation of Φ: $\Phi(t,x) = e^{i\omega t}\varphi(x)$, where ω denotes the energy of the particle. Then the Dirac equation (21) will turn into:

$$\left[i\omega\gamma_0 - \sum_{n=1}^{3}\gamma_n\partial_m + im + \gamma_0 u(x_1)\right]\varphi(x) = 0, \qquad (22)$$

Also, we will use the pair of matrix operators $\mathbf{A}$ and $\mathbf{A}^{-1}$, presented in [6]:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix},$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix}, \qquad (23)$$

because by applying them according to the expression:

$$\mathbf{A}\gamma_1\gamma_2\gamma_3\left[\gamma_0\partial_t - \sum_{n=1}^{3}\gamma_n\partial_n + im + \gamma_0 u(x_1)\right]\mathbf{A}^{-1},$$

we will obtain a biquaternionic Dirac equation of the form (4):

$$\left(D - M^{g\mathbf{e}_1 + m\mathbf{e}_2}\right) f = 0.$$

Here g contains the particle energy as well electric potential.

$$g = iu(x_1) + i\omega,$$

and

$$f = \mathbf{A}\varphi(x).$$

Notice the use of $\mathbf{A}$ and $\mathbf{A}^{-1}$ provokes the reflection of the x_3-axis [6]. This is:

$$\mathbf{A}^{-1}\varphi(x_1,x_2,x_3) \to f(x_1,x_2,-x_3),$$

as well

$$\mathbf{A}f(x_1,x_2,x_3) \to \varphi(x_1,x_2,-x_3).$$

B. Numerical approaching of the solutions for the quaternionic Dirac equation.

To analyze the class of solutions obtained according the Section II, considering that u is an arbitrary function, it is mandatory the use of numerical analysis, since the parametric integrals of the formulas (20) can not be solved, in general, in exact form.

Because of that we will employ a numerical approximation to evaluate these integrals, and in consequence, to analyze the probability distributions obtained from the solutions of the Dirac equation.

We will focus our attention into four kinds of electric potentials, as it was done in [2]:

$$u = B,\ B \in \mathbb{R}, \qquad (24)$$

$$u = B\ x_1,\ B \in \mathbb{R}, \qquad (25)$$

$$u = B\ e^{Cx_1},\ B, C \in \mathbb{R}, \qquad (26)$$

$$u = B\cos(Cx_1),\ B, C \in \mathbb{R}, \qquad (27)$$

Specifically, we will approach $N = 10$ formal powers $Z_0^{(n)}(1,0;z)$ for every case, assuming that the curves Γ of (20) are straight lines (radii) with length $R = 1$, converging at the (x_1,x_3)-plane origin, and whose slopes are equally distributed along the angle interval $[0, 2\pi)$. This means we will approach the formal powers within the unit circle.

Also, we will consider $R = 1000$ radii, each one sectioned into 1000 equal segments, producing $P = 1001$ points over which the formal powers will be approached. To approach a set of $P = 1001$ values for each radius, $r = 0,1,...,1000$; corresponding to the numerical formal powers $Z_0^{(n)}(x_1[r], x_3[r])$; we use a variation of the computational method presented in [3], whose recursive discrete formulas can be summarized as follows:

$$Z_0^{(n+1)}(x_1[r+1], x_3[r+1]) =$$
$$= \frac{1}{2}F_0(x_1[r], x_3[r])\cdot$$
$$\cdot\mathbf{Sc}\ G_0^*(x_1[r+1], x_3[r+1])\cdot$$

$$\cdot Z_1^{(n)}\left(x_1\left[r+1\right], x_3\left[r+1\right]\right) dz\left[r\right]+$$
$$+\frac{1}{2}F_0\left(x_1\left[r\right], x_3\left[r\right]\right)\cdot$$
$$\cdot\mathbf{Sc}\ G_0^*\left(x_1\left[r\right], x_3\left[r\right]\right) Z_1^{(n)}\left(x_1\left[r\right], x_3\left[r\right]\right) dz\left[r\right]+$$
$$+\frac{1}{2}G_0\left(x_1\left[r\right], x_3\left[r\right]\right)\cdot$$
$$\cdot\mathbf{Sc}\ F_0^*\left(x_1\left[r+1\right], x_3\left[r+1\right]\right)\cdot$$
$$\cdot Z_1^{(n)}\left(x_1\left[r+1\right], x_3\left[r+1\right]\right) dz\left[r\right]+$$
$$+\frac{1}{2}G_0\left(x_1\left[r\right], x_3\left[r\right]\right)\cdot$$
$$\cdot\mathbf{Sc}\ F_0^*\left(x_1\left[r\right], x_3\left[r\right]\right) Z_1^{(n)}\left(x_1\left[r\right], x_3\left[r\right]\right) dz\left[r\right]+$$
$$+Z_0^{(n+1)}\left(x_1\left[r\right], x_3\left[r\right]\right);$$

where we have that:

$$x_1\left[r\right]=\frac{r}{P-1}\cos\theta\left[l\right],$$
$$x_3\left[r\right]=\frac{r}{P-1}\sin\theta\left[l\right],$$
$$dz\left[r\right]=x_1\left[r+1\right]+x_3\left[r+1\right]\mathbf{e}_1-x_1\left[r\right]-x_3\left[r\right]\mathbf{e}_1,$$

being

$$r=0,1,...,P-1;$$

whereas

$$\theta\left[l\right]=\frac{l\cdot 2\pi}{N};\ l=0,1,...,N-1.$$

When the complete numerical calculations are finished, we will posses $N\times P=100100$ values for every numerically approached formal power $Z_0^{(n)}(1,0;z)$, $n=0,1,...,10$. Thus, considering now the relation (14):

$$W=q_1-q_3\mathbf{e}_1,$$

where W is a solution of the biquaternionic Vekua equation (13), we obtain solutions for the biquaternionic Dirac equation (4) according to (5):

$$f=\alpha Q=Ke^{-mx_2}\left(q_1\mathbf{e}_1+q_3\mathbf{e}_3\right),$$

where K is an arbitrary real constant. Because every single formal power is solution of the Vekua equation (13), these solutions will have the form:

$$f=Ke^{-mx_2}\left(q_1\mathbf{e}_1+q_3\mathbf{e}_3\right)=$$
$$=Ke^{-mx_2}\left(\mathbf{e}_1\mathbf{Sc}\ Z_0^{(n)}(1,0;z)-\mathbf{e}_3\mathbf{Vec}\ Z_0^{(n)}(1,0;z)\right).$$

Thus we can apply the matrix transformation $\mathbf{A}^{-1}$ described in (23) to the solutions f, to obtain solutions for the time-harmonic Dirac equation (22):

$$\varphi=\begin{pmatrix}-\mathbf{Sc}\ Z_0^{(n)}(1,0;x_1,-x_3)\\ \mathbf{Vec}\ Z_0^{(n)}(1,0;x_1,-x_3)\\ \mathbf{Vec}\ Z_0^{(n)}(1,0;x_1,-x_3)\\ \mathbf{Sc}\ Z_0^{(n)}(1,0;x_1,-x_3)\end{pmatrix}Ke^{-mx_2}. \quad (28)$$

Now an standard procedure will allow us to use the solutions of the Dirac equation to build the probabilistic function describing the mechanics of the quantum particle within a circular domain (see *e.g.* [4]).

Once more, as mentioned in [2], the methods previously posed can be directly referred to a cylindrical domain, but it should be more convenient to focus our attention into the behavior of the functions in a fixed section of the cylinder, this is, an unit circle. We now display the illustrations of the probabilistic functions $\mathcal{P}$ provided by the relations:

$$\mathcal{P}=\|\mathbf{Sc}\ Z_0^{(n)}(1,0;x_1,-x_3)\|^2+$$
$$+\|\mathbf{Vec}\ Z_0^{(n)}(1,0;x_1,-x_3)\|^2,$$

where:

$$\|\mathbf{Sc}\ Z_0^{(n)}(1,0;x_1,-x_3)\|^2=$$
$$\left(\mathbf{Re}\ \mathbf{Sc}\ Z_0^{(n)}(1,0;x_1,-x_3)\right)^2+$$
$$+\left(\mathbf{Im}\ \mathbf{Sc}\ Z_0^{(n)}(1,0;x_1,-x_3)\right)^2,$$

as well as

$$\|\mathbf{Vec}\ Z_0^{(n)}(1,0;x_1,-x_3)\|^2=$$
$$\left(\mathbf{Re}\ \mathbf{Vec}\ Z_0^{(n)}(1,0;x_1,-x_3)\right)^2+$$
$$+\left(\mathbf{Im}\ \mathbf{Vec}\ Z_0^{(n)}(1,0;x_1,-x_3)\right)^2.$$

Please notice that on behalf of the space we did not normalize the probability functions $\mathcal{P}$, limiting our plots to a scale adjustment so a better comparison between the different cases is possible. All presented plots correspond to the case when $K=1$ and $x_2=0$.

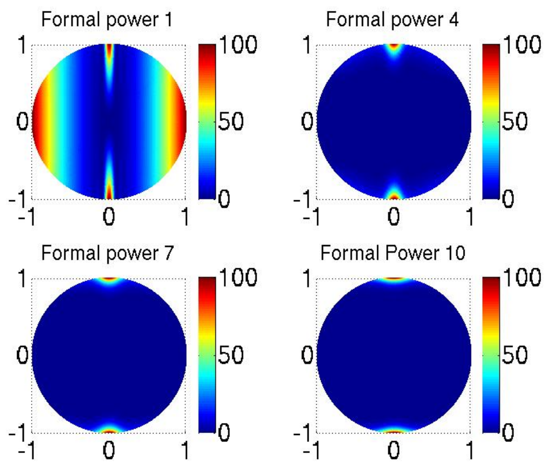

Fig. 1: Probability functions $\mathcal{P}$ generated by the formal powers Z_0^1, Z_0^4, Z_0^7 and Z_0^{10} when $u=5\pi$ and $w=5\pi$.

Observing these figures, it shows up that only the first formal power of the all cases (24), (25), (26) and (27) provides a more intricate dynamics of the probability functions $\mathcal{P}$. The upper powers ($n>1$) already show the clustering of the highest values of $\mathcal{P}$ near the boundary and over the x_3-axis.

It is indeed an unexpected behavior, given the very different classes of electric potentials that provoke them.

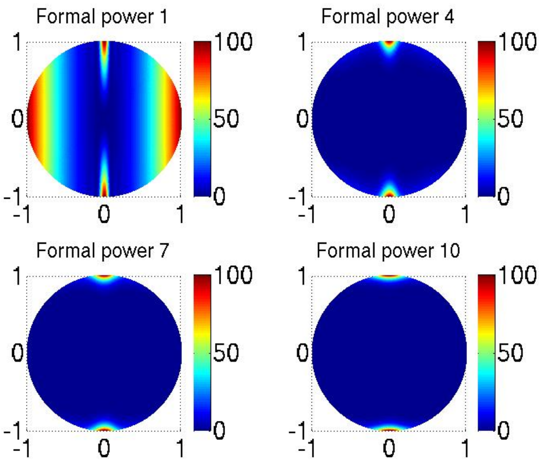

Fig. 2: Probability functions $\mathcal{P}$ generated by the formal powers Z_0^1, Z_0^4, Z_0^7 and Z_0^{10} when $u = 20\ x_1$ and $w = 10\pi$.

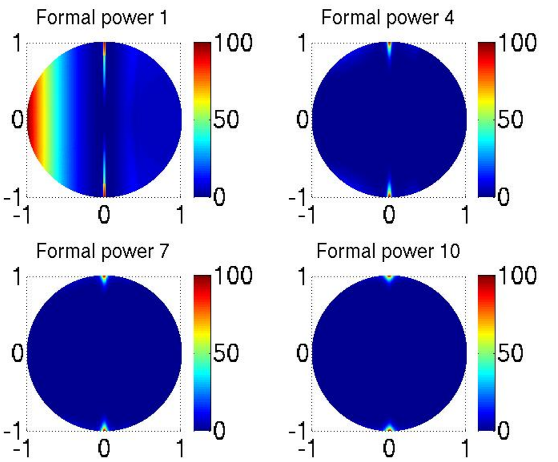

Fig. 3: Probability functions $\mathcal{P}$ generated by the formal powers Z_0^1, Z_0^4, Z_0^7 and Z_0^{10} when $u = 10\ e^{10\ x_1}$ and $w = 10\pi$.

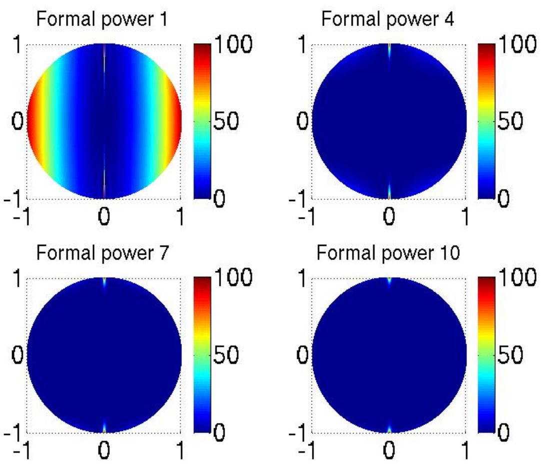

Fig. 4: Probability functions $\mathcal{P}$ generated by the formal powers Z_0^1, Z_0^4, Z_0^7 and Z_0^{10} when $u = -100\cos(10\pi\ x_1)$ and $w = 10\pi$.

C. Results of a second decoupling.

Let us consider once more the equation (6)

$$D\alpha \cdot Q + \alpha DQ - \alpha Qg\mathbf{e}_1 - \alpha Qm\mathbf{e}_2 = 0,$$

employing the same decoupling idea:

$$DQ - Qg\mathbf{e}_1 = 0, \tag{29}$$

$$D\alpha \cdot Q - \alpha Qm\mathbf{e}_2 = 0, \tag{30}$$

only this time, as it was posed in [8], beside considering again Q as a non-zero divisor, we suppose it to possess the form:

$$Q = q_0 + q_2\mathbf{e}_2.$$

The pair of corresponding equations will be:

$$\partial_2\alpha - \alpha m = 0;$$

whose general solution is:

$$\alpha = Ke^{mx_2};$$

being K a purely real constant; and (29) can be written as:

$$\partial_{\overline{z}}W - \frac{\partial_{\overline{z}}p}{p}\overline{W} = 0, \tag{31}$$

where

$$\partial_{\overline{z}} = \partial_1 + \mathbf{e}_3\partial_3,$$

$$W = q_0 + q_2\mathbf{e}_3, \quad \overline{W} = q_0 - q_2\mathbf{e}_3$$

and p possesses, again, the form (15):

$$p = e^{\int g dx_1}.$$

Thus once more we have arrived to a biquaternionic Vekua equation, only this time the quaternionic variable is $z = x_1 + x_3\mathbf{e}_3$.

Therefor the solutions for this case can be constructed employing the very same procedure indicated in the previous paragraphs. And once more, despite the proper changes on the coordinates, when the construction of the probability functions $\mathcal{P}$ is performed, considering all the same four cases of electric potentials, the patterns displayed on the resulting illustrations will exhibit a high similarity with those displayed in the Figures 1, 2, 3 and 4.

This is, the unexpected behavior of the probability functions appears again when considering solutions upcoming from formal powers with formal exponent $n > 1$.

On behalf of the space, and given the strong similarity indicated before, we will not include the set of sixteen graphics reached by the Vekua equation (31).

D. Results of a third decoupling.

The last case we will consider comes from the work [7], which contains the original idea of the decoupling employed in these pages. The work itself does not clarify that the biquaternionic Vekua equation can be obtained immediately from the main partial differential systems studied in the work.

But id does contain a very interesting result that shows the total independence of the probability functions $\mathcal{P}$ from the electric potential, in the sense that will be explained now.

Consider once more the equation (6)

$$D\alpha \cdot Q + \alpha DQ - \alpha Qg\mathbf{e}_1 - \alpha Qm\mathbf{e}_2 = 0,$$

But this time assume the following decoupling:

$$D\alpha \cdot Q - \alpha Qg\mathbf{e}_1 = 0,$$

$$\alpha DQ - \alpha Qm\mathbf{e}_2 = 0,$$

and also that $Q = q_0 + q_1\mathbf{e}_1$ is not a zero divisor. We will immediately obtain that the first of these two equations is equivalent to:

$$\partial_1\alpha - \alpha g = 0,$$

whose general solution is:

$$\alpha = Ke^{\int g dx_1},$$

where K is a constant. This implies that the second equation can be written down as:

$$\partial_{\overline{z}}W - \frac{\partial_{\overline{z}}p}{p}\overline{W} = 0, \tag{32}$$

where

$$\partial_{\overline{z}} = \partial_2 + \mathbf{e}_3\partial_3,$$

$$W = q_0 - q_1\mathbf{e}_3, \quad \overline{W} = q_0 + q_1\mathbf{e}_3$$

and p is:

$$p = e^{mx_2}.$$

This biquaternionic Vekua equation is defined for the quaternionic variable $z = x_2 + x_3\mathbf{e}_3$.

Of course, this equation can be also solved by the methods explained before. Yet the important fact here is that the Vekua equation does not contain the purely imaginary function $g = iu + i\omega$. Therefor, the solutions of this Vekua equation will not depend on the energy of the particle ω neither on the electric potential u.

Moreover, when the solutions of the time-harmonic Dirac equation are written according to the expression (28), adequately adapted for the formal powers solutions of the equation (32), the exponential function:

$$\alpha = Ke^{\int g dx_1},$$

can be factorized at each of the four components of the vector solution (28). This, at the very moment of approaching the probability function $\mathcal{P}$, will provoke that such exponential function will not affect the function $\mathcal{P}$, since

$$\|e^{\int gx_1}\|^2 = \|e^{i\int u(x_1)x_1 + \omega dx_1}\|^2 = 1.$$

Summarizing, this case presents a very particular class of solutions of the time-harmonic Dirac equation (21), since every solution constructed according to the decoupling shown in this last section, will provoke a probability function $\mathcal{P}$ completely independent from the electric potential as well from the energy of the particle.

IV. Discussion

This work compiles the results previously published in [2], [7] (slightly adapted to be exposed in this work), and in [8], but focuses the attention into a particular behavior of the probability functions $\mathcal{P}$ constructed employing a special class of solutions for the time-harmonic Dirac equation (21).

The special behavior is clearly illustrated by the results presented along the lats section of the work: A special class of solutions of the Dirac equation for massive particles, under the influence of an arbitrary electric potential depending upon only one spatial variable, that surprisingly provoke probability functions fully independent of the electric potential.

This fact seems to prevail, even when the mathematical techniques are adapted to ensure that the solutions of the biquaternionic Vekua equations, strongly related with the construction of this class of solutions, take into account the electric potentials.

The unexpected behavior is heuristically detected when numerically computing the probability functions, employing the solutions of the cited Vekua equations, since only one probability function for each different example of electric potential, evidences the influence the potential, whereas the rest barely display any influence, or its different nature (see the Figures 1, 2, 3 and 4).

The authors do not consider to possess an adequate explanation for such behavior, therefor we propose a further discussion before venturing any conclusion, since this results need to be examined very carefully according to the modern principles of Relativistic Quantum Mechanics. Yet, such analysis is far beyond the scope of this paper.

Acknowledgments 1: **A. Bucio R.** would like to thank the support of the project SIP 20150203, IPN, Mexico. **R. Castillo P.** would like to thank the support of the SIBE and EDI programs of the IPN as well as that of the project SIP20150203.

References

[1] L. Bers (1953), *Theory of Pseudoanalytic Functions*,IMM, New York University.

[2] A. Bucio R., V. Martinez-Garza-Garcia, M. P. Ramirez T. and R. Castillo-Perez (2015), *Analysis of the Numerical Solutions for the Massive Dirac Equation with Electric Potential Employing Biquaternionic Functions*, Proceedings of the World Congress on Engineering and Computer Science 2015 Vol I, WCECS 2015, San Francisco, USA, pp. 270-275.

[3] A. Bucio R., R. Castillo-Perez, M.P. Ramirez T. (2011), *On the Numerical Construction of Formal Powers and their Application to the Electrical Impedance Equation*, 8th International Conference on Electrical Engineering, Computing Science and Automatic Control, IEEE Catalog Number: CFP11827-ART, ISBN:978-1-4577-1013-1, pp. 769-774.

[4] W. Greiner (2013), *Relativistic Quantum Mechanics. Wave Equations*, Springer, Germany.

[5] V. V. Kravchenko (2009), *Applied Pseudoanalytic Function Theory*, Series: Frontiers in Mathematics, ISBN: 978-3-0346-0003-3.

[6] V. Kravchenko (2003), *Applied Quaternionic Analysis*, Heldermann.

[7] V. Kravchenko, M. P. Ramirez T. (2000), *New Exact Solutions for the Massive Dirac Equation with Electric or Scalar Potential*, Mathematical Methods in the Applied Sciences, Vol. 23, 769-776.

[8] M. P. Ramirez T., V. Martinez-Garza-Garcia (2015), *Solutions for the massive Dirac equation with electric potential, employing a biquaternionic Vekua equation*, Transactions on Engineering Technologies: World Congress on Engineering and Computer Science 2014, pp. 117-127.

[9] I. N. Vekua (1962), *Generalized Analytic Functions,* International Series of Monographs on Pure and Applied Mathematics, Pergamon Press.