Solution of Exterior Problems using Elliptical Arc Artificial Boundary

Yajun Chen, and Qikui Du

Abstract—**In this paper, the artificial boundary method for Poisson problem in an infinite domain with a concave angle is investigated. The exact and approximate elliptical arc artificial boundary conditions are given. The finite element approximation is formulated in a bounded domain using the approximate artificial boundary condition and error estimates are obtained. Finally, some numerical examples show the effectiveness of this method.**

Index Terms—**artificial boundary method, elliptical arc, exterior problem.**

I. Introduction

THE problems in unbounded domains are encountered in many fields of scientific and engineering computing. To solve such problems numerically, there is a variety of numerical methods. One commonly method is the artificial boundary method [1]-[2], which is also called coupling method with natural boundary reduction [3]-[5] or DtN method [6]-[7]. The circular and spheroidal artificial boundaries were used for exterior problems in early years [8]-[10], and the ellipsoid and ellipsoidal artificial boundaries were generalized later to reduce the computational cost [11]-[14]. For problems in concave angle domains, the circular arc artificial boundary was often chosen [15]-[17]. Other related works can also be found from [18]-[22].

In this paper, a new artificial boundary method using elliptical arc artificial boundary is devised for the numerical solution of Poisson problem in an infinite domain with a concave angle. Let Ω be an exterior concave angle domain with angle α, and $0<\alpha\leq 2\pi$. The boundary of domain Ω is decomposed into three disjoint parts: Γ, Γ_0 and Γ_α(see Fig. 1), i.e. $\partial\Omega=\overline{\Gamma\cup\Gamma_0\cup\Gamma_\alpha}$, $\Gamma_0\cap\Gamma_\alpha=\emptyset$, $\Gamma\cap\Gamma_0=\emptyset$, $\Gamma\cap\Gamma_\alpha=\emptyset$. The boundary Γ is a simple smooth curve part, Γ_0 and Γ_α are two half lines.

We consider the Poisson problem in two cases:

$$\begin{cases} -\Delta u=f, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \Gamma_0\cup\Gamma_\alpha,\\ \dfrac{\partial u}{\partial n}=g, \quad \text{on } \Gamma,\\ u \text{ is vanish at infinity,} \end{cases} \tag{1}$$

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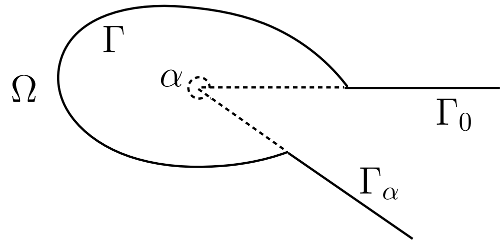

Fig. 1. The Illustration of Domain Ω

and

$$\begin{cases} -\Delta u=f, \quad \text{in } \Omega,\\ \dfrac{\partial u}{\partial n}=0, \quad \text{on } \Gamma_0\cup\Gamma_\alpha,\\ u=k, \quad \text{on } \Gamma,\\ u \text{ is bounded at infinity,} \end{cases} \tag{2}$$

where u is the unknown function, $f\in L^2(\Omega)$ and $g,k\in L^2(\Gamma)$ are given functions, supp(f) is compact.

The rest of the paper is organized as follows. In section 2, we obtain the exact elliptical arc artificial boundary condition. In section 3, we discuss the finite element approximation. Finally, in section 4 we give some numerical examples to show the effectiveness of our method.

II. The exact elliptical arc artificial boundary condition

We introduce an elliptical arc artificial boundary $\Gamma_{\mu_1}=\{(x,y)|\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,(x,y)\in\Omega\}, a>b>0$, which divides Ω into a bounded computational domain Ω_i and an unbounded domain Ω_e(see Fig. 2). Furthermore, we suppose that supp(f) is in Ω_i. Let f_0 denote the half distance between the two foci of an ellipse. We introduce an elliptic system of co-ordinates (μ,φ) such that the artificial boundary Γ_{μ_1} coincides with the elliptical arc $\{(\mu,\varphi)|\mu=\mu_1, 0<\varphi<\alpha\}$, where $f_0=\sqrt{a^2-b^2}$ and $\mu_1=\ln\frac{a+b}{\sqrt{a^2-b^2}}$. Thus, the Cartesian co-ordinates (x,y) are related to the elliptic co-ordinates (μ,φ) , that is $x=f_0\cosh\mu\cos\varphi$, $y=f_0\sinh\mu\sin\varphi$, where cosh and sinh are the hyperbolic cosine and hyperbolic sine, respectively.

In the first case, problem (1) confines in Ω_e is

$$\begin{cases} -\Delta u=0, \quad \text{in } \Omega_e,\\ u=0, \quad \text{on } \Gamma_{0e}\cup\Gamma_{\alpha e},\\ u \text{ is vanish at infinity,} \end{cases} \tag{3}$$

where $\Gamma_{0e}=\Gamma_0\cap\Omega_e$, $\Gamma_{\alpha e}=\Gamma_\alpha\cap\Omega_e$. By separation of variables, we know that the solution of problem (3) has the form

$$u(\mu,\varphi)=\sum_{n=1}^{+\infty} b_n e^{(\mu_1-\mu)\frac{n\pi}{\alpha}}\sin\frac{n\pi\varphi}{\alpha}, \tag{4}$$

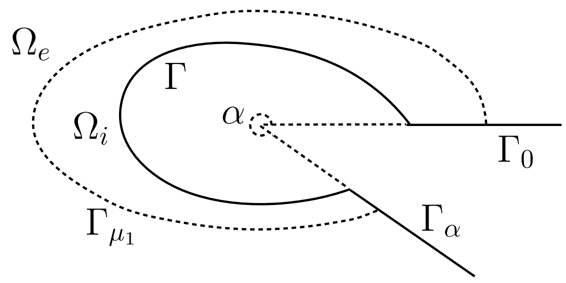

Fig. 2. The Illustration of Domain Ω_i and Ω_e

where

$$b_n = \frac{2}{\alpha}\int_0^\alpha u(\mu_1,\phi)\sin\frac{n\pi\phi}{\alpha}d\phi, \quad n=1,2,\cdots\cdots. \tag{5}$$

We differentiate (4) with respect to μ and set $\mu=\mu_1$ to obtain

$$\frac{\partial u}{\partial \mu} = -\frac{2\pi}{\alpha^2}\sum_{n=1}^{+\infty} n\int_0^\alpha u(\mu_1,\phi)\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\phi}{\alpha}d\phi. \tag{6}$$

Since

$$\frac{\partial u}{\partial n} = -\frac{1}{\sqrt{J}}\frac{\partial u}{\partial \mu},$$

where $J = f_0^2(\cosh^2\mu_1 - \cos^2\varphi)$, we obtain the exact artificial boundary condition on Γ_{μ_1}

$$\begin{aligned}\frac{\partial u}{\partial n} &= \frac{2\pi}{\alpha^2\sqrt{J}}\sum_{n=1}^{+\infty} n\int_0^\alpha u(\mu_1,\phi)\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\phi}{\alpha}d\phi \\ &\triangleq \mathcal{K}_1(\mu_1,\varphi).\end{aligned} \tag{7}$$

For the second case, the solution of problem (2) in the domain Ω_e has the form

$$u(\mu,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{(\mu_1-\mu)\frac{n\pi}{\alpha}}\cos\frac{n\pi\varphi}{\alpha}, \tag{8}$$

where

$$a_n = \frac{2}{\alpha}\int_0^\alpha u(\mu_1,\phi)\cos\frac{n\pi\phi}{\alpha}d\phi, \quad n=0,1,2,\cdots\cdots. \tag{9}$$

A similar computation shows that

$$\begin{aligned}\frac{\partial u}{\partial n} &= \frac{2\pi}{\alpha^2\sqrt{J}}\sum_{n=1}^{+\infty} n\int_0^\alpha u(\mu_1,\phi)\cos\frac{n\pi\varphi}{\alpha}\cos\frac{n\pi\phi}{\alpha}d\phi \\ &\triangleq \mathcal{K}_2(\mu_1,\varphi).\end{aligned} \tag{10}$$

By the exact artificial boundary condition (7) and (10), the original problem (1) confines in Ω_i is

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega_i, \\ u = 0, \quad \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i}, \\ \dfrac{\partial u}{\partial n} = g, \quad \text{on } \Gamma, \\ \dfrac{\partial u}{\partial n} = \mathcal{K}_1(\mu_1,\varphi), \quad \text{on } \Gamma_{\mu_1}, \end{cases} \tag{11}$$

the original problem (2) confines in Ω_i is

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega_i, \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i}, \\ u = k, \quad \text{on } \Gamma, \\ \dfrac{\partial u}{\partial n} = \mathcal{K}_2(\mu_1,\varphi), \quad \text{on } \Gamma_{\mu_1}, \end{cases} \tag{12}$$

where $\Gamma_{0i} = \Gamma_0\cap\Omega_i$, $\Gamma_{\alpha i} = \Gamma_\alpha\cap\Omega_i$.

III. Finite Element Approximation

In this section, we just consider the finite element approximation of problem (11), we can obtain corresponding result of problem (12) in the same way.

Let $V = \{v\in H^1(\Omega_i), v|_{\Gamma_{0i}\cup\Gamma_{\alpha i}} = 0\}$, then the problem (11) is equivalent to the following variational problem

$$\begin{cases} \text{Find } u\in V, \text{ such that} \\ a(u,v) + b(u,v) = f(v), \quad \forall v\in V, \end{cases} \tag{13}$$

where

$$a(u,v) = \int_{\Omega_i}\nabla u\cdot\nabla v dx, \tag{14}$$

$$\begin{aligned} b(u,v) = &\sum_{n=1}^{+\infty}\frac{2}{n\pi} \\ &\cdot\int_0^\alpha\int_0^\alpha \frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \varphi}\cos\frac{n\pi\phi}{\alpha}\cos\frac{n\pi\varphi}{\alpha}d\phi d\varphi,\end{aligned} \tag{15}$$

$$f(v) = \int_{\Omega_i} fv dx + \int_\Gamma gv ds. \tag{16}$$

In practice, we need to truncate the above infinite series by finite terms, let

$$\mathcal{K}_1^N = \frac{2\pi}{\alpha^2\sqrt{J}}\sum_{n=1}^{N} n\int_0^\alpha u(\mu_1,\phi)\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\phi}{\alpha}d\phi. \tag{17}$$

Consider the following approximation problem

$$\begin{cases} -\Delta u^N = f, \quad \text{in } \Omega_i, \\ u^N = 0, \quad \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i}, \\ \dfrac{\partial u^N}{\partial n} = g, \quad \text{on } \Gamma, \\ \dfrac{\partial u^N}{\partial n} = \mathcal{K}_1^N, \quad \text{on } \Gamma_{\mu_1}. \end{cases} \tag{18}$$

This problem is equivalent to the following variational problem

$$\begin{cases} \text{Find } u^N\in V, \text{ such that} \\ a(u^N,v) + b_N(u^N,v) = f(v), \quad \forall v\in V, \end{cases} \tag{19}$$

where

$$\begin{aligned} b_N(u^N,v) = &\sum_{n=1}^{N}\frac{2}{n\pi} \\ &\cdot\int_0^\alpha\int_0^\alpha \frac{\partial u^N}{\partial \phi}\frac{\partial v}{\partial \varphi}\cos\frac{n\pi\phi}{\alpha}\cos\frac{n\pi\varphi}{\alpha}d\phi d\varphi.\end{aligned} \tag{20}$$

For any real number s, we have the equivalent definition of Sobolev spaces $H^s(\Gamma_{\mu_1})$ as follows [19]:

$$\forall v\in H^s(\Gamma_{\mu_1}) \Leftrightarrow v(\mu_1,\varphi) = \sum_{n=1}^{+\infty} d_n\sin\frac{n\pi\varphi}{\alpha},$$

$$\text{and} \sum_{n=1}^{+\infty}(1+n^2)^s d_n^2 < \infty.$$

The norm of $H^s(\Gamma_{\mu_1})$ can be defined as follows:

$$\|v(\mu_1,\varphi)\|_{s,\Gamma_{\mu_1}} = [\sum_{n=1}^{+\infty}(1+n^2)^s d_n^2]^{\frac{1}{2}}.$$

Then we have the following results.

Lemma 1. $b(u,v)$ and $b_N(u,v)$ are both a symmetric, semi-definite and continuous bilinear form on $V\times V$.

Proof. Let

$$u(\mu_1,\phi) = \sum_{n=1}^{+\infty} b_n \sin\frac{n\pi\phi}{\alpha},$$

$$v(\mu_1,\varphi) = \sum_{n=1}^{+\infty} d_n \sin\frac{n\pi\varphi}{\alpha},$$

taking the derivative with respect to ϕ and φ we have

$$\frac{\partial u(\mu_1,\phi)}{\partial\phi} = \sum_{n=1}^{+\infty}\frac{n\pi}{\alpha} b_n\cos\frac{n\pi\phi}{\alpha},$$

$$\frac{\partial v(\mu_1,\varphi)}{\partial\varphi} = \sum_{n=1}^{+\infty}\frac{n\pi}{\alpha} d_n\cos\frac{n\pi\varphi}{\alpha},$$

then we have

$$b(u,v) = \sum_{n=1}^{+\infty}\frac{n\pi}{2} b_n d_n,$$

and

$$\begin{aligned}|b(u,v)| &\le \frac{\pi}{2}\sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}} b_n d_n\\ &\le \frac{\pi}{2}\|u\|_{\frac{1}{2},\Gamma_{\mu_1}}\|v\|_{\frac{1}{2},\Gamma_{\mu_1}}\\ &\le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i}.\end{aligned}$$

In the same way, we obtain

$$|b_N(u,v)| = \sum_{n=1}^{N}\frac{n\pi}{2} b_n d_n \le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},$$

$$|b(u,u)| = \sum_{n=1}^{+\infty}\frac{n\pi}{2} b_n^2 \ge 0,$$

$$|b_N(u,u)| = \sum_{n=1}^{N}\frac{n\pi}{2} b_n^2 \ge 0.$$

By using this lemma we have the following theorem.

Theorem 1. The variational problem (13) and (19) have a unique solution on V, respectively.

Proof. It is easy to see that $a(u,v)$ is a symmetric, continuous and V-elliptic bilinear form on $V\times V$. Note that $f(v)$ is a continuous linear function on V and lemma 1, we completed the prove of this theorem by Lax-Milgram theorem.

Assume that $\mathcal{J}_h$ is a regular and quasi-uniform triangulation of Ω_i such that

$$\Omega_i = \bigcup_{K\in\mathcal{J}_h} K,$$

where K is a (curved) triangle and h is the maximal diameter of the triangles. Let

$$V_h = \{v\in V, v|_K \text{ is a linear polynomial, } \forall K\in\mathcal{J}_h\}.$$

We consider the approximation problem of (19)

$$\begin{cases}\text{Find } u_h^N\in V_h, \text{ such that}\\ a(u_h^N,v)+b_N(u_h^N,v) = f(v), \quad \forall v\in V_h.\end{cases} \tag{21}$$

Similar with theorem 1, we have

Theorem 2. The variational problem (21) has a unique solution $u_h^N\in V_h$.

For u and u_h^N, we have the following lemma.

Lemma 2. There exists a constant C independent of h, N and μ_1 such that

$$\begin{aligned}&\|u-u_h^N\|_{1,\Omega_i}\\ &\le C(\inf_{v\in V_h}\|u-v\|_{1,\Omega_i} + \sup_{w\in V}\frac{|b_N(u,w)-b(u,w)|}{\|w\|_{1,\Omega_i}}).\end{aligned} \tag{22}$$

Proof. From variational problem (13) we have

$$\begin{aligned}&a(u,v)+b_N(u,v)\\ &= b_N(u,v)-b(u,v)+f(v), \quad \forall v\in V_h.\end{aligned}$$

Then form variational problem (19) we obtain

$$\begin{aligned}&a(u-u_h^N,v)+b_N(u-u_h^N,v)\\ &= b_N(u,v)-b(u,v), \quad \forall v\in V_h.\end{aligned}$$

For $\forall v\in V_h$ we have

$$\begin{aligned}&\|u_h^N-v\|_{1,\Omega_i}^2\\ &\le C(a(u_h^N-v,u_h^N-v)+b_N(u_h^N-v,u_h^N-v))\\ &= C(a(u-v,u_h^N-v)+b_N(u-v,u_h^N-v)\\ &\quad + b(u,u_h^N-v)-b_N(u,u_h^N-v))\\ &\le C(\|u-v\|_{1,\Omega_i}\|u_h^N-v\|_{1,\Omega_i}\\ &\quad + |b(u,u_h^N-v)-b_N(u,u_h^N-v)|).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|u_h^N-v\|_{1,\Omega_i}\\ &\le C(\|u-v\|_{1,\Omega_i}+\sup_{w\in V}\frac{|b_N(u,w)-b(u,w)|}{\|w\|_{1,\Omega_i}}), \quad \forall v\in V_h.\end{aligned}$$

The proof follows immediately by the triangle inequality.

Let $\Gamma_{\mu_0} = \{(\mu_0,\varphi)|0<\varphi<\alpha\}$ be the smallest elliptical arc to enclose the support of f, we have the following results:

Lemma 3. Suppose $u\in H^1(\Omega_i)$ is a solution of problem (1), $u|_{\Gamma_{\mu_0}}\in H^{k-\frac{1}{2}}(\Gamma_{\mu_0})$ ($k\ge 1$, $k\in Z$), then for any $w\in V$ we have

$$\begin{aligned}&|b_N(u,w)-b(u,w)|\\ &\le C\frac{e^{(\mu_0-\mu_1)\frac{(N+1)\pi}{\alpha}}}{(N+1)^{k-1}}\|u\|_{k-\frac{1}{2},\Gamma_{\mu_0}}\|w\|_{1,\Omega_i},\end{aligned} \tag{23}$$

where C is a constant independent of h, N and μ_1.

Proof. By the formula (4) we have

$$u(\mu_0,\phi) = \sum_{n=1}^{+\infty} b_n e^{(\mu_0-\mu_1)\frac{n\pi}{\alpha}}\sin\frac{n\pi\phi}{\phi}.$$

For any $w\in V$, let

$$w|_{\Gamma_{\mu_1}} = w(\mu_1,\varphi) = \sum_{n=1}^{+\infty} f_n\sin\frac{n\pi\varphi}{\varphi}.$$

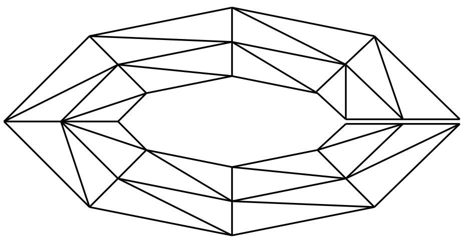

Fig. 3. Mesh h of Subdomain Ω_i for Example 1

Then we have

$$
\begin{aligned}
&|b_N(u,w) - b(u,w)| \\
&= |\sum_{n=N+1}^{+\infty} \frac{2}{n\pi} \int_0^\alpha \int_0^\alpha \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial \varphi} \cos\frac{n\pi\phi}{\alpha} \cos\frac{n\pi\varphi}{\alpha} d\phi d\varphi| \\
&= |\sum_{n=N+1}^{+\infty} \frac{n\pi}{2} b_n e^{(\mu_0-\mu_1)\frac{n\pi}{\alpha}} f_n| \\
&\leq \frac{\pi e^{(\mu_0-\mu_1)\frac{(N+1)\pi}{\alpha}}}{2(N+1)^{k-1}} |\sum_{n=N+1}^{+\infty} n^k b_n f_n| \\
&\leq C \frac{e^{(\mu_0-\mu_1)\frac{(N+1)\pi}{\alpha}}}{(N+1)^{k-1}} \|u\|_{k-\frac{1}{2},\Gamma_{\mu_0}} \|w\|_{1,\Omega_i}.
\end{aligned}
$$

Theorem 3. Suppose $u \in H^2(\Omega_i)$ is a solution of problem (1), $u|_{\Gamma_{\mu_0}} \in H^{k-\frac{1}{2}}(\Gamma_{\mu_0})$ $(k \geq 1,\ k \in Z)$, $u_h^N \in V_h$ is the solution of problem (21), the following error estimate holds

$$
\begin{aligned}
&\|u - u_h^N\|_{1,\Omega_i} \\
&\leq C(h\|u\|_{2,\Omega_i} + \frac{e^{(\mu_0-\mu_1)\frac{(N+1)\pi}{\alpha}}}{(N+1)^{k-1}} \|u\|_{k-\frac{1}{2},\Gamma_{\mu_0}}),
\end{aligned}
\tag{24}
$$

where C is a constant independent of h, N and μ_1.

Proof. By lemma 2 and lemma 3, for the first term we have

$$\inf_{v \in V_h} \|u - v\|_{1,\Omega_i} \leq Ch\|u\|_{2,\Omega_i}.$$

For the second term we have

$$\sup_{w \in V} \frac{|b_N(u,w) - b(u,w)|}{\|w\|_{1,\Omega_i}} \leq C \frac{e^{(\mu_0-\mu_1)\frac{(N+1)\pi}{\alpha}}}{(N+1)^{k-1}} \|u\|_{k-\frac{1}{2},\Gamma_{\mu_0}}.$$

So the error estimate follows.

IV. Numerical Examples

We computed two numerical examples using the method developed in Section 2 and 3 to test the effectiveness of the method. The finite element method with linear elements is used in the computation.

Example 1. We consider problem (1), where $\Omega = \{(\mu,\varphi)|\mu > 1, 0 < \varphi < 2\pi\}$, $\Gamma = \{(1,\varphi)|0 < \varphi < 2\pi\}$, $\Gamma_0 = \{(\mu,0)|\mu > 1\}, \Gamma_\alpha = \{(\mu,2\pi)|\mu > 1\}$ and $f_0 = 2$. Let $u(\mu,\varphi) = \frac{2\sinh\mu\sin\varphi}{f_0(\cosh 2\mu + \cos 2\varphi)}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$. Let $\Gamma_{\mu_1} = \{(2,\varphi)|0 < \varphi < 2\pi\}$ be the artificial boundary. Fig. 3 shows the Mesh h of subdomain Ω_i, Table 1 shows $L^2(\Omega_i)$ and $L^\infty(\Omega_i)$ errors with different Mesh($N = 20$), Fig. 4 shows $L^\infty(\Omega_i)$ errors with different N.

Example 2. We consider problem (2), where $\Omega = \{(\mu,\varphi)|\mu > \mu_0, 0 < \varphi < \frac{3\pi}{2}\}$, $\Gamma = \{(\mu_0,\varphi)|0 < \varphi < \frac{3\pi}{2}\}$, $\Gamma_0 = \{(\mu,0)|\mu > \mu_0\}$, $\Gamma_\alpha = \{(\mu,\frac{3\pi}{2})|\mu > \mu_0\}$, $f_0 = 2$ and $\mu_0 = 1$. Let $u = \frac{4(\cosh^2\mu\cos^2\varphi - \sinh^2\mu\sin^2\varphi)}{f_0(\cosh 2\mu + \cos 2\varphi)}$ be the exact solution of original problem and $k = u|_\Gamma$. Let $\Gamma_{\mu_1} = \{(\mu_1,\varphi)|\mu_1 > \mu_0, 0 < \varphi < \frac{3\pi}{2})\}$ be the artificial boundary. Fig. 4 shows the Mesh h of subdomain Ω_i, Table 2 shows $L^2(\Omega_i)$ and $L^\infty(\Omega_i)$ errors with different Mesh($N = 20$, $\mu_1 = 2$), Fig. 6 shows $L^\infty(\Omega_i)$ errors with different $N(\mu_1 = 2)$, Fig. 7 shows $L^\infty(\Omega_i)$ errors with different $\mu_1(N = 20)$.

The numerical results show that the numerical errors can be affected by the finite element mesh, the truncation terms of the series and the location of artificial boundary. Numerical results are identical with the theoretical analysis and show that our method is very effective.

TABLE I
The Errors with Different Mesh for Example 1

Mesh	$L^2(\Omega_i)$ Error	Ratio	$L^\infty(\Omega_i)$ Error	Ratio
h	1.75135E-1		1.21421E-1	
$h/2$	4.45494E-2	3.931	3.90953E-2	3.106
$h/4$	1.07941E-2	4.127	1.07895E-2	3.623
$h/8$	2.61521E-3	4.127	2.78209E-3	3.878

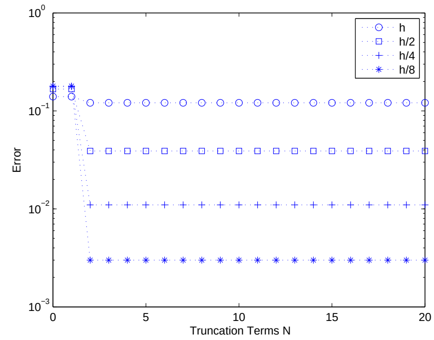

Fig. 4. $L^\infty(\Omega_i)$ Errors with Different N for Example 1

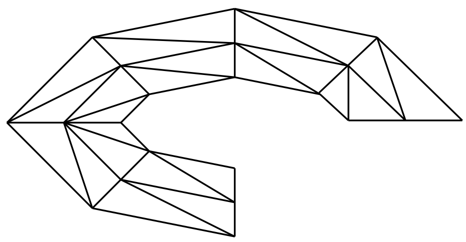

Fig. 5. Mesh h of Subdomain Ω_i for Example 2

TABLE II
The Errors with Different Mesh for Example 2

Mesh	$L^2(\Omega_i)$ Error	Ratio	$L^\infty(\Omega_i)$ Error	Ratio
h	3.65653E-2		1.90028E-2	
$h/2$	6.54475E-3	5.587	4.03139E-3	4.714
$h/4$	1.49451E-3	4.379	9.99498E-4	4.033
$h/8$	3.58623E-4	4.167	2.49612E-4	4.004

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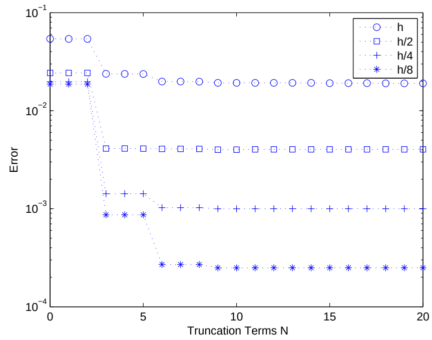

Fig. 6. $L^{\infty}(\Omega_i)$ Errors with Different N for Example 2

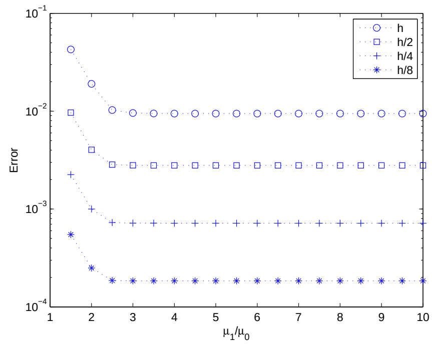

Fig. 7. $L^{\infty}(\Omega_i)$ Errors with Different μ_1 for Example 2

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