Critical Exponent for Nonlinear Parabolic Equation and System with Spatio-temporal Fractional Derivatives

Brahim Tellab and Kamel Haouam

***Abstract*—In this paper we discuss the existence of a critical exponent of Fujita type for the Cauchy problems with a nonlinear fractional equation**

$$\begin{cases} D_{0/t}^{\alpha}u+(-\Delta)^{\beta/2}(u^m)=|u|^p+f(x,t),(x,t)\in Q_T, \\ u(x,0)=u_0(x)\geq 0, x\in\mathbb{R}^N, \end{cases} \quad (1)$$

where $\alpha\in]0,1[,\beta\in]0,2[$ **and the Cauchy problem with a nonlinear parabolic fractional system**

$$\begin{cases} D_{0/t}^{\alpha}u+(-\Delta)^{\beta/2}(u^m)=|v|^p+f(x,t),(x,t)\in Q_T, \\ D_{0/t}^{\delta}v+(-\Delta)^{\gamma/2}(v^m)=|u|^q+g(x,t),(x,t)\in Q_T, \\ u(x,0)=u_0(x)\geq 0, x\in\mathbb{R}^N, \\ v(x,0)=v_0(x)\geq 0, x\in\mathbb{R}^N, \end{cases} \quad (2)$$

with $0\prec\alpha,\delta\prec 1$ **and** $0\prec\gamma,\beta\prec 2$.

Index Terms—Fractional derivatives, test-function, critical exponent.

I. Introduction

FRACTIONAL calculus is a mathematical analysis field where notions of integrals and derivatives of arbitrary order are used or applied. Establishment of sufficient and necessary conditions for local and global existence of solutions of fractional derivatives equations is a subject of topicality and it is the interest of many authors, so it was found so much in literature. See for example [2], [3], [4], [5], [8], [9], [12], [13], [15] and [20].

In this paper we follow idea treated in reference cited above by adding a function $f(x,t)$ to the term which wefind habitually in the second term in a standard form as $|u|^p$ or $h(x,t)|u|^p$ and discuss the influence (the impact) of the last added function on conditions leading to a nonlinear evolution parabolic equations and systems with spatio-temporal fractional derivatives.

In fact, in his pioneering paper [3], Fujita considered the following Cauchy problem

$$\begin{cases} u_t=\Delta u+|u|^{1+\tilde{p}},(x,t)\in\mathbb{R}^N\times\mathbb{R}^+=Q, \\ u(x,0)=a(x)\geq 0, x\in\mathbb{R}^N, \end{cases}$$

where $0\prec\tilde{p}$. If $p_c=2/N$ (c for critical), he proved that:

(I) If $0\prec\tilde{p}\prec p$ and $a(x_0)\succ 0$ for some x_0. then any solution to this Cauchy problem blow-up in a finite time.

(II) If $p\succ p_c$, then there exist solutions on Q as well as solutions which exist on $\mathbb{R}^N\times(0,T)$ for some finite T, but not on Q. (For this p, not all solutions are global, indeed if $\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2dx-\frac{1}{p+1}\int_{\mathbb{R}^N}u_0dx\prec 0$), the solution cannot be global [11].

The critical case $p=p_c$, was decided later by Hayakawa [7] for $N=1,2$ and by Kobayashi, Sirao and Tanaka [22] for $N\geq 3$.

Later on Nagasawa and Sirao [12], Sujitani [20] and Guedda and Kirane [4] considered the problem

$$\begin{cases} u_t+(-\Delta)^{\beta/2}(u)=c(x,t)|u|^{1+\tilde{p}},(x,t)\in Q, \\ u(x,0)=u_0(x)\geq 0, x\in\mathbb{R}^N, \end{cases}$$

Nagasawa and Sirao have taken $c(x,t)=c(x)$, Sujitani $c(x,t)=1$, while Guedda and Kirane [4] studied the case $c(x,t)=c(t)$. The method of proof in [12] is probabilistic while in [4] and [19], the approach analytic in a more recent paper, Guedda and Kirane [5] extended the previous results to the equation:

$$u_t+(-\Delta)^{\beta/2}(u)=h(x,t)|u|^{1+\tilde{p}},(x,t)\in Q,$$

where $h(x,t)=O\left(t^{\sigma}|x|^{\rho}\right)$ for large $|x|$.

Finally, Kirane and Qafsaoui [8] treated the more general equation:

$$u_t+(-\Delta)^{\beta/2}(u^m)+a(x,t).\nabla u^q=f(x,t)|u|^{1+\tilde{p}},$$

$(x,t)\in Q$,

which covers in particular the equation considered by S.Q. Zhang [21]

$$u_t-\Delta(u^m)=|x|^{\sigma}t^s|u|^{1+\tilde{p}},(x,t)\in Q$$

B. Tellab Department of Mathematics, Ouargla University. Algeria. E-mail: brahimtel@yahoo.fr

K. Haouam Department of Mathematics and Informatics, LAMIS Laboratory, University of Tebessa. Algeria. E-mail: haouam@yahoo.fr

(Advance online publication: 18 May 2016)

To ensure that the problem (1) is well posed, the fractional derivative has been interpreted in the Caputo sense [14] (also cf. [16] for a justification of the choice of Caputo derivatives for a nonlinear ordinary differential equation with fractional derivatives).

Our theorems are reduced to the assertion on the nonexistence of solutions. If an existence result of solutions to the Cauchy problem holds, then the nonexistence of solutions means that every nonnegative solution blow-up in finite time.

To have an idea about its posed problems, one is referred to the important contributions [1], [6], [10], [12], [16], [18], [21].

Results that we obtain for the following equations and systems can be considered as extension of results founded in the previous referenced paper.

Part one concern with the following Cauchy problem

$$\begin{cases} D_{0/t}^{\alpha}u+(-\Delta)^{\beta/2}\left(u^m\right)=|u|^p+f\,(x,t),(x,t)\in Q_T, \\ u\,(x,0)=u_0\,(x)\geq 0, x\in\mathbb{R}^N, \end{cases}$$

Where $D_{0/t}^{\alpha}$ denotes the time-derivatives of arbitrary or $\alpha\in\left]0,1\right[$ in the sense of Caputo see [14], $(-\Delta)^{\beta/2}$ where $\beta\in\left]0,2\right[$, is the $\frac{\beta}{2}$-fractional power of the Laplacian $(-\Delta)$ defined by $(-\Delta)^{\beta/2}v\,(x,t)=\Im^{-1}\left(|\zeta|^{\beta}\Im(v)(\zeta)\right)(x,t)$, where $\Im$ denotes the Fourier transform and $\Im^{-1}$ its inverse.

$Q_T=\mathbb{R}^N\times(0,T)$, $L_{loc}^p\left(Q_T,dxdt\right)$ Is the space of functions $v:Q_T\to\mathbb{R}$ such that $\int_K|v|^p dxdt\prec\infty,$ for all compact K in Q_T.

Remark 1: When $\alpha=1,\beta=1,$ the problem (1) is reduced to the classical heat one.

In part two, attention is paid to the evolution system in order to extend result of one equation to a coupled system which that looks like a reaction-diffusion one

$$\begin{cases} D_{0/t}^{\alpha}u+(-\Delta)^{\beta/2}\left(u^m\right)=|v|^p+f\,(x,t),(x,t)\in Q_T, \\ D_{0/t}^{\delta}v+(-\Delta)^{\gamma/2}\left(v^m\right)=|u|^q+g\,(x,t),(x,t)\in Q_T, \\ u\,(x,0)=u_0\,(x)\geq 0, x\in\mathbb{R}^N, \\ v\,(x,0)=v_0\,(x)\geq 0, x\in\mathbb{R}^N, \end{cases}$$

Remark 2: Well position of the problem (1) and (2) is ensured if the fractional derivative has been considered in the Caputo sense [14] (also cf. [19] for a justification of the choice of Caputo derivatives for a nonlinear ordinary differential equation with fractional derivatives).

Remark 3: Nonexistence of solutions means that every nonnegative solution blows-up in finite time.

II. PRELIMINARIES

Some definitions of fractional derivatives needed for further work are given follows:

Definition 1: Left and right Riemann-Liouville derivatives for $\phi\in L^1(0,T)$, are defined respectively as follows

$$D_{0/t}^{\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\phi(\sigma)}{(t-\sigma)^{\alpha}}d\sigma,$$

and $$D_{0/t}^{\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\phi(\sigma)}{(t-\sigma)^{\alpha}}d\sigma,$$

the symbol Γ is the usual Euler gamma function. .

The Caputo derivative is given by

$\mathbf{D}_{0/t}^{\alpha}\phi(t)=\frac{1}{\Gamma(1-\sigma)}\frac{d}{dt}\int_0^t\frac{\phi'(\sigma)}{(t-\sigma)^{\alpha}}.d\sigma.$ So one can write

$$\mathbf{D}_{0/t}^{\alpha}g\,(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{g\,(0)}{t^{\alpha}}+\int_0^t\frac{g'(t)}{(t-\sigma)^{\alpha}}d\sigma\right].$$

and $$\mathbf{D}_{t/T}^{\alpha}f\,(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f\,(t)}{(T-t)^{\alpha}}+\int_t^T\frac{f'(\sigma)}{(\sigma-t)^{\alpha}}d\sigma\right].$$

This lead to a relation between Caputo and Riemann-Liouville derivative and it is written as

$$\mathbf{D}_{0/t}^{\alpha}\phi(t)=D_{0/t}^{\alpha}\left[\phi(t)-\phi(0)\right].$$

Also integration by parts gives:

$$\int_0^T f\,(t)\left(D_{0/t}^{\alpha}g\right)(t)dt=\int_0^T\left(D_{t/T}^{\alpha}f\right)(t)g\,(t)dt.$$

Now we present some definitions concerning weak formulations to problems (1) and (2).

Definition 2: If we denote by Q_T a set such that $Q_T=(0,T)\times\mathbb{R}^N$, a function $u\in L_{loc}^1(Q_T)$ is a local weak solution to problem (1) defined on $Q_T, 0\prec T\prec+\infty,$ if $u\in L_{loc}^p(Q_T)$ and

$$\begin{aligned} &\int_{Q_T}u_0\,(x)D_{t/T}^{\alpha}\varphi(x,t)dxdt+\int_{Q_T}|u|^p\varphi(x,t)dxdt \\ &+\int_{Q_T}\varphi(x,t)f\,(x,t)dxdt \\ &=\int_{Q_T}(u)^m(-\Delta)^{\beta/2}\varphi(x,t)dxdt \\ &+\int_{Q_T}uD_{t/T}^{\alpha}\varphi(x,t)dxdt \end{aligned} \tag{3}$$

Definition 3: Also we can define weak formulation for a system (2) on Q_T such that $Q_T=(0,T)\times\mathbb{R}^N$ as

$$\begin{aligned} &\int_{Q_{TR}}u_0\,(x)D_{t/TR}^{\alpha}\varphi_1(x,t)dxdt+\int_{Q_{TR}}|v|^p\varphi_1(x,t)dxdt \\ &+\int_{Q_{TR}}\varphi_1(x,t)f\,(x,t)dxdt \\ &=\int_{Q_{TR}}(u)^m(-\Delta)^{\beta/2}\varphi_1(x,t)dxdt \\ &+\int_{Q_{TR}}uD_{t/TR}^{\alpha}\varphi_1(x,t)dxdt \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\int_{Q_{TR}}v_0\,(x)D_{t/TR}^{\delta}\varphi_2(x,t)dxdt+\int_{Q_{TR}}|u|^q\varphi_2(x,t)dxdt \\ &+\int_{Q_{TR}}\varphi_2(x,t)g\,(x,t)dxdt \\ &=\int_{Q_{TR}}(v)^m(-\Delta)^{\delta/2}\varphi_2(x,t)dxdt \\ &+\int_{Q_{TR}}uD_{t/TR}^{\alpha}\varphi_2(x,t)dxdt. \end{aligned} \tag{5}$$

In the two definitions cited in the above we define a test function $\varphi \in C^{2,1}_{x,t}(Q_T)$ such that $\varphi(x,t)=0,$ and we suppose that integrals to be convergent.

Remark 4: If $T=+\infty,$ the solutions of problems (1) and (2) are said to be global.

III. AIM AND RESULTS

Now we can announce our first result as.

Theorem 1: Let $N \succ 1,\ 1 \le m \prec p$ and $\int_{Q_T} f(x,t)dxdt \succ 0.$ If: $1 \prec p \le p_c = \dfrac{\beta+\alpha mN}{\alpha N+\beta(1-\alpha)},$ then the problem (1) does not admit nontrivial global weak solution.

Proof: Contradiction is the game on which is based the demonstration of the theorem 1.

Suppose that u is nontrivial nonnegative solution which exists globally in time. That is u exists in $(0,T^*)$ for any arbitrary $T^* \succ 0.$ Let T and R be two real numbers such that $0 \prec TR^{\beta(p-1)/\alpha(p-m)} \prec T^*.$

Let $\phi \in C^2_0(\mathbb{R}_+),$ a decreased function such that:

$$\phi(y)=\begin{cases}1 & if \quad 0 \le y \le 1,\\ 0 & if \quad y \ge 2,\end{cases} \quad \text{and } 0 \le \phi \le 1.$$

We choose $\varphi(x,t)=\phi\left(\dfrac{|x|^2+t^\theta}{R^2}\right)$ such that

$$\int_{Q_T}\left|(-\Delta)^{\beta/2}\varphi\right|^{p/p-m}\varphi^{-m/p-m} \prec +\infty \text{ and}$$

$$\int_{Q_T}\left|D^\alpha_{t/T}\varphi\right|^{p'}\varphi^{-p'/p} \prec +\infty.$$

In order to estimate the right-handed side of the relation (3) on $Q_{TR^{2/\theta}}$, we use Young inequality and then we find for the conjugate exponents p/m and $p/p-m,$

$$\begin{aligned}&\int_{Q_{TR^{2/\theta}}}(u)^m(-\Delta)^{\beta/2}\varphi dxdt\\ &=\int_{Q_{TR^{2/\theta}}}(u)^m\varphi^{m/p}\left((-\Delta)^{\beta/2}\varphi\right)^{-m/p}dxdt\\ &\le \varepsilon\int_{Q_{TR^{2/\theta}}}|u|^p\varphi dxdt\\ &+C(\varepsilon)\int_{Q_{TR^{2/\theta}}}\left|(-\Delta)^{\beta/2}\varphi\right|^{p/p-m}\varphi^{-m/p-m}dxdt.\end{aligned} \tag{6}$$

Similarly

$$\begin{aligned}&\int_{Q_{TR^{2/\theta}}}uD^\alpha_{t/TR^{2/\theta}}\varphi dxdt\\ &=\int_{Q_{TR^{2/\theta}}}u\varphi^{1/p}\left(D^\alpha_{t/TR^{2/\theta}}\varphi\right)\varphi^{-1/p}dxdt\\ &\le \varepsilon\int_{Q_{TR^{2/\theta}}}|u|^p\varphi dxdt\\ &+C(\varepsilon)\int_{Q_{TR^{2/\theta}}}\left|D^\alpha_{t/TR^{2/\theta}}\varphi\right|^{p'}\varphi^{-p'/p}dxdt.\end{aligned} \tag{7}$$

For ε enough small, (3), (6) and (7) gives:

$$\begin{aligned}&\int_{Q_{TR^{2/\theta}}}f\varphi dxdt+\int_{Q_{TR^{2/\theta}}}|u|^p\varphi dxdt\\ &\le C(\varepsilon)\left(\int_{Q_{TR^{2/\theta}}}\left|(-\Delta)^{\beta/2}\varphi\right|^{p/p-m}\varphi^{-m/p-m}dxdt\right.\\ &\left.+\int_{Q_{TR^{2/\theta}}}\left|D^\alpha_{t/TR^{2/\theta}}\varphi\right|^{p'}\varphi^{-p'/p}dxdt\right).\end{aligned} \tag{8}$$

Now we set $\varphi(x,t)=\varphi(Ry,R^{2/\theta}\tau)=\chi(y,\tau)$ with the variables change

$$t=R^{2/\theta}\tau,\quad x=Ry,\quad dxdt=R^{N+\frac{2}{\theta}}dyd\tau$$

and define the set Ω as

$$\Omega=\left\{(y,\tau)\in\mathbb{R}^N\times\mathbb{R}_+,|y|^2+\tau^\theta \prec 2\right\}.$$

In inequality (8), we estimate the first term of the left side, i.e. $\int_{Q_{TR^{2/\theta}}}\left|(-\Delta)^{\beta/2}\varphi\right|^{-p/p-m}\varphi^{-m/p-m}dxdt.$ We know that

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+.......+\frac{\partial^2\varphi}{\partial x_N^2} \quad \text{and}$$

$$\frac{\partial\varphi}{\partial x_1}=\frac{\partial\chi}{\partial y_1}\times\frac{\partial y_1}{\partial x_1}=\frac{1}{R}\times\frac{\partial\chi}{\partial y_1}. \quad \text{Then,}$$

$$\begin{aligned}\frac{\partial^2\varphi}{\partial x_1^2}&=\frac{\partial}{\partial x_1}\left(\frac{\partial\varphi}{\partial x_1}\right)=\frac{\partial}{\partial y_1}\times\frac{\partial y_1}{\partial x_1}\left(\frac{\partial x}{\partial y_1}\times\frac{\partial y_1}{\partial x_1}\right)\\ &=\frac{1}{R^2}\times\frac{\partial^2\chi}{\partial y_1^2}=R^{-2}\times\frac{\partial^2\chi}{\partial y_1^2}.\end{aligned}$$

Similarly calculus gives

$$\frac{\partial^2\varphi}{\partial x_2^2}=R^{-2}\times\frac{\partial^2\chi}{\partial y_2^2},........,\frac{\partial^2\varphi}{\partial x_N^2}=R^{-2}\times\frac{\partial^2\chi}{\partial y_N^2}$$

then,

$$\begin{aligned}\Delta\varphi&=\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+.......+\frac{\partial^2\varphi}{\partial x_N^2}\\ &=R^{-2}\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}+.......+\frac{\partial^2\chi}{\partial y_N^2}\right) \text{ i.e.}\end{aligned}$$

$\Delta\varphi=R^{-2}\Delta\chi.$ So we have

$(-\Delta)^{\beta/2}\varphi=R^{-\beta}(-\Delta)^{\beta/2}\chi.$ Substitution gives:

$$\begin{aligned}&\int_{Q_{TR^{2/\theta}}}\left|(-\Delta)^{\beta/2}\varphi\right|^{p/p-m}\varphi^{-m/p-m}dxdt\\ &=\int_\Omega R^{-\beta p/p-m}\left|(-\Delta)^{\beta/2}\chi\right|^{p/p-m}\varphi^{-m/p-m}R^{N+\frac{2}{\theta}}dyd\tau\\ &=R^{\frac{-\beta p}{p-m}+N+\frac{2}{\theta}}\int_\Omega\left|(-\Delta)^{\beta/2}\chi\right|^{p/p-m}\varphi^{-m/p-m}dyd\tau.\end{aligned} \tag{9}$$

Now, passing to the second term in the left side of inequality (8), i.e.

$$\int_{Q_{TR^{2/\theta}}}\left|D^\alpha_{t/TR^{2/\theta}}\varphi\right|^{p'}\varphi^{-p'/p}dxdt$$

and from definition of fractional derivatives, we obtain

$$D^\alpha_{t/TR^{2/\theta}}\varphi=\frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{TR^{2/\theta}}\frac{\varphi(\sigma)}{(\sigma-t)^\alpha}d\sigma. \tag{10}$$

Make the following variables change $\sigma = uR^{2/\theta}$, then $d\sigma = R^{2/\theta}du$. Substitution of the last changes in (10) gives:

$$D^{\alpha}_{t/TR^{2/\theta}}\varphi = \frac{-1}{\Gamma(1-\alpha)}\frac{1}{R^{2/\theta}}\frac{d}{d\tau}\int_{\tau}^{T}\frac{\varphi(u)}{\left(R^{2/\theta}u - R^{2/\theta}\tau\right)^{\alpha}}R^{2/\theta}du$$

$$= \frac{-1}{\Gamma(1-\alpha)}\frac{1}{R^{2/\theta}}\frac{1}{R^{2\alpha/\theta}}R^{2/\theta}\frac{d}{d\tau}\int_{\tau}^{T}\frac{\varphi(u)}{(u-\tau)^{\alpha}}du$$

$$= R^{-2\alpha/\theta}\frac{-1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\int_{\tau}^{T}\frac{\varphi(u)}{(u-\tau)^{\alpha}}du$$

$$= R^{-2\alpha/\theta}D^{\alpha}_{\tau/T}\chi.$$

So

$$\int_{Q_{TR^{2/\theta}}}\left|D^{\alpha}_{t/_{TR^{2/\theta}}}\varphi\right|^{p'}\varphi^{-p'/p}dxdt$$

$$= \int_{\Omega}R^{-2\alpha p'/\theta}\left|D^{\alpha}_{\tau/T}\chi\right|^{p'}\varphi^{-p'/p}R^{N+\frac{2}{\theta}}dyd\tau \tag{11}$$

$$= R^{\frac{-2\alpha p'}{\theta}+N+\frac{2}{\theta}}\int_{\Omega}\left|D^{\alpha}_{\tau/T}\chi\right|^{p'}\varphi^{-p'/p}dyd\tau.$$

Equate the two powers of R in (9) and (11) we find

$$\frac{\beta p}{p-m} = \frac{2\alpha p'}{\theta},\ \text{i.e. } \theta = \frac{2\alpha(p-m)}{\beta(p-1)}.$$

Substitute (9) and (11) in (8) it come

$$\int_{Q_{TR^{2/\theta}}} f\,\varphi + \int_{Q_{TR^{2/\theta}}}|u|^{p}\varphi \le CR^{\gamma}, \tag{12}$$

where,

$$C = C(\varepsilon)\times$$

$$\int_{\Omega}\left\{\left|(-\Delta)^{\beta/2}\varphi\right|^{p/p-m}\varphi^{-m/p-m} + \left|D^{\alpha}_{\tau/T}\chi\right|^{p'}\varphi^{-p'/p}\right\}dydt,\ \text{and}$$

$$\gamma = \frac{-\beta p}{p-m} + N + \frac{\beta(p-1)}{\alpha(p-m)}.$$

Remark that from the last expression come the valid critical exponent for the equation (1), which is

$$p_c = \frac{\beta + \alpha mN}{\alpha N + \beta(1-\alpha)}.$$

For the last value we can distinguish two cases

First case:

For $\gamma \prec 0$, i.e., $p \prec p_c$, letting R going to $+\infty$, and while applying (12) one get $\int_{\mathbb{R}^N\times\mathbb{R}_+} f + \int_{\mathbb{R}^N\times\mathbb{R}_+}|u|^p \le 0$, and this lead to contradiction with the hypothesis $\int_{\mathbb{R}^N\times\mathbb{R}_+} f \succ 0$.

Second case:

For $\gamma = 0$, i.e., $p = p_c$, the relation (12) become $\int_{Q_{TR^{2/\theta}}} f\,\varphi + \int_{Q_{TR^{2/\theta}}}|u|^p\varphi \le C$, i.e., $\int_{Q_{TR^{2/\theta}}}|u|^p\varphi \le C$,

so convergence of the integral $\int_{Q_{TR^{2/\theta}}}|u|^p\varphi$ is ensured.

Now if we put

$$C_R = \left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}_+ : R^2 \le |x|^2 + t^{\theta} \le 2R^2\right\},$$

so $$\lim_{R\to\infty}\int_{C_R}|u|^p\varphi dxdt = 0. \tag{13}$$

Using expression (3) we get

$$\int_{Q_{TR^{2/\theta}}} f\,\varphi + \int_{Q_{TR^{2/\theta}}}|u|^p\varphi \le \int_{Q_{TR^{2/\theta}}}|u|^m\left|(-\Delta)^{\beta/2}\varphi\right|dxdt$$
$$+\int_{Q_{TR^{2/\theta}}}|u|\left|D^{\alpha}_{t/TR^{2/\theta}}\varphi\right|dxdt. \tag{14}$$

By Holder inequality, we arrive to the next expression

$$\int_{Q_{TR^{2/\theta}}}|u|^m\left|(-\Delta)^{\beta/2}\varphi\right|dxdt =$$

$$\int_{Q_{TR^{2/\theta}}}|u|^m\varphi^{m/p}\left|(-\Delta)^{\beta/2}\varphi\right|\varphi^{-m/p}dxdt \le$$

$$\left(\int_{Q_{TR^{2/\theta}}}|u|^p\varphi dxdt\right)^{m/p}\times$$

$$\left(\int_{Q_{TR^{2/\theta}}}\left|(-\Delta)^{\beta/2}\varphi\right|^{p/p-m}\varphi^{-m/p-m}dxdt\right)^{p-m/p}.$$

Under another form of writing, the last expression become

$$\int_{Q_{TR^{2/\theta}}}|u|^m\left|(-\Delta)^{\beta/2}\varphi\right|dxdt \le$$

$$\left(\int_{C_R}|u|^p\varphi\right)^{m/p}\times \tag{15}$$

$$\left(\int_{\Omega_1}\left|(-\Delta)^{\beta/2}\chi\right|^{p/p-m}\chi^{-m/p-m}dyd\tau\right)^{p-m/p}.$$

Apply in second time Holder inequality, we get

$$\int_{Q_{TR^{2/\theta}}}|u|\left|D^{\alpha}_{t/_{TR^{2/\theta}}}\varphi\right|dxdt$$

$$= \int_{Q_{TR^{2/\theta}}}|u|\varphi^{1/p}\left|D^{\alpha}_{t/_{TR^{2/\theta}}}\varphi\right|\varphi^{-1/p}dxdt$$

$$\le\left(\int_{Q_{TR^{2/\theta}}}|u|^p\varphi\right)^{1/p}\left(\int_{Q_{TR^{2/\theta}}}\left|D^{\alpha}_{t/_{TR^{2/\theta}}}\varphi\right|^{p'}\varphi^{-p'/p}dxdt\right)^{p-m/p}$$

i.e.

$$\int_{Q_{TR^{2/\theta}}}|u|\left|D^{\alpha}_{t/_{TR^{2/\theta}}}\varphi\right|dxdt$$
$$\le\left(\int_{C_R}|u|^p\varphi\right)^{1/p}\left(\int_{\Omega_1}\left|D^{\alpha}_{t/T}\chi\right|^{p'}\chi^{-p'/p}dyd\tau\right)^{p-m/p} \tag{16}$$

Where $\Omega_1 = \left\{(y,\tau)\in\mathbb{R}^N\times\mathbb{R}_+ : 1 \le |y|^2 + \tau^{\theta} \le 2\right\}$,

by substitution of (15) and (16) in (14) it come

$$\int_{Q_{TR^{2/\theta}}} f\,\varphi + \int_{Q_{TR^{2/\theta}}}|u|^p\varphi \le$$

$$\left(\int_{C_R}|u|^p\varphi\right)^{m/p}\times\left(\int_{\Omega_1}\left|(-\Delta)^{\beta/2}\chi\right|^{p/p-m}\chi^{-m/p-m}dyd\tau\right)^{p-m/p}$$

$$+\left(\int_{C_R}|u|^p\varphi\right)^{1/p}\left(\int_{\Omega_1}\left|D^{\alpha}_{t/T}\chi\right|^{p'}\chi^{-p'/p}dyd\tau\right)^{p-m/p}.$$

Applying (13) to the last expression and let $R\to+\infty$ we get $\int_{\mathbb{R}^N\times\mathbb{R}_+} f + \int_{\mathbb{R}^N\times\mathbb{R}_+}|u|^p \le C$, then, $\int_{\mathbb{R}^N\times\mathbb{R}_+} f \le 0$, this is a contradiction with hypothesis $\int_{\mathbb{R}^N\times\mathbb{R}_+} f \succ 0$, and this ends the proof ■

IV. SYSTEM OF FRACTIONAL EQUATIONS

In this section we are interested by solving problem (2) concerning system which containing parabolic fractional equations as

$$\begin{cases} D_{0/t}^{\alpha}u+(-\Delta)^{\beta/2}\left(u^{m}\right)=|v|^{p}+f\left(x,t\right),\left(x,t\right)\in Q_{T},\\ D_{0/t}^{\delta}v+(-\Delta)^{\gamma/2}\left(v^{m}\right)=|u|^{q}+g\left(x,t\right),\left(x,t\right)\in Q_{T},\end{cases}$$

with initial conditions

$u\left(x,0\right)=u_{0}\left(x\right)\geq 0,\quad v\left(x,0\right)=v_{0}\left(x\right)\geq 0, x\in\mathbb{R}^{N}$,

where $0\prec\alpha,\delta\prec 1,\ 0\prec\gamma,\beta\prec 2,$ and f, g are two function verifying

$\int_{Q_T} f\left(x,t\right)dxdt\succ 0$ and $\int_{Q_T} g\left(x,t\right)dxdt\succ 0.$

Now we are able to cite our second result

Theorem 2: Let $N\succ 1,\ 1\leq m\prec p.$ If

$$1\prec N\leq\max\left\{\frac{\dfrac{m\delta}{q}+\alpha-\left(1-\dfrac{m^{2}}{pq}\right)}{\dfrac{m\left(p-m\right)\delta}{pq\gamma}+\dfrac{\left(q-m\right)\alpha}{q\beta}},\frac{\dfrac{m\alpha}{p}+\delta-\left(1-\dfrac{m^{2}}{pq}\right)}{\dfrac{m\left(q-m\right)\alpha}{pq\beta}+\dfrac{\left(p-m\right)\delta}{p\gamma}}\right\}.$$

Then the system (2) does not admit local nontrivial weak solution.

Proof: We proceed always by contradiction. Put

$$\varphi_{j}\left(x,t\right)=\phi\left(\frac{t^{2}+|x|^{2\theta_{j}}}{R^{2}}\right),\quad j=1,2,$$

where $R\succ 0,\ \theta_{1}=\dfrac{\beta}{\alpha}$ and $\theta_{2}=\dfrac{\gamma}{\delta}.$ Applying Holder in-equality to the weak formulations (4) and (5), it come

$$\int_{Q_{TR}}u^{m}\left|(-\Delta)^{\beta/2}\varphi_{1}\right|\leq$$
$$\left(\int_{Q_{TR}}|u|^{q}\varphi_{2}\right)^{m/q}\left(\int_{Q_{TR}}\left|(-\Delta)^{\beta/2}\varphi_{1}\right|^{q/q-m}\varphi_{2}^{-m/q-m}\right)^{q-m/q}$$

and

$$\int_{Q_{TR}}u\left|D_{t/_{TR}}^{\alpha}\varphi_{1}\right|\leq$$
$$\left(\int_{Q_{TR}}|u|^{q}\varphi_{2}\right)^{m/q}\left(\int_{Q_{TR}}\left|D_{t/_{TR}}^{\alpha}\varphi_{1}\right|^{q/q-m}\varphi_{2}^{-m/q-m}\right)^{q-m/q},$$

Consequently

$$\int_{Q_{TR}}|v|^{p}\varphi_{1}+\int_{Q_{TR}}f\varphi_{1}\leq\left(\int_{Q_{TR}}|u|^{q}\varphi_{2}\right)^{m/q}.\ A,\tag{17}$$

where

$$A=\left(\int_{Q_{TR}}\left|(-\Delta)^{\beta/2}\varphi_{1}\right|^{q/q-m}\varphi_{2}^{-m/q-m}\right)^{q-m/q}$$
$$+\left(\int_{Q_{TR}}\left|D_{t/TR}^{\alpha}\varphi_{1}\right|^{q/q-m}\varphi_{2}^{-m/q-m}\right)^{q-m/q},$$

the same way give us the next estimate

$$\int_{Q_{TR}}|u|^{q}\varphi_{2}+\int_{Q_{TR}}g\varphi_{2}\leq\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{m/p}.\ B\ .\tag{18}$$

Where

$$B=\left(\int_{Q_{TR}}\left|(-\Delta)^{\gamma/2}\varphi_{2}\right|^{p/p-m}\varphi_{1}^{-m/p-m}\right)^{p-m/p}$$
$$+\left(\int_{Q_{TR}}\left|D_{t/TR}^{\delta}\varphi_{2}\right|^{p/p-m}\varphi_{1}^{-m/p-m}\right)^{p-m/p},$$

Using inequalities (17) and (18) we can write

$$\int_{Q_{TR}}|v|^{p}\varphi_{1}+\int_{Q_{TR}}f\varphi_{1}\leq\left(\int_{Q_{TR}}|u|^{q}\varphi_{2}\right)^{m/q}.\ A$$
$$\leq\left[\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{m/p}.B\right]^{m/q}.\ A$$
$$=\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{m^{2}/pq}.\ B^{m/q}.\ A.$$

So

$$\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{1-\frac{m^{2}}{pq}}+\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{-\frac{m^{2}}{pq}}\int_{Q_{TR}}f\varphi_{1}\leq B^{m/q}.A\tag{19}$$

also

$$\left(\int_{Q_{TR}}|u|^{q}\varphi_{2}\right)^{1-\frac{m^{2}}{pq}}+\left(\int_{Q_{TR}}|u|^{q}\varphi_{2}\right)^{-\frac{m^{2}}{pq}}\int_{Q_{TR}}g\varphi_{2}\leq A^{m/p}.B\tag{20}$$

Using the variables changes $t=R\tau,\ x=R^{\alpha/\beta}y$, in A and $t=R\tau,\ x=R^{\delta/\gamma}$, then with the help of (19) we can arrive to the estimate

$$\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{1-\frac{m^{2}}{pq}}+\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{-\frac{m^{2}}{pq}}\int_{Q_{TR}}f\varphi_{1}\leq C\left(R^{-l_{1}}\right)^{m/q}R^{-l_{2}}$$

i.e.

$$\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{1-\frac{m^{2}}{pq}}+\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{-\frac{m^{2}}{pq}}\int_{Q_{TR}}f\varphi_{1}\leq CR^{-\left(\frac{ml_{1}}{q}+l_{2}\right)},\tag{21}$$

where

$$l_{1}=\delta-\frac{p-m}{p}\left(\frac{\delta}{\gamma}N+1\right),\ l_{2}=\alpha-\frac{q-m}{q}\left(\frac{\alpha}{\beta}N+1\right).$$

When $\dfrac{ml_{1}}{q}+l_{2}\geq 0,$ we found the critical value for the first equation of the system (2).

$$N\leq\frac{\dfrac{m\delta}{q}+\alpha-\left(1-\dfrac{m^{2}}{pq}\right)}{\dfrac{m\left(p-m\right)\delta}{pq\gamma}+\dfrac{\left(q-m\right)\alpha}{q\beta}}.\tag{22}$$

Similarly with using (20) we obtain the second critical value for the second equation of (2)

$$N\leq\frac{\dfrac{m\alpha}{p}+\delta-\left(1-\dfrac{m^{2}}{pq}\right)}{\dfrac{m\left(q-m\right)\alpha}{pq\beta}+\dfrac{\left(p-m\right)\delta}{p\gamma}}.\tag{23}$$

Under the above conditions (critical value) and letting R going to $+\infty$ in (21), we get

$$\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{1-\frac{m^{2}}{pq}}+\left(\int_{Q_{TR}}|v|^{p}\varphi_{1}\right)^{-\frac{m^{2}}{pq}}\int_{Q_{TR}}f\varphi_{1}\leq 0\tag{24}$$

with the same method and considering the condition

$\frac{ml_2}{p}+l_1 \geq 0$, we get also

$$\left(\int_{Q_{TR}} |u|^q \varphi_2\right)^{1-\frac{m^2}{pq}} + \left(\int_{Q_{TR}} |u|^q \varphi_2\right)^{-\frac{m^2}{pq}} \int_{Q_{TR}} g\varphi_2 \leq 0. \qquad (25)$$

Inequality (24) leads to contradiction $\int_{Q_{TR}} f \leq 0$, and inequality leads to contradiction $\int_{Q_{TR}} g \leq 0$ which mean that the system (2) can not admit local weak solution other the trivial one.

Remark 5: Both the two critical values can be presented as mentioned in theorem 2

$$1 \prec N \leq \max \left\{ \frac{\frac{m\delta}{q}+\alpha-\left(1-\frac{m^2}{pq}\right)}{\frac{m(p-m)\delta}{pq\gamma}+\frac{(q-m)\alpha}{q\beta}}, \frac{\frac{m\alpha}{p}+\delta-\left(1-\frac{m^2}{pq}\right)}{\frac{m(q-m)\alpha}{pq\beta}+\frac{(p-m)\delta}{p\gamma}} \right\}.$$

Acknowledgment. The authors thank the referee for his/ her useful comments.

REFERENCES

[1] K. Deng, H. A. Levine, "The role of critical exponents in blow-up theorems," The sequel, J. Math. Anal. Appl. 243 (2000) ,85-126. http://www.journals.elsevier.com/journal-of-mathematical-analysis-and-applications/

[2] M. Escobedo, M. A. Herrero, "Boundedness and blow-up for a semilinear reaction-diffusion equation ," J. Diff. Equa. 89 (1991), 176-202. http://www.journals.elsevier.com/journal-of-differential-equations/

[3] H. Fujita, "On the blowing up of solutions of the Cauchy problem, " J.Fac. Sci. Univ. Tokyo Sect. 113 (1966), 109-124.

[4] M. Guedda, M. Kirane, "A note on nonexistence of global solutions to a nonlinear integral equation ," Bull. Belg. Math. Soc. Simon Steven. 6 (1999), 491-497.

[5] M. Guedda, M. Kirane, "Critcality for some evolution equations ," J. Diff. Equa. 37 (2001), 540-550.

[6] K. Haouam, M. Sfaxi, "Critical exponent for nonlinear hyperbolic system with spatio-temporal fractional derivatives ," International Journal of Applied Mathematics, 6 (2011), 661-871.

[7] K. Hayakawa, "On nonexistence of global solutions of some differential equations," Proc. Japan. Acad. 49 (1973), 503-505.

[8] M. Kirane, M. Qafsaoui, "Global nonexistence for the Cauchy problem of some nonlinear reaction-diffusion systems ," J. Math. Anal. Appl. 268 (2002), 217-243.

[9] H. Kuiper, " Life span of nonegative solutions to certain quasilinear parabolic Cauchy problems ," Electron. J. Differential Equation, 66 (2003), 1-11.

[10] G. G. Laptev, "Nonexistence results for higher-order evolution partial differential inequalities ," Proc. Amer. Math. Soc. 131 (2003), 415-423. http://www.ams.org/publications/journals/journalsframework/proc

[11] H. A. Levine, "Some nonexistence and instability theorems for solutions of formally parabolic equations ," Arch. Rational Mech. Anal. 51 (1973), 371-386.

[12] M. Nagasawa, T. Sirao, "Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation ," Trans. Amer. Math. Soc. 139 (1969), 301-310.

[13] L. E. Payne, "Improperly posed problems in Partial Differential Equations ," Regional Confernce Series in Applied Mathematics, Vol 22, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1975.

[14] I. Podlubny, "Fractional Differential Equations," Math. Sci. Energ, Vol. 198, Academic Press, New York, 1999.

[15] G. Rudolf, F. Mainardi, "Essential of Fractional Calculus," Maphysto, 2000.

[16] A. A. Samarski, V. A. Galaktinov, S. P. Kurdyumov, " Blowups in problems for quasilinear parabolic equations ," Gruyter Expositions in Mathematics, Vol. 19, de Gruyterm Berlin, 1995.

[17] E. Scalas, R. Gorenflo, F. Mainardi, "Fractional calculus and continuous-time finance," Phys. A. 284 (2000), 376-384.

[18] M. Seredynska, A. Hanyga, "Nonlinear Hamiltonian equations with fractional damping," J. Math. Phys. 41 (2000), 2135-2156.

[19] B. Straughan, "Explosive instabilities in Mechanics," Springer-Verlag, Berlin, 1998.

[20] S. Sugitani, "On nonexistence of global solutions for some nonlinear integral equations," Osaka J. Math. 12 (1975), 45-51.

[21] Q. S. Zhang, "A blow-up result for a nonlinear wave equation with damping the critical case," C. R. Acad. Sci. Paris 333 (2001), 109-114.

[22] K. Kobayashi, T. Sirao, H. Tanaka, "On the growing up problem for semilinear reaction-diffusion systems ," J. Math. Anal. Appl. 268 (2002), 217-243.