Performance Analysis of Open Queueing Networks Subject to Breakdowns and Repairs

Yu-Li Tsai*, *Member IAENG*, Daichi Yanagisawa, Katsuhiro Nishinari

***Abstract*—In this paper, we study an open queueing network with operating service stations suffering breakdowns. Stochastic modeling of Markov chains is applied to describe the steady-state probabilities of the system. The steady-state probabilities are evaluated by matrix-geometric method. Performance measures including mean number in the system, mean waiting time in the system, blocking probability, failure probability of working stations and reliable probability of the system are defined. Stability conditions of the system are derived in closed-form formulae. Disposition strategies and sensitivity analysis of the system with variation of specific parameters are investigated to make the system operate more efficiently.**

***Keywords*—Performance Analysis, Matrix-geometric method, Disposition Strategy, Sensitivity Analysis**

I. Introduction

Series configuration open queueing networks are very popular in modern manufacturing systems (e.g. automobile assembly line). Traditional studies focused on "perfect working stations". However, many cases show that the happening of breakdowns of working stations is frequent in real industrial applications. With the development of so called big data analytics and internet of things (IOT) in the 21th century, having deep understanding about the characteristic of performances of this kind of queueing systems is beneficial for further designing high efficient automated production systems. In addition, we can imagine that each service station in this kind of system can reflect its statuses of average service rate by applying IOT technology. Furthermore, based on the information of mean arrival rate, mean service rate of each service station and other important system parameters, it is expected to design a controlling center to make the queueing system work more efficiently and reflect real-time situation of operations. Therefore, quantitatively evaluating performance measures of this kind of system is a prerequisite to make the system work smartly.

Traditional analyses of open queueing networks focus on deriving exact result of the steady-state probabilities. Exact results are useful to do further analysis, but it is difficult to obtain exact iterative relations of the steady-state probabilities when the situations of the system become more complex (e.g. breakdowns or other complicated situations). The most important results in steady-state analysis are stability conditions. If the calculated steady-state probabilities not obey the stability conditions composed of parameters of the system, the numerical value of the steady-state probabilities are wrong. Furthermore, evaluating related significant performance measures of the system become impossible. It hardly derives exact formulae by means of traditional ways.

Fortunately, we can apply matrix-geometric method to analyze the system and obtain abundant results include numerical result of the steady-state probabilities, exact formulae of stability conditions, significant performance measures etc. In this research, we show how to apply matrix-geometric method to obtain theses results and make better decisions to increase operational efficiency through numerical simulations.

Neuts and Lucantoni [1] have first considered a queueing system with N servers subject to breakdowns and repairs by means of Markovian chain. They discussed the stationary distributions of various waiting times and presented the effect of utilizing interactive computation in answering questions on the behavior, design and control of certain service systems. Neuts [2] systematically studied the matrix-geometric method in terms of algorithmic analysis and computational ways. Papadopoulos and Heavey [3] reviewed the works about the design and analysis of manufacturing system by queueing networks. Gray, Wang and Scott [4] discussed a queueing model with multiple-vacation and server breakdowns. Bierbooms et al. [6] proposed approximate methods for fluid flow production lines with multi-server workstations and finite buffers. Their method is suitable for the estimations of characteristics of longer production lines. An approximation method to determine the throughput and mean sojourn time of single server tandem queues with general service times and finite buffers by decomposition method was developed by Bierbooms et al. [7]. Zhou et al. [8] investigated a two-stage tandem queueing network with Markovian arrival process inputs and buffer sharing. The buffer sharing policy is more flexible when the inputs have large variant and are correlated was suggested. Hillier [9] studied the optimal design of unpaced assembly lines. He discovered that the allocation of work to the stations and the allocation of buffer storage space between service stations are two major key points for designing an unpaced assembly production line. Sakuma et al. [10] applied Whitt's approximation to obtain the stationary distribution of an assembly-like queueing system with generally distributed time-constraint. Shin et al. [11]

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proposed an approximation method for throughput in tandem queues with multiple independent reliable servers at each stage and finite buffers between service stations. Complete reviews for the topics about unbalanced unpaced serial production lines were collected by Hudson et al. [12]. Sani and Daman [13] studied a queueing system consisting of two service stations with an exponential server and a general service under a controlled queue discipline. The steady-state distribution for the number of customers in the system, mean waiting time and blocking probability of the system are derived in closed form. Ramasamy et al. [14] discussed the steady-state analysis of heterogeneous services of a queueing system, called Geo/G/2. They applied embedded method and supplementary variable technique to investigate the system performances.

General disposition strategies of open queueing networks with multiple working service stations were proposed by Tsai, Yanagisawa and Nishinari [15]. They successfully evaluated steady-state probability and derived exact results of stability conditions for the system consisting of two, three and four service stations by matrix-geometric method.

Our major contributions are following:

- Methodological.

We analyze open queueing networks with blocking phenomena subject to breakdowns and repairs, in particular:

1. Constructing steady-state structured generator matrix equations of the queueing system with two service stations.
2. Deriving exact formulae of stability conditions consist of system parameters.
3. Numerically solving the steady-state probabilities with different conditions of system parameters.
4. Providing insights through numerical simulations to suggest disposition strategies for the system working more smartly and efficiently.

- Practical.

We theoretically model and analyze the performance measures of the system in various scenarios under the control of service rates, failure rates and repair rates for each service station of the system from the viewpoint of practitioners.

1. Important performance measures of the system can be quantitatively described by our model, such as mean number in the system, mean waiting time in the system, blocking probability of the station-1, reliable probability of the system, failure probability of each service station.
2. Controlling finite resources and adjusting related parameters based on the information to increase the operational efficiency of the system in applications.

Paper Outline: The rest of the paper is organized as follows. First, we introduce assumptions and notations used in our model in the beginning of next section. Section 3. contains details of matrix-geometric method derived for the system. Furthermore, the stability conditions and major performance metrics are also included in this section. In section 4., we perform numerical experiments and propose disposition strategies for the system through case studies. Finally, conclusions and discussions of our works and indications of possible directions for future research are included in Section 5.

II. Problem Formulation and Notations

In our analysis, the queueing system consists of two independent service stations disposed in series configuration and operates simultaneously with server breakdowns. For the simplicity of modeling work, we assume that the service stations cannot become breakdown simultaneously in this study. Customers arrive at the system in accordance with Poisson arrival process with mean arrival rate λ. In each station, the average time to serve a customer follows exponential distribution with mean $\frac{1}{\mu}$. The service stations can break down and the breakdown times are exponentially distributed with breakdown rate α. Concurrently, the repair times are assumed to be exponential with mean repair time $\frac{1}{\beta}$. A complete service is defined as customers passing through all of the service stations in order and finishes the final service in the terminal station. There is no queue between service stations. A queue with infinite capacity in front of the first station is allowed. In addition, each service station can only serve a customer at a time while the service rate is independent of the number of customers. The service of the system obeys the first come first serve (FCFS) discipline.

- **Notations**

In this section, the notations used in our model framework are introduced. In steady-state, the following notations are used.

λ, mean arrival rate of the customers

μ_1, mean service rate of the station-1

μ_2, mean service rate of the station-2

α_1, mean failure rate of the station-1

α_2, mean failure rate of the station-2

β_1, mean repair rate of the station-1

β_2, mean repair rate of the station-2

$P_{0;n_1,n_2,n_3}$, steady-state probability in working states (both service stations work concurrently). $P_{1;n_1,n_2,n_3}$, steady-state probability in failure states of the station-1 (only station-2 works), $P_{2;n_1,n_2,n_3}$, steady-state probability in failure states of the station-2 (only station-1 works). Note that the notation $P_{0;n_1,n_2,n_3}$ is used to denote the steady-state probability in working states $P_{0;n_1,n_2,n_3}$ of n_1 customer in the station-2 and n_2 customer in the station-1 and n_3 customer in the queue. For instance, the notation $P_{0;1,b,3}$ of steady-state probability means that in working states, there is a customer receiving the service in the station-2 and a customer who is blocked in the station-1 by the customer in the station-2 and three customers waiting to secure services in the queue.

III. Modeling Framework

- **Matrix-Geometric Method**

Let $\mathbf{P}=[\mathbf{P_0},\mathbf{P_1},\mathbf{P_2},...]$ denote as steady-state probability vector corresponding to the transition matrix Q. The steady-state probability vector contains steady-state probabilities of the quasi-death-birth process in working states, failure states of the station-1 and the station-2. The compositions of the sub-matrices of the transition matrix Q for the system are shown in Appendix. The steady-state equations of the quasi-birth-death process in vector form with the transition matrix can bewritten as $\mathbf{P}\mathrm{Q}=\mathbf{0}$, while $\mathbf{P1}=1$ is the normalization condition of the steady-state probability. Then, the global steady-state equations of the quasi-birth-death process can be described as

$$\mathbf{P_0}\mathrm{B}_{0,0}+\mathbf{P_1}\mathrm{B}_{1,0}=\mathbf{0}, \tag{1}$$

$$\mathbf{P_0}\mathrm{B}_{0,1}+\mathbf{P_1}\mathrm{A}_1+\mathbf{P}_2\mathrm{A}_2=\mathbf{0}, \tag{2}$$

$$\mathbf{P}_i\mathrm{A}_0+\mathbf{P}_{i+1}\mathrm{A}_1+\mathbf{P}_{i+2}\mathrm{A}_2=\mathbf{0}, \qquad i\ge 1. \tag{3}$$

A rate matrix R exists, and the follows the recurrence relations

$$\mathbf{P}_i=\mathbf{P}_{i-1}\mathrm{R}=\mathbf{P}_1\mathrm{R}^{i-1}, \qquad i\ge 1. \tag{4}$$

We substitute (4) into (3), and simplify to quadratic matrix equation in order to solve the rate matrix R

$$\mathrm{A}_0+\mathrm{R}\mathrm{A}_1+\mathrm{R}^2\mathrm{A}_2=0. \tag{5}$$

The simplified matrix equations of (1) and (2) can be represented as

$$\mathbf{P_0}\mathrm{B}_{0,0}+\mathbf{P_1}\mathrm{B}_{1,0}=\mathbf{0}, \tag{6}$$

$$\mathbf{P_0}\mathrm{B}_{0,1}+\mathbf{P_1}(\mathrm{A}_1+\mathrm{R}\mathrm{A}_2)=\mathbf{0}. \tag{7}$$

The normalization condition equation that only involve $\mathbf{P_0}$ and $\mathbf{P_1}$ can be referred in Bloch et al. [5],

$$\mathbf{P_0}\mathbf{1}+\mathbf{P_1}(\mathrm{I}-\mathrm{R})^{-1}\mathbf{1}=1, \tag{8}$$

where I is the identity matrix with same size as the rate matrix R.

We employ logarithmic reduction method to solve the rate matrix R from (5). Then, we collect (6), (7) and (8) together, the steady-state probability vector of $\mathbf{P_0}$ and $\mathbf{P_1}$ can be obtained by solving following matrix equation

$$(\mathbf{P_0},\mathbf{P_1})\begin{pmatrix}\mathrm{B}_{0,0} & \mathrm{B}_{0,1}^{*} & \mathbf{1}\\ \mathrm{B}_{1,0} & (\mathrm{A}_1+\mathrm{R}\mathrm{A}_2)^{*} & (\mathrm{I}-\mathrm{R})^{-1}\mathbf{1}\end{pmatrix}=(\mathbf{0},1). \tag{9}$$

where $(.)^{*}$ indicates that the last column of the included matrix is removed to avoid linear dependency.

- **Stability Conditions**

According to Neuts [2], the stability conditions of the system can be derived from following inequality:

$$\mathbf{P}_\mathrm{A}\mathrm{A}_0\mathbf{1}<\mathbf{P}_\mathrm{A}\mathrm{A}_2\mathbf{1}, \tag{10}$$

where $\mathbf{P}_\mathrm{A}$ is the steady-state probability vector corresponding to the conservative stable matrix A.

The conservative stable matrix is defined to be

$$\mathrm{A}=\mathrm{A}_0+\mathrm{A}_1+\mathrm{A}_2. \tag{11}$$

We can obtain the steady-state probability $\mathbf{P_A}$ by solving the following system equations with normalization condition

$$\mathbf{P_A}\mathrm{A}=\mathbf{0}, \tag{12}$$

$$\sum_{i=0}^{2}\mathrm{P}_{\mathrm{A},i}=1. \tag{13}$$

- **Performance Metrics and Exact Results**

In this section, the performance metrics for the series configuration system consisting of two service stations subject to breakdowns and repairs are defined. Performance measures include mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue and blocking probability of the station-1, failure probability of each service station and reliable probability of the system. Exact formulae of stability conditions for the system with special and general case are also given in the section.

Theorem 1. The following inequalities are necessary and sufficient conditions for the system to be stable.

(1) Special case: $\mu_1=\mu_2=\mu$, $\alpha_1=\alpha_2=\alpha$, and $\beta_1=\beta_2=\beta$,

$$\lambda < \frac{2\beta\mu(\alpha+\beta+\mu)^2}{(2\alpha+\beta)(3\alpha\beta+3\beta^2+2\alpha\mu+6\beta\mu+3\mu^2)}. \tag{14}$$

(2) General case: $\mu_1\ne\mu_2$, $\alpha_1\ne\alpha_2$, $\beta_1\ne\beta_2$,

$$\lambda < \frac{\mathrm{N}}{\mathrm{D}}, \tag{15}$$

where

$$\begin{aligned}\mathrm{N}=\ &\beta_1\beta_2\mu_1\mu_2(\alpha_1\alpha_2^2\beta_1\mu_1+\alpha_2^2\beta_1^2\mu_1+2\alpha_1\alpha_2\beta_1\beta_2\mu_1+2\alpha_2\beta_1^2\beta_2\mu_1+\alpha_1\beta_1\beta_2^2\mu_1+\beta_1^2\beta_2^2\mu_1\\ &+2\alpha_1\alpha_2\beta_1\mu_1^2+2\alpha_2\beta_1^2\mu_1^2+2\alpha_1\beta_1\beta_2\mu_1^2+2\beta_1^2\beta_2\mu_1^2+\alpha_1\beta_1\mu_1^3+\beta_1^2\mu_1^3+\alpha_1^2\alpha_2\beta_2\mu_2\\ &+2\alpha_1\alpha_2\beta_1\beta_2\mu_2+\alpha_2\beta_1^2\beta_2\mu_2+\alpha_1^2\beta_2^2\mu_2+2\alpha_1\beta_1\beta_2^2\mu_2+\beta_1^2\beta_2^2\mu_2+2\alpha_1^2\alpha_2\mu_1\mu_2+2\alpha_1\alpha_2^2\mu_1\mu_2\\ &+4\alpha_1\alpha_2\beta_1\mu_1\mu_2+2\alpha_2^2\beta_1\mu_1\mu_2+2\alpha_2\beta_1^2\mu_1\mu_2+2\alpha_1^2\beta_2\mu_1\mu_2+4\alpha_1\alpha_2\beta_2\mu_1\mu_2+4\alpha_1\beta_1\beta_2\mu_1\mu_2\\ &+4\alpha_2\beta_1\beta_2\mu_1\mu_2+2\beta_1^2\beta_2\mu_1\mu_2+2\alpha_1\beta_2^2\mu_1\mu_2+2\beta_1\beta_2^2\mu_1\mu_2+\alpha_1^2\mu_1^2\mu_2+4\alpha_1\alpha_2\mu_1^2\mu_2+2\alpha_1\beta_1\mu_1^2\mu_2\\ &+4\alpha_2\beta_1\mu_1^2\mu_2+\beta_1^2\mu_1^2\mu_2+4\alpha_1\beta_2\mu_1^2\mu_2+4\beta_1\beta_2\mu_1^2\mu_2+2\alpha_1\mu_1^3\mu_2+2\beta_1\mu_1^3\mu_2+2\alpha_1\alpha_2\beta_2\mu_2^2\\ &+2\alpha_2\beta_1\beta_2\mu_2^2+2\alpha_1\beta_2^2\mu_2^2+2\beta_1\beta_2^2\mu_2^2+4\alpha_1\alpha_2\mu_1\mu_2^2+\alpha_2^2\mu_1\mu_2^2+4\alpha_2\beta_1\mu_1\mu_2^2+4\alpha_1\beta_2\mu_1\mu_2^2\\ &+2\alpha_2\beta_2\mu_1\mu_2^2+4\beta_1\beta_2\mu_1\mu_2^2+\beta_2^2\mu_1\mu_2^2+2\alpha_1\mu_1^2\mu_2^2+2\alpha_2\mu_1^2\mu_2^2+2\beta_1\mu_1^2\mu_2^2+2\beta_2\mu_1^2\mu_2^2\\ &+\mu_1^3\mu_2^2+\alpha_2\beta_2\mu_2^3+\beta_2^2\mu_2^3+2\alpha_2\mu_1\mu_2^3+2\beta_2\mu_1\mu_2^3+\mu_1^2\mu_2^3)\end{aligned}$$

and

$$\begin{aligned}\mathrm{D}=\ &(\alpha_2\beta_1+\alpha_1\beta_2+\beta_1\beta_2)(\alpha_2^2\beta_1^2\mu_1^2+2\alpha_2\beta_1^2\beta_2\mu_1^2+\beta_1^2\beta_2^2\mu_1^2+2\alpha_2\beta_1^2\mu_1^3+2\beta_1^2\beta_2\mu_1^3+\beta_1^2\mu_1^4\\ &+\alpha_1\alpha_2\beta_1\beta_2\mu_1\mu_2+\alpha_2\beta_1^2\beta_2\mu_1\mu_2+\alpha_1\beta_1\beta_2^2\mu_1\mu_2+\beta_1^2\beta_2^2\mu_1\mu_2+2\alpha_1\alpha_2\beta_1\mu_1^2\mu_2+2\alpha_2^2\beta_1\mu_1^2\mu_2\\ &+2\alpha_2\beta_1^2\mu_1^2\mu_2+2\alpha_1\beta_1\beta_2\mu_1^2\mu_2+4\alpha_2\beta_1\beta_2\mu_1^2\mu_2+2\beta_1^2\beta_2\mu_1^2\mu_2+2\beta_1\beta_2^2\mu_1^2\mu_2+\alpha_1\beta_1\mu_1^3\mu_2\\ &+4\alpha_2\beta_1\mu_1^3\mu_2+\beta_1^2\mu_1^3\mu_2+4\beta_1\beta_2\mu_1^3\mu_2+2\beta_1\mu_1^4\mu_2+\alpha_1^2\beta_2^2\mu_2^2+2\alpha_1\beta_1\beta_2^2\mu_2^2+\beta_1^2\beta_2^2\mu_2^2\\ &+2\alpha_1^2\beta_2\mu_1\mu_2^2+2\alpha_1\alpha_2\beta_2\mu_1\mu_2^2+4\alpha_1\beta_1\beta_2\mu_1\mu_2^2+2\alpha_2\beta_1\beta_2\mu_1\mu_2^2+2\beta_1^2\beta_2\mu_1\mu_2^2+2\alpha_1\beta_2^2\mu_1\mu_2^2\\ &+2\beta_1\beta_2^2\mu_1\mu_2^2+\alpha_1^2\mu_1^2\mu_2^2+2\alpha_1\alpha_2\mu_1^2\mu_2^2+\alpha_2^2\mu_1^2\mu_2^2+2\alpha_1\beta_1\mu_1^2\mu_2^2+4\alpha_2\beta_1\mu_1^2\mu_2^2+\beta_1^2\mu_1^2\mu_2^2\\ &+4\alpha_1\beta_2\mu_1^2\mu_2^2+2\alpha_2\beta_2\mu_1^2\mu_2^2+4\beta_1\beta_2\mu_1^2\mu_2^2+\beta_2^2\mu_1^2\mu_2^2+2\alpha_1\mu_1^3\mu_2^2+2\alpha_2\mu_1^3\mu_2^2+2\beta_1\mu_1^3\mu_2^2\\ &+2\beta_2\mu_1^3\mu_2^2+\mu_1^4\mu_2^2+2\alpha_1\beta_2^2\mu_2^3+2\beta_1\beta_2^2\mu_2^3+4\alpha_1\beta_2\mu_1\mu_2^3+\alpha_2\beta_2\mu_1\mu_2^3+4\beta_1\beta_2\mu_1\mu_2^3+\beta_2^2\mu_1\mu_2^3\\ &+2\alpha_1\mu_1^2\mu_2^3+2\alpha_2\mu_1^2\mu_2^3+2\beta_1\mu_1^2\mu_2^3+2\beta_2\mu_1^2\mu_2^3+\mu_1^3\mu_2^3+\beta_2^2\mu_2^4+2\beta_2\mu_1\mu_2^4+\mu_1^2\mu_2^4\end{aligned}$$

- **Performance metrics**

Performance measures for the system consisting of two service stations subject to breakdowns and repairs are defined by

(1) Mean number of customers in the system

$$L = (P_{0;1,0,0} + P_{0;0,1,0} + P_{0;1,b,0}) + \sum_{n=2}^{\infty}(P_{0;1,b,n-1} + P_{0;1,1,n-2} + P_{0;0,1,n-1}) \cdot n$$
$$+ (P_{1;1,0,0} + P_{1;0,1,0} + P_{1;1,b,0}) + \sum_{n=2}^{\infty}(P_{1;1,b,n-1} + P_{1;1,1,n-2} + P_{1;0,1,n-1}) \cdot n$$
$$+ (P_{2;1,0,0} + P_{2;0,1,0} + P_{2;1,b,0}) + \sum_{n=2}^{\infty}(P_{2;1,b,n-1} + P_{2;1,1,n-2} + P_{2;0,1,n-1}) \cdot n. \quad (16)$$

(2) Mean number of customers in the queue

$$L_q = \sum_{n=1}^{\infty}(P_{0;1,b,n} + P_{0;1,1,n} + P_{0;0,1,n}) \cdot n + \sum_{n=1}^{\infty}(P_{1;1,b,n} + P_{1;1,1,n} + P_{1;0,1,n}) \cdot n$$
$$+ \sum_{n=1}^{\infty}(P_{2;1,b,n} + P_{2;1,1,n} + P_{2;0,1,n}) \cdot n. \quad (17)$$

(3) Mean waiting time in the system (Little's Law)

$$W = \frac{L}{\lambda}. \quad (18)$$

(4) Mean waiting time in the queue (Little's Law)

$$W_q = \frac{L_q}{\lambda}. \quad (19)$$

(5) Blocking probability of the customer in the station-1

$$P_b = \sum_{n=0}^{\infty} P_{0;1,b,n} + \sum_{n=0}^{\infty} P_{1;1,b,n} + \sum_{n=0}^{\infty} P_{2;1,b,n}. \quad (20)$$

(6) Failure probability of the station-1

$$P_{f,1} = (P_{1;0,0,0} + P_{1;1,0,0} + P_{1;1,b,0})$$
$$+ \sum_{n=1}^{\infty} P_{1;0,1,n-1} + P_{1;1,1,n-1} + P_{1;1,b,n}. \quad (21)$$

(7) Failure probability of the station-2

$$P_{f,2} = (P_{2;0,0,0} + P_{2;1,0,0} + P_{2;1,b,0})$$
$$+ \sum_{n=1}^{\infty} P_{2;0,1,n-1} + P_{2;1,1,n-1} + P_{2;1,b,n}. \quad (22)$$

(8) Reliable probability of the system

$$P_r = (P_{0;0,0,0} + P_{0;1,0,0} + P_{0;1,b,0})$$
$$+ \sum_{n=1}^{\infty} P_{0;0,1,n-1} + P_{0;1,1,n-1} + P_{0;1,b,n}$$
$$= 1 - (P_{f,1} + P_{f,2}). \quad (23)$$

IV. Numerical Results

In this section, we perform numerical experiments for the queueing system consisting of two service stations subject to breakdowns and repairs to study the effects of various parameters on mean waiting time in the system, blocking probability, failure probability and reliable probability. In addition, sensitivity analysis clearly shows how performance measures vary with specific parameters. We will suggest better disposition strategies and methods of controlling parameters to increase operational efficiency for the system according to the results of simulation.

- **Validation of stability conditions by numerical computations**

First, we validate the consistency of stability conditions through numerical computations. We fix $\mu_1 = \mu_2 = 1$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, and present the trends of mean number in the system and blocking probabilities of the station-1 as the mean arrival rate λ varies from 0.01 to 0.352. It is noted that mean number in the system increases as λ increases, as shown in Figure 1. The upper bound of the stability condition in this case ($\frac{6}{17} \approx 0.352$) also validates the exact results we derived in the section 2.5. It is observed that the blocking probability of the station-1 increases as the values of λ increases and the maximum value of the blocking probability of the station-1 approaches to 0.37, as shown in Figure 2.

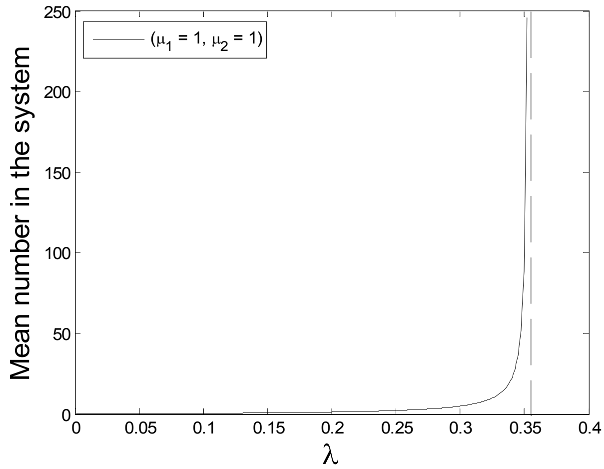

Fig 1. Mean number in the system

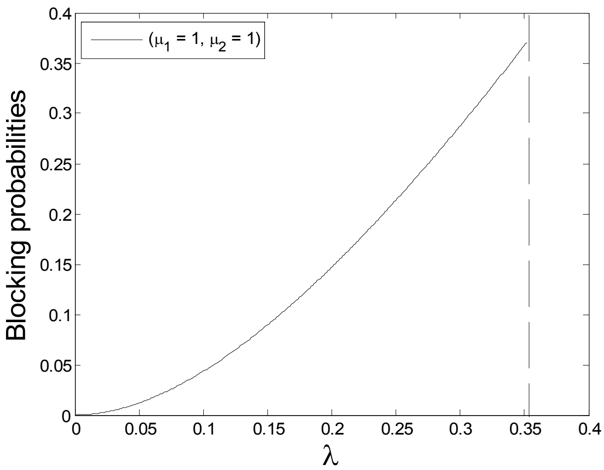

Fig 2. Blocking probability

- **Sensitivity Analysis of Performance Metrics**

In this section, we examine the impact of mean failure rate of service stations in working states α on the mean waiting time in the system and blocking probability. We set $\mu_1 = \mu_2 = 1$, $\beta_1 = \beta_2 = 0.8$ with various numbers of mean failure rate which increases from 0.2 to 0.6 and vary the mean arrival rate λ from 0.01 to 0.35. It is discovered that both mean waiting time in the system and blocking probability increases as the values of α increases, as shown in Figure 3 and Figure 4, respectively.

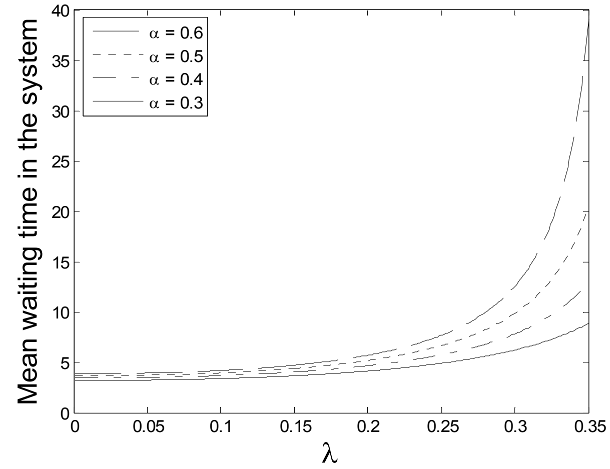

Fig 3. Mean waiting time in the system with respect to failure rates

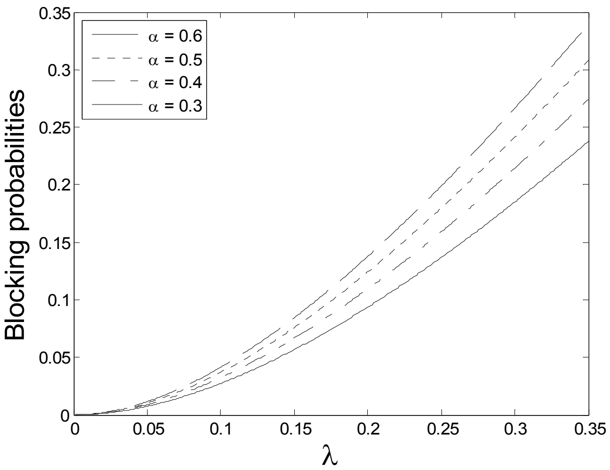

Fig 4. Blocking probability with respect to failure rates

On the other hand, we investigate the effect of mean repair rate of service stations either in failure states of the station-1 or of the station-2 on mean waiting time in the system and blocking probability. We fix $\mu_1 = \mu_2 = 1$, $\alpha_1 = \alpha_2 = 1$ with different numbers of mean repair rate which increase from 0.4 to 0.8 and vary the mean arrival rate λ from 0.01 to 0.35. It is investigated that the mean waiting time in the system and the blocking probability decreases as β increases, as shown in Figure 5 and Figure 6, respectively.

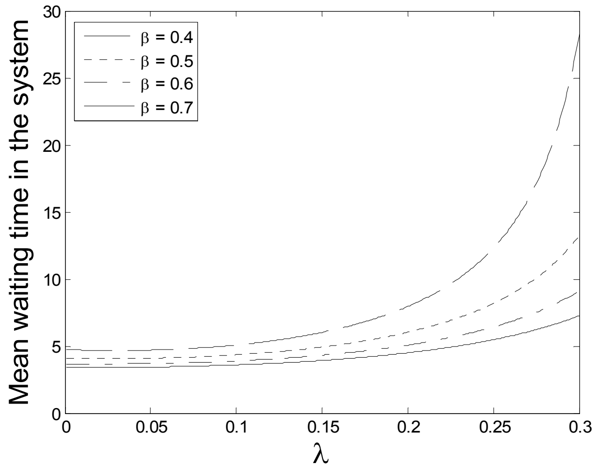

Fig 5. Mean waiting time in the system with respect to repair rates

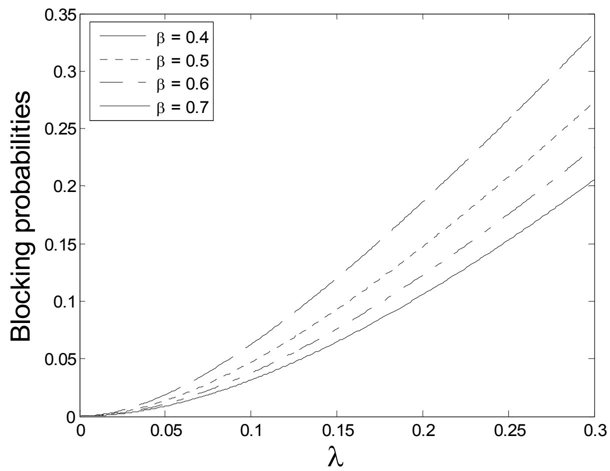

Fig 6. Blocking probability with respect to repair rates

Next, we study the sensitivity performance of failure probability of the station-1 and the reliable probability of the system with respect to mean failure rate of the station-2 α_2 and vary the mean failure rate of the station-1 α_1 from 0.01 to 1. It is clear that the failure probability of the station-1 and the reliable probability of the system decreases as α_2 increases, as shown in Figure 7 and Figure 8, respectively. These results mean that in practice, we can control the failure probabilities of the station-1 and the station-2 and reliable probabilities of the system by adjusting the ratio of the mean failure rate of the station-2 α_2 to the mean failure rates of the station-1 α_1 and vice versa.

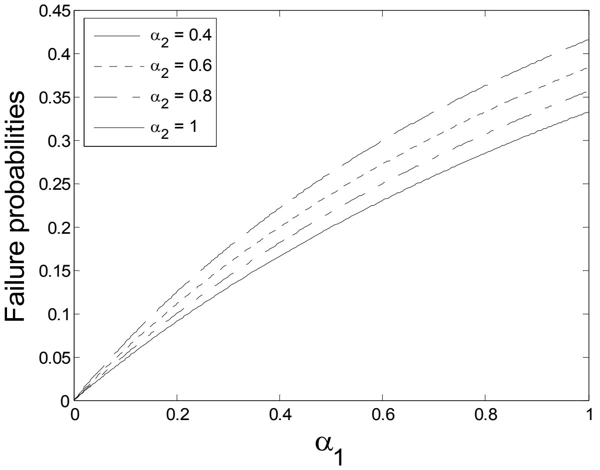

Fig 7. Failure probability of the station-1 with respect to failure rates

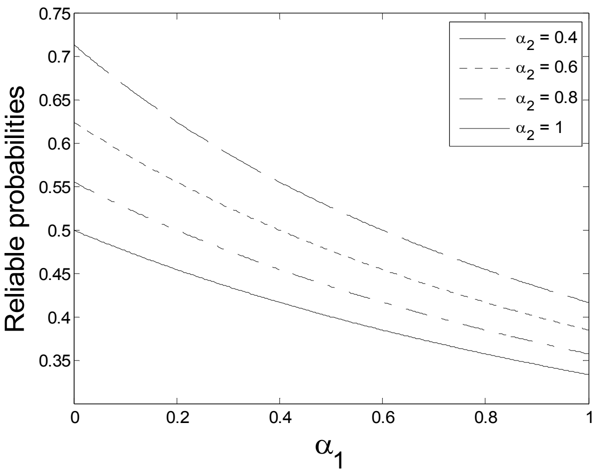

Fig 8. Reliable probability of the system with respect to failure rates

The cross effects of the failure rate of the station-1 α_1 and the failure rate of the station-2 α_2 on failure probability of the station-1, failure probability of the station-2 and reliable probabilities of the system are studied. We fix the parameters at $\lambda = 0.2$, $\mu_1 = \mu_2 = 1$, and $\beta_1 = \beta_2 = 1$. Figure 9 and Figure 10 show how failure probabilities of the station-1 and failure probabilities of the station-2 change as α_1 and α_2 vary from 0.01 to 1, respectively. The three dimensional surface presents that the failure probabilities of the station-1 and the failure probabilities of the station-2 increase as α_1 and α_2 increase. The cross effects of the failure rate of the station-1 α_1 and the failure rate of the station-2 α_2 on reliable probabilities of the system as shown in Figure 11. It is investigated that the reliable probabilities decrease as α_1 and α_2 decrease.

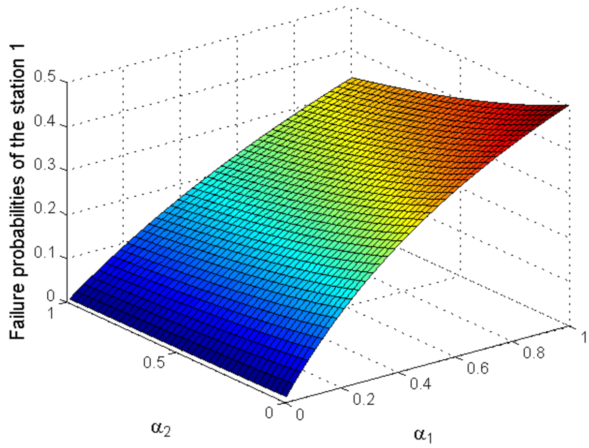

Fig 9. Failure probability of the station-1

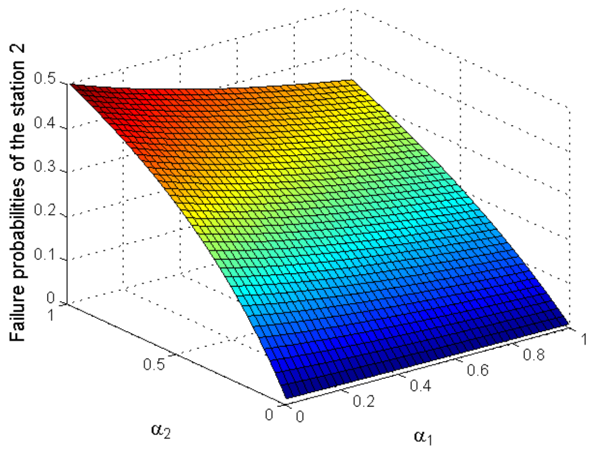

Fig 10. Failure probability of the station-2

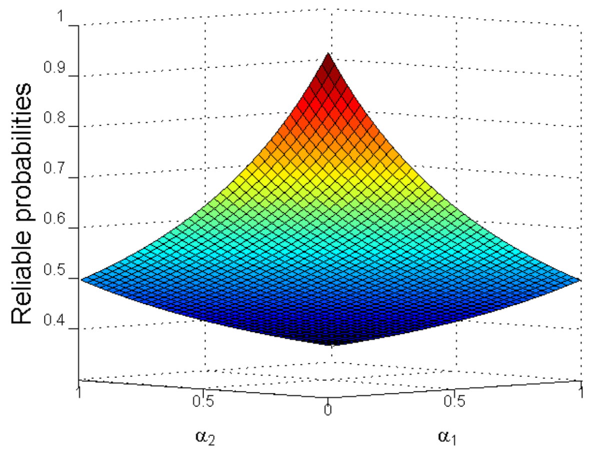

Fig 11. Reliable probability of the system

- **Disposition Strategies**

We consider the influence of disposition strategies of mean service rate, mean failure rate and mean repair rate on the performance of mean waiting time in the system. We set $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0.8$, and study the cases with different service rates $\mu_1 = 1,\ \mu_2 = 2$, and $\mu_1 = 2,\ \mu_2 = 1$, then vary the mean arrival rate λ from 0.01 to 0.35. It is observed that setting higher service rate for the station-1 increases the operational efficiency for the system subject to breakdowns and repairs, as shown in Figure 12.

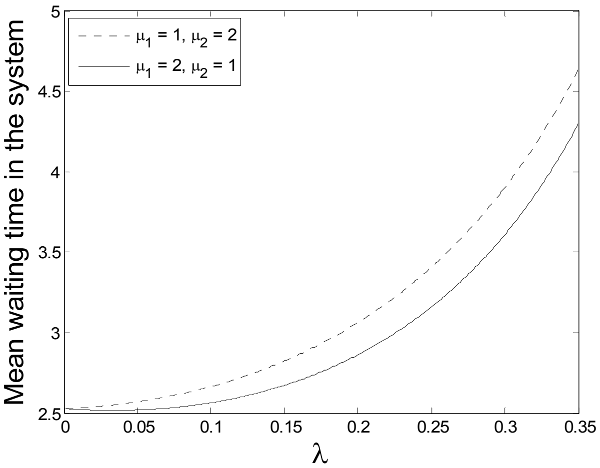

Fig 12. Mean waiting time in the system with different service rates

Next, we fix the parameters $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0.8$ and investigate the cases with different failure rates of service stations in working states $\alpha_1 = 0.6,\ \alpha_2 = 0.3$, and $\alpha_1 = 0.3,\ \alpha_2 = 0.6$, then vary the mean arrival rate λ from 0.01 to 0.25. It can be seen that if the system with higher failure rate of the station-1, the mean waiting time of the system is higher than that of the case with lower failure rate of the station-1, as shown in Figure 13.

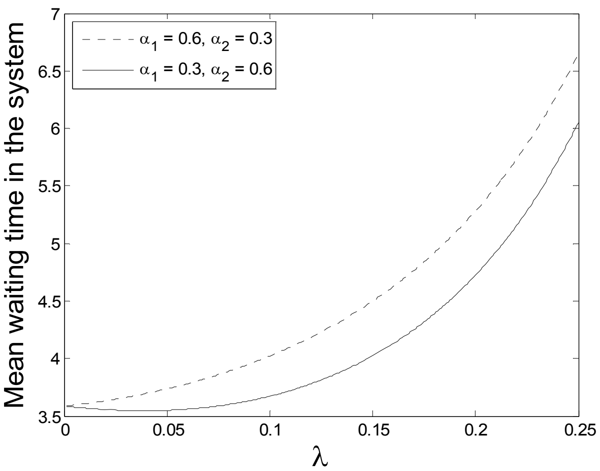

Fig 13. Mean waiting time in the system with different failure rate

Finally, the system parameters with different repair rates are fixed at $\alpha_1 = \alpha_2 = 0.3$, $\mu_1 = \mu_2 = 1$. We present the cases $\beta_1 = 0.4,\ \beta_2 = 0.8$, and $\beta_1 = 0.8,\ \beta_2 = 0.4$, and vary the mean arrival rate λ from 0.01 to 0.25. It is noted that disposing higher repair rate for the station-1 can make the system work in a higher performance, as shown in Figure 14. These results show the fact that the bottleneck of the working stations for the series configuration queueing system with two service stations is the station-1 which affects the whole operational performances of the system with perfect working states, referred as Tsai [15], and the system subject to breakdowns and repairs. We suggest dispose higher service rate and repair rate and keep lower failure rate for the station-1 of the system in order to increase operational efficiency of the system.

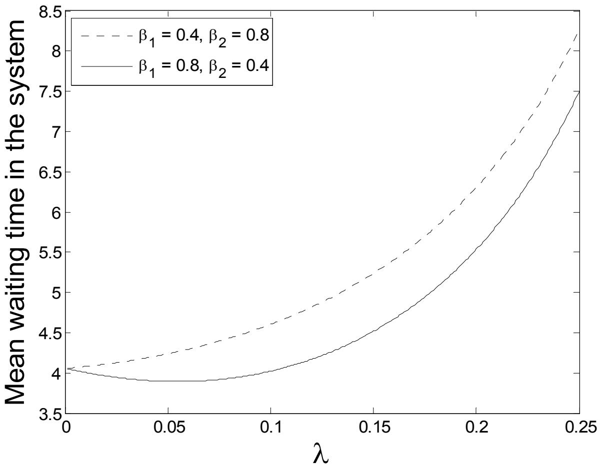

Fig 14. Mean waiting time in the system with different repair rate

V. CONCLUSIONS

We study a series configuration queueing system consisting of two service stations subject to breakdowns and repairs. The service stations in this kind of system might fail and contain repair procedures for the failure service stations We successfully demonstrate matrix-geometric method is still a powerful tool for further investigating and understanding characteristics of series configurations queueing systems as described in our previous works, Tsai [15], even for more complex extensions of the mathematical models and conditions close to the real applications in industries.

Numerical results validate the correctness of the exact formulae of stability conditions. We also discover the fact that the bottleneck of the working stations for the series configuration queueing system with two service stations is the station-1 which significantly influences the whole operational performances of the system with perfect working states and the system subject to breakdowns and repairs. We suggest that setting higher service rates, repair rate and sustain lower failure rate for the station-1 of the system to make the system work more efficiently according to the numerical experiments.

Future research will focus on conducting statistical analysis for real industrial application of manufacturing systems and validate the propositions of the analysis with our theoretical results developed in this research. Transient analysis and reliability analysis of the system would be considered further.

APPENDIX

The structure of the transition matrix Q and its sub-matrices for the system with two service stations subject to breakdowns and repairs

We provide transition matrix of the series configuration queueing system with two service stations subject to breakdowns and repairs as

$$Q = \begin{bmatrix} B_{0,0} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The details of sub-matrices of the composition of the transition matrix corresponding to the quasi-birth-death process for the system with two service stations are given by

$$B_{0,0} = \begin{bmatrix} -(\lambda+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ \mu_2 & -(\lambda+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ \beta_1 & 0 & 0 & -(\lambda+\beta_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) \end{bmatrix}$$

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu_1+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ \mu_2 & -(\lambda+\mu_1+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ \beta_1 & 0 & 0 & -(\lambda+\beta_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\beta_2) & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\beta_2) & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) \end{bmatrix}$$

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