A Copula-based Degradation Modeling and Reliability Assessment

Chunping Li, Huibing Hao

Abstract—**In this paper, a novel system reliability assessment model with two dependent performance characteristics (PCs) is proposed via the marginal distribution functions and copula theory. Firstly, the best fitted marginal distribution function is obtained by using the approximation method and the Anderson Darling test statistic. Secondly, the dependent of the two PCs is described by different copula functions, and the unknown parameters of the proposed model are obtained by using an inference for margins method. Finally, a numerical example about an actual train wheels wear degradation data is given to demonstrate the usefulness and validity of the proposed model and method. Numerical results show that ignoring the dependence between two PCs may result in different reliability conclusion.**

Index Terms—**Bivariate degradation model, Copula function, the inference for margins method**

I. Introduction

Considering that modern products have long life and high reliability, it is difficult to assess the reliability of these products by using life test or accelerated life test method, degradation data can provide useful reliability information to assess the reliability of those products. In the last decades, degradation data has played a more important role in reliability assessment than ever before.

There are some important references about the degradation reliability assessment. Nelson[1] has reviewed two methods for modeling the degradation data: the general degradation path approach and the stochastic process approach, and the degradation path method has been widely used for degradation modeling, such as Su et al.[2], Gebraeel et al.[3-4], Lu and Meeker[5], Wu and Shao[6], and so on. Another method is the stochastic process method, such as Markov chain[7], Gamma process[8], and Wiener process[9].

Most of the previous research about the reliability evaluation just thinks about one performance characteristic (PC). In practice, modern products usually have multiple PCs due to complex structure, which means that multiple degradation mechanisms may be involved. In such situations, if PCs are independent with each other, it is easy to deal with the issue of the reliability assessment. When the two PCs are dependent, it creates a challenging problem to accurately analyze the system reliability.

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C. P. Li is with the Department of Mathematics, Hubei Engineering University, Hubei, 432000, China. e-mail: lichunping315@163.com.

H. B. Hao is the corresponding author with the Department of Mathematics, Hubei Engineering University, Hubei, 432000, China. e-mail: haohuibing1979@163.com.

There is little literature dealing with the system reliability of bivariate or multivariate degradation data. Related work can be found in Ref. [10-16]. However, those multiple PCs usually are assumed to be independent with each other, or to be dependent with a multivariate normal distribution. From a practical point of view, these assumptions may be not match the engineering practice.

In this paper, we assume that a system has two PCs, and each PC is treated as a linear degradation model. By extrapolating the degradation path to a certain critical value, the marginal distributions of the pseudo failure lifetime are obtained. Moreover, we assume that the dependency of PCs is described by different copula functions, such as Gumbel copula、Clayton copula and Frank copula. Via the copula functions and the marginal distributions, the system reliability assessment model is proposed, and the inference for margins method is used to estimate the unknown parameters. As an illustration of the proposed model, a numerical example about an actual train wheels wear degradation data is presented.

The rest of the paper is organized as follows. In Section 2, copula function is described. Then, the system reliability model via the marginal distributions and copula functions is introduced in Section 3. In Section 4, the estimation of unknown parameter based on the inference for margins method is obtained. A numerical example is given in Section 5. Finally, some conclusions are made in Section 6.

II. Copula function brief introduction

Copula is a function that couples the joint distribution function and their marginal distribution functions together, which is a powerful tool to model the dependence of multivariate (see in Nelson[18]).

Definition [18] A two dimensional copula function, $C(\cdot,\cdot)$, has the following properties:

(1) The domain of definition is $[0,1]^2$.

(2) Function, $C(\cdot,\cdot)$, has zero-grounded and it is two increasing.

(3) Any variables u, $v\in[0,1]$, then $C(u,1)= u$ and $C(1,v)= v$.

Theorem[18] Let X and Y be random variables with continuous distribution $F(x)$ and $G(y)$, respectively, and $H(x, y)$ be the two dimensional cumulative distribution function. Then, there exists a two dimensional copula $C(\cdot,\cdot)$ such that for all $x, y\in(-\infty,+\infty)$,

$$H\left(x,y\right)=C\left(F(x),G(y)\right) \tag{1}$$

Archimedean copulas have a wide range of applications because they have many nice properties, such as they can be constructed easily, they can be easily extended from 2-dimension to m-dimension when satisfying some

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conditions, and so on. In this paper, the Gumbel copula, Clayton copula and Frank copula are used to depict the dependence among multiple PCs, where those three copulas are belong to Archimedean copula family.

The definition of the Gumbel Copula function is

$$C(u,v;\theta)=\exp\left\{-\left[\left(-\log u\right)^{\frac{1}{\theta}}+\left(-\log v\right)^{\frac{1}{\theta}}\right]^{\theta}\right\} \quad (2)$$

where θ is the Gumbel copula parameter and $\theta\in(0,1]$. And the relationship between Kendall's τ and the Gumbel copula parameter θ is given by

$$\tau=1-\theta \quad (3)$$

The definition of the Clayton Copula function is

$$C(u,v;\gamma)=\max((u^{-\gamma}+v^{-\gamma}-1)^{-1/\gamma},0) \quad (4)$$

where γ is the Clayton copula parameter and $\gamma\in[-1,+\infty)\setminus\{0\}$. And the relationship between Kendall's τ and the Clayton copula parameter θ is given by

$$\tau=\frac{\gamma}{2+\gamma} \quad (5)$$

The definition of the Frank Copula function is

$$C(u,v)=-\frac{1}{\alpha}\ln\left\{1+\frac{[\exp(-\alpha u)-1][\exp(-\alpha v)-1]}{\exp(-\alpha)-1}\right\} \quad (6)$$

where α is the Frank copula parameter and $\alpha\in(-\infty,0)\bigcup(0,+\infty)$. And the relationship between Kendall's τ and the Frank copula parameter α is given by

$$\tau=1+4\frac{D_1(\alpha)-1}{\alpha} \quad (7)$$

where $D_1(\alpha)=(1/\alpha)\int_0^{\alpha}\left(t/(e^t-1)\right)dt$ is a Debye function.

III. System reliability model via the marginal distributions and copula function

A. Marginal reliability model based on the approximation approach

The approximation approach comprises two steps. In the first step, when the unit's degradation path reaches the failure threshold, the pseudo failure time can be predicted. In the second step, the failure lifetime distribution can be obtained by fitting these pseudo lifetime data.

Suppose that a product has two PCs, and each PC is treated as a linear degradation path model $X_k(t)=D_k(t;\alpha,\beta)+\varepsilon$, $\varepsilon\sim N(0,\sigma^2)$, $k=1,2$. Let ξ_k be the threshold value of the kth PC, then the formally approximation method is consists of the following steps:

a) Based on the kth degradation path $X_k(t)=D_k(t;\alpha,\beta)+\varepsilon$, by using different estimation methods to obtain the estimate of parameter α_k and β_k, say as $\hat{\alpha}_k$ and $\hat{\beta}_k$.
b) By solving the equation $D_k(t;\hat{\alpha}_k,\hat{\beta}_k)=\xi_k$, the pseudo failure time for each unit in the kth PC can be obtained.
c) Some distributions are used to fit these pseudo lifetime data, such as Exponential distribution, Normal distribution and Weibull distribution. The best fitted distribution $F_k(t)$ can be determined by using the Anderson-Darling test statistic.
d) By using the fitted distribution $F_k(t)$, the reliability function $R_k(t)$ of the kth PC can be obtained.

B. Degradation model based on bivariate degradation data

Copula function is a powerful tool to model the dependence structure among multiple PCs. One advantage of copula function is that the joint distribution function can be modeled directly through the univariate marginal distribution function.

Suppose that $F_k(t)=1-R_k(t)$ is the distribution function of lifetime T_k for each PC, and let $H(t_1,t_2)$ be the joint copula of the marginal distribution function. According to the Theorem, there exists a unique copula C such that

$$P(T_1\le t,T_2\le t)=H(t_1,t_2)=C\left(F_1(t),F_2(t);\theta\right)$$

where θ is the parameter vector of the copula function.

It is noticed that the product is considered to be failed if any one PC reaches its corresponding failure threshold. Therefore, the product still works when each PC keeps below its failure thresholds. Given the failure time T_k of the kth PC, suppose that the lifetime of the system is T, and $T=\min(T_1, T_2)$. Then, the product reliability can be written as follows

$$\begin{aligned}R(t)&=P(T>t)=P(T_1>t,T_2>t)\\&=1-P(T_1\le t)-P(T_2\le t)+P(T_1\le t,T_2\le t)\\&=1-F_1(t)-F_2(t)+C\left(F_1(t),F_2(t);\theta\right)\\&=R_1(t)+R_2(t)-1+C\left(F_1(t),F_2(t);\theta\right)\end{aligned} \quad (8)$$

If the product has two PCs linked by Gumbel copula function in Equation (2), then, we can obtain the system reliability function as

$$\begin{aligned}R(t)&=R_1(t)+R_2(t)-1\\&-\exp\left\{-\left[\left(-\log F_1(t)\right)^{\frac{1}{\theta}}+\left(-\log F_2(t)\right)^{\frac{1}{\theta}}\right]^{\theta}\right\}\end{aligned} \quad (9)$$

If the two degradation failure mechanisms are assumed to be independent, the product reliability in Equation (4) can be rewritten as

$$\begin{aligned}R(t)&=\Pr(T_1>t,T_2>t)=\Pr\left(T_1>t\right)\times\Pr\left(T_2>t\right)\\&=(1-F_1(t))\times(1-F_2(t))=R_1(t)\times R_2(t)\end{aligned} \quad (10)$$

IV. Statistical inferential methods for unknown parameters

The inference for margins (IFM) approach is effect parameter estimation method for copula function. The IFM approach first estimates the marginal distribution parameters separately, and then estimates the parameters of the copula function via the estimated marginal parameters. More details information can be found in Cherubini et al. (2004).

Let $X=\{x_{1t},x_{2t},\cdots,x_{Kt}\}_{t=1}^{T}$ be the data matrix, drawn from the marginal distribution function $F_i(x_i;\lambda_i)$ with copula C. The density function for the joint distribution can be obtained as

$$f\left(x_1,x_2,\cdots,x_K;\lambda\right)=c\left(F_1(x_1;\lambda_1),\cdots,F_K(x_K;\lambda_K);\lambda_c\right)\cdot\prod_{i=1}^{K}f_i\left(x_i;\lambda_i\right) \quad (11)$$

where

$$c\left(F_1(x_1;\lambda_1), F_2(x_2;\lambda_2), \cdots, F_K(x_K;\lambda_K);\lambda_c\right) = \frac{\partial^K C\left(F_1(x_1;\lambda_1), F_2(x_2;\lambda_2), \cdots, F_K(x_K;\lambda_K);\lambda_c\right)}{\partial F_1(x_1;\lambda_1)\partial F_2(x_2;\lambda_2)\cdots\partial F_K(x_K;\lambda_K)} \tag{12}$$

Therefore, the log-likelihood function is in the form of

$$l(\lambda^*) = \sum_{t=1}^{T} \ln c\left(F_1(x_1;\lambda_1), F_2(x_2;\lambda_2), \cdots, F_K(x_K;\lambda_K);\lambda_c\right) + \sum_{t=1}^{T}\sum_{i=1}^{K} \ln f_i(x_{it};\lambda_i) \tag{13}$$

where λ^* is composed of all the parameters, and $\lambda^*=(\lambda_1, \lambda_2, \cdots, \lambda_K, \lambda_c)$.

The IFM method comprises two stages. The first stage is to calculate the parameters in marginal functions; and in the second stage, the maximum likelihood method is used to estimate the parameters of the joint copula function.

Stage 1

$$\begin{aligned} \hat{\lambda}_1 &= \arg\max_{\lambda_1} \sum_{t=1}^{T} \ln f_1(x_{1t};\lambda_1) \\ \hat{\lambda}_2 &= \arg\max_{\lambda_2} \sum_{t=1}^{T} \ln f_2(x_{2t};\lambda_2) \\ &\vdots \\ \hat{\lambda}_K &= \arg\max_{\lambda_K} \sum_{t=1}^{T} \ln f_K(x_{Kt};\lambda_K) \end{aligned} \tag{14}$$

Stage 2

$$\hat{\lambda}_c = \arg\max_{\lambda_c} \sum_{t=1}^{T} \ln c\left(F_1(x_{1t};\hat{\lambda}_1), \cdots, F_K(x_{Kt};\hat{\lambda}_K);\lambda_c\right) \tag{15}$$

It is noticed that, by using the IFM method, it is easy to estimate the unknown parameters in the copula function, especially for the one-parameter copula function.

V. NUMERICAL EXAMPLE

Train wheel failures will lead the catastrophic events and tremendous economic loss. To avoid catastrophic events, many railway companies are periodically monitoring the wear degradation of wheels for the purpose of preventive maintenance scheduling, and the wear degradation database is store for the maintenance actions performed on their trains. In this paper, the degradation data is collected from a study by a Brazilian railway company.

The actual train wheels wear degradation data are taken from Freitas et al. [17]. The diameter of the wheel is an important performance for train wheel. Usually when the diameter exceeds to a predefined threshold level, the train wheel is considered to be failed. The diameter of a new train wheel is 966 mm. When the diameter reaches 889 mm, the train wheel is replaced by a new one. In the original data, 14 samples are tested for train wheels wear degradation data and the measurements are taken at the same measurement times. The measured frequency of its mileage is 50,000km. For demonstrating the bivariate degradation model, we choose 14 samples and the data will be treated as if half of it is the first PC (left wheel) and the other half represents the second PC (right wheel). The used data are the data measured only until 600,000km, or until the wheel fails, whichever comes first. Fig.1 and Fig.2 show the cumulative degradation of the train wheels. Instead of plotting the diameters itself, the curves are constructed by using the degradation observed at time t (i.e., 966-[observed diameter measure at time t]). Failure of the train wheel is then defined to occur when the degradation reaches the failure threshold level 77mm.

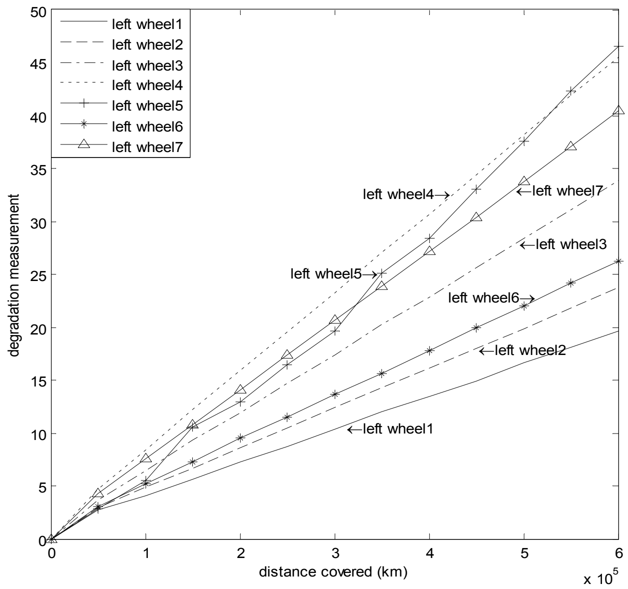

Fig.1. The degradation data of left wheel

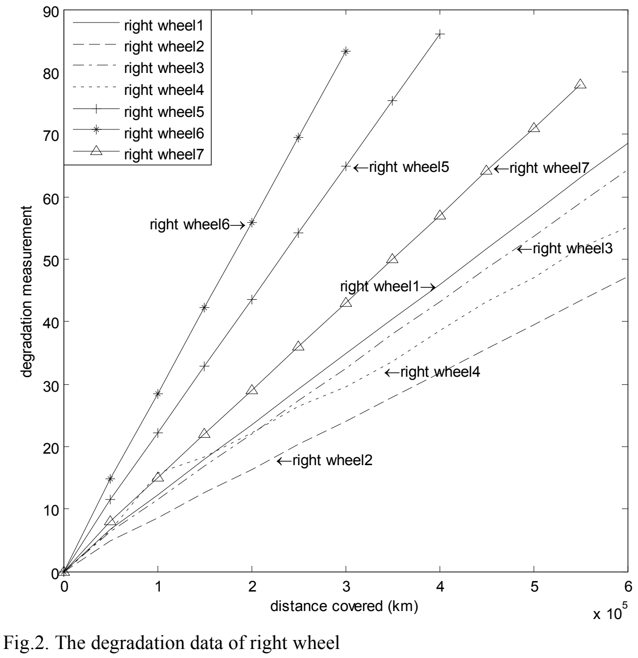

Fig.2. The degradation data of right wheel

A. Marginal distribution via the approximation method

From the Fig.1 and Fig.2, we choose linear degradation path model to describe the degradation path of train wheel wear data. Suppose that train wheel system has two PCs, and each PC is treated as a linear degradation path model. Let $X_k(t_{ij})$ be the degradation observed at time t_{ij} of the unit i in the kth PC, then the degradation path model can be given as

$$X_k(t_{ij}) = D_k(t_{ij};\alpha_k,\beta_k) + \varepsilon = \alpha_k + \beta_k t_{ij} + \varepsilon,\ k = 1, 2. \tag{16}$$

By using the least squares estimator method, parameters (α_k, β_k) of the kth degradation path model can be estimated. By extrapolating the model of each unit to the critical failure threshold, the pseudo failure times the kth PC can be obtained as Table I and Table II, where the Table I displays the calculation result of the PC1 (left wheel) and Table II displays the result of the PC2 (right wheel).

TABLE I
THE PSEUDO FAILURE TIME OF LEFT WHEELS

wheel	α_1	β_1	Pseudo failure time (km)
1	3.1754E-5	0.7522	2.4011e+006
2	3.8408E-5	0.7737	1.9845e+006
3	5.5429E-5	0.7161	1.3762e+006
4	7.4905E-5	0.7322	1.0181e+006
5	7.8219E-5	-1.8736	1.0083e+006
6	4.2659E-5	0.7367	1.7877e+006
7	6.6296E-5	0.6882	1.1511e+006

TABLE II
THE PSEUDO FAILURE TIME OF RIGHT WHEELS

wheel	α_2	β_2	Pseudo failure time (km)
1	1.1328E-4	0.7288	6.7325e+5
2	0.7777E-4	0.6923	9.8111e+5
3	1.0616E-4	0.7051	7.1867e+5
4	2.1381E-4	0.6186	3.5723e+5
5	2.7638E-4	0.5562	2.7658e+5
6	0.8756E-4	3.5934	8.3835e+5
7	1.4073E-4	0.7034	5.4213e+5

The next step of the approximation method is to obtain the probability distribution of the pseudo failure times. In this paper, three type distribution functions (Exponential, Normal and Weibull) are selected to fit the pseudo failure times, and the fitting results are justified by the Anderson-Darling (A-D) test statistic. The A-D test statistic is quantitatively defined as follows[20]:

$$A_n = n\int_{-\infty}^{+\infty}\frac{[G_n(x)-G(x)]^2}{G(x)[1-G(x)]}dG(x) \tag{17}$$

where n is the number of sample, $G_n(x)$ is the empirical distribution function of sample, and $G(x)$ is the hypothesized continuous distribution function. The judgment standard of this method is to compare the A-D value of proposed fitting models and the lowest A-D value means the best distribution fitting. According to the A-D value shown in Table III and Table IV, normal distribution with lowest A-D value presents the best fitting performance among those distributions, and by using the maximum likelihood method, the parameter estimation results of the corresponding distribution are shown in Table III and Table IV.

TABLE III
THE PARAMETER ESTIMATION RESULTS OF LEFT WHEELS

Model	Exponential	Normal	Weibull
Parameters	Scale:1532470	Local:153247 Scale:537472	Shape:3.35161 Scale:1711620
p-value	0.020	0.403	0.250
A-D value	1.552	0.330	0.348

TABLE IV
THE PARAMETER ESTIMATION RESULTS RIGHT WHEELS

Model	Exponential	Normal	Weibull
Parameters	Scale:626763	Local:626763 Scale:252775	Shape:3.02439 Scale:704139
p-value	0.048	0.901	0.254
A-D value	1.236	0.164	0.259

B. Estimation of unknown parameters

From Table III and Table IV, we know that the marginal distributions of PCs follow Normal distribution with local parameter μ_i and scale parameter σ_i, and the cumulative distribution function as follow

$$F_1(t)=\Phi\left(\frac{t-\mu_1}{\sigma_1}\right)=\Phi\left(\frac{t-1532470}{537472}\right),$$

$$F_2(t)=\Phi\left(\frac{t-\mu_2}{\sigma_2}\right)=\Phi\left(\frac{t-626763}{252775}\right)$$

From the Equation (9), we know that the probability density function of Gumbel Copula C can be obtained as

$$\begin{aligned}c(u_1,u_2;\theta)=&\frac{\left[\log F_1(t)\cdot\log F_2(t)\right]^{\frac{1}{\theta}-1}}{F_1(t)\cdot F_2(t)\left[\left(-\log F_1(t)\right)^{\frac{1}{\theta}}+\left(-\log F_2(t)\right)^{\frac{1}{\theta}}\right]^{2-\theta}}\\&\times\exp\left\{-\left[\left(-\log F_1(t)\right)^{\frac{1}{\theta}}+\left(-\log F_2(t)\right)^{\frac{1}{\theta}}\right]\right\}^{\theta}\\&\times\left\{\left[\left(-\log F_1(t)\right)^{\frac{1}{\theta}}+\left(-\log F_2(t)\right)^{\frac{1}{\theta}}\right]^{\theta}+\frac{1}{\theta}-1\right\}\end{aligned}$$

Then, the log-likelihood function can be given as

$$\begin{aligned}\log L(\underset{j=1,2,\cdots,w}{t_{1j}},\underset{j=1,2,\cdots,w}{t_{2j}};\theta)=&\left(\frac{1}{\theta}-1\right)\sum_{j=1}^{w}\log\left\{\left[-\prod_{i=1}^{2}\log F_i(t_{ij})\right]\right\}-\sum_{j=1}^{w}\log F_1(t_{1j})\\&-\sum_{j=1}^{w}\log F_2(t_{2j})+\log\left\{\left[\sum_{i=1}^{2}(-\log F_i(t_{ij}))^{\frac{1}{\theta}}\right]^{\theta}+\frac{1}{\theta}-1\right\}\\&-\sum_{j=1}^{w}\left\{\sum_{i=1}^{2}\left[-\log F_i(t_{ij})\right]^{\frac{1}{\theta}}\right\}^{\theta}-\sum_{j=1}^{w}\log\left[\sum_{i=1}^{2}(-\log F_i(t_{ij}))^{\frac{1}{\theta}}\right]^{2-\theta}\end{aligned}$$

By using the maximum likelihood method, the estimator of unknown parameter θ in the Gumbel copula function can be obtained.

TABLE V
THE PARAMETER ESTIMATION OF COPULAS

Copula	Unknown parameter
Gumbel Copula	$\hat{\theta}=0.1371$
Clayton Copula	$\hat{\gamma}=1.3354$
Frank Copula	$\hat{\alpha}=243.75$

Similarly, the unknown parameter γ in the Clayton copula and the unknown parameter α in the Frank copula can be also obtained, and the estimated results are presented in Table V.

C. Train wheels system reliability assessment

Based on the estimated results of the unknown parameters, the reliability curves of PC1 (left wheel) and PC2 (right wheel) are presented in Fig.3. It can be concluded from the Fig.3 that there is larger difference between the reliability of left wheel and the right wheel.

Furthermore, the system reliability curves under the independent and dependent case (with Gumbel copula、Clayton copula and Frank copula assumption) are plotted in Fig.4. From the Fig.4, we can obtain the following two conclusions. Firstly, when the dependent of two PCs is described by three different copula functions, the system has same reliability curves. This means that the wheel system is not sensitive to the choice of different copula functions. Secondly, there are some differences between the dependent and independent cases. That is to say that ignoring the dependence between left wheel and right wheel may result in different reliability conclusion. Therefore, it is necessary to analyze the possibility of the failure mechanisms dependency and perform the dependent reliability analysis.

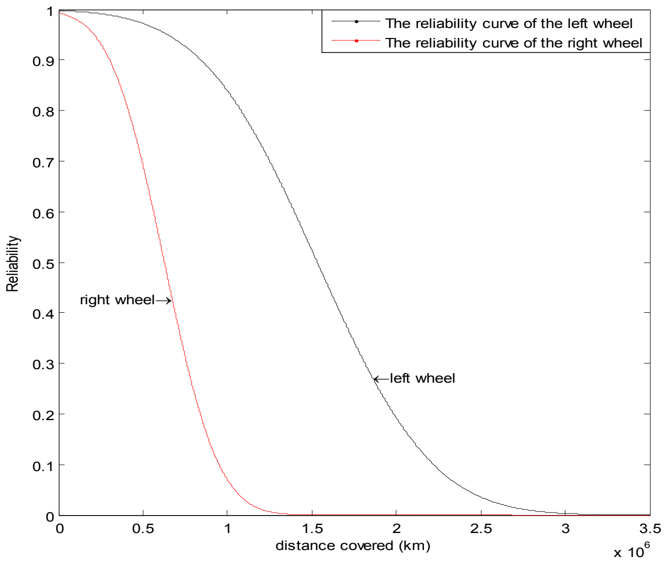

Fig.3. Reliability curves of PC1 (left wheel) and PC2 (right wheel)

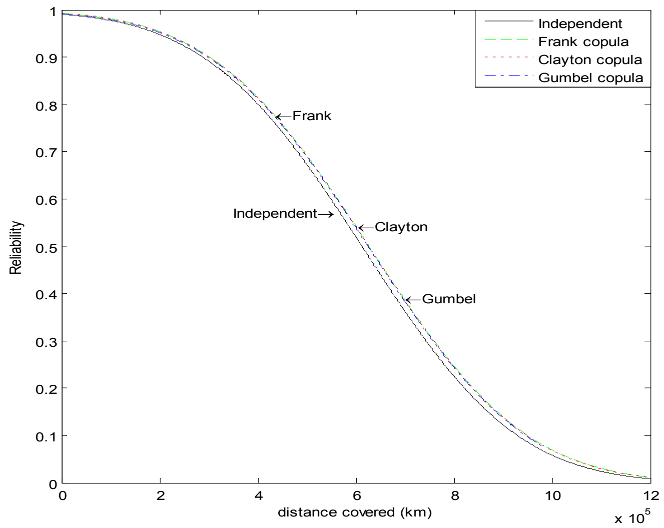

Fig.4. Reliability curves of system based on dependent and independent cases

VI. Conclusion

In this paper, we establish a reliability assessment model for the train wheel system with two PCs, and each PC is treated as a linear degradation model. We suppose that the two PCs are dependent and the dependency is described by different copula functions. The inference for margins method is used to obtain the estimator of unknown parameters. From the numerical example of Section 5, we know that ignoring the dependence between PCs may result in different reliability conclusion.

In this work, we have only considered the bivariate Normal distributions, but the results can be extended to other bivariate distributions. For example, one PC could be modeled by a Normal distribution, while the other could be described by a Weibull distribution. Moreover, from the practice point of view, how to make effective maintenance decisions for the products with two PCs based on the proposed estimation results is necessary to be studied in the future.

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