Two-grid Method for Characteristics Finite Volume Element of Nonlinear Convection-dominated Diffusion Equations

Lingzhi Qian, Huiping Cai*

Abstract—**A characteristics finite volume element discretization technique based on two subspaces is presented for a nonlinear convection-dominated diffusion equations. The solution of a nonlinear system on the fine space is composed of solving one small (nonlinear) system on the coarse space and a linear system on the fine space. Error estimates are derived and numerical experiments are performed to validate the accuracy and efficiency of our present scheme. It is shown both theoretically and numerically, that the new scheme is efficient to the nonlinear convection-dominated diffusion equations.**

Index Terms—**Characteristics finite volume element; Two-grid; Newton iterative; Error estimates.**

I. Introduction

CONVECTION -dominated diffusion problems often occur in underground fluid flow, exploiting oil-gas resources, heat transfer problems and enviorment science. In actual numerical computation, the nonlinear convection-dominated diffusion equations are very important. In this paper, we consider the nonlinear partial differential equations with initial-boundary value problem of the form

$$c(x)\frac{\partial u}{\partial t}+b(x)\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)$$
$$=f(u,x,t)\ \ (x,t)\in I\times(0,T], \tag{1a}$$
$$u\,(x,t)=0\ \ (x,t)\in\partial I\times(0,T], \tag{1b}$$
$$u\,(x,t)=u_0\ \ (x)\,x\in I. \tag{1c}$$

The convection-dominated diffusion equation has strong hyperbolic characteristics, therefor the numerical simulation is very difficult in mechanics and mathematics. Douglas and Russell considered combining the method of characteristics with finite element or finite difference techniques to overcome oscillation and faults likely to occur in the traditional finite difference or finite element method [10].

On the other hand, finite volume element method has been widely used in the approximation for the conservation laws, computational fluid dynamics and nonlinear convection-diffusion problems, see *e.g.* [2], [6], [11], [12], [15], [20], [23]. The method is more popular due to its conservation property of the original problem locally. The finite volume element method uses a volume integral formulation of the original problem and a finite element partition of the spatial domain to discrete the equations [18]. The finite volume element method for convection-diffusion problems has been studied by many authors [13], [14], [16], [24]. Recently, P. Chatzipantelidis has studied the finite volume element method for elliptic and parabolic problems and derived optimal H^1 and L^2 error estimate [4], [5]. The high order locally conservative finite volume element method is considered by M. Plexousakis et al. But the extension to two dimensional case of the current research is the "battle field" [22]. The basic tools and the theoretical framework for the finite volume element method are described in [7], [8], [17], [18], [19] and references therein for details.

When using the characteristics finite volume element method to discrete in space for the nonlinear equation, we can obtain the nonlinear system. The solution of a nonlinear system on the fine space is much expensive. Inspired by Xu [21], [25], [26], [27] for a technique to solve nonlinear equations, we employ two-grid approach in our discretization schemes. The solution of a nonlinear system on the fine space is reduced to the solution of a nonlinear system on the coarse space and a linear system on the fine space.

The remainder of the paper is organized as follows. In section 2, we introduce notations and preliminaries. The algorithm is described in section 3. Section 4 is devoted to the error estimates for the new scheme. Numerical experiments confirming the theoretical results are provided in section 5. Finally, the last section presents the conclusions and future research directions.

Throughout this paper, C denotes a generic positive constant independent of $\Delta t, h, H$ which may be different at different occurrences.

II. Notations and Preliminaries

Given a domain $\Omega\subset R$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space [1]. The norm of $v\in W^{m,p}(\Omega)$ is defined as follows

$$\|v\|_{m,p,\Omega}=\left(\sum_{j\le m}\|D^j v\|^p_{L^p(\Omega)}\right)^{1/p}\ \ 1\le p\le\infty,$$

and

$$\|v\|_{m,\infty,\Omega}=\max_{j\le m}\ esssup|D^j v|\ \ p=\infty.$$

Denote

$$W^{m,2}(\Omega)=H^m(\Omega),\|v\|_{H^m(\Omega)}=\|v\|_m,\|v\|_{L^2(\Omega)}=\|v\|.$$

Manuscript received 8 November, 2015; revised 25 April, 2016. This work was supported in part by the Jiangsu Province Ordinary University Graduate Student Scientific Research Innovation Projects (KYLX15_0720), the Science and Technology Research and Development Applied Basic Research Youth project of Shihezi University (2014ZRKXYQ-LH06) and the Training Plan for Youth Backbone Teacher of Shihezi University.

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We assume that the coefficients a, b, c are bounded and the solution u of (1a)–(1c) satisfies:

$$0 \le a_0 \le a(x) \le \infty, 0 \le c(x) \le \infty, \text{and}$$
$$\mid \frac{b(x)}{c(x)} \mid + \mid \frac{d}{dx}(\frac{b(x)}{c(x)}) \mid \le C, x \in I, \tag{2a}$$
$$\mid \frac{\partial f}{\partial u} \mid + \mid \frac{\partial^2 f}{\partial u^2} \mid \le C, x \in I. \tag{2b}$$
$$u \in L^\infty(0,T;H^q(I)), \tag{2c}$$
$$\frac{\partial u}{\partial t} \in L^2(0,T;H^{q-1+\theta}(I)),$$
$$\theta = 1 \text{ if } q = 2, \text{ and } \theta = 0 \text{ if } q > 2, \tag{2d}$$
$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(I)). \tag{2e}$$

Let T_h be a partition of the interval $I = [a, b]$ such that

$$T_h : a = x_0 \le x_1 \le x_2 \le \cdots \le x_N = b,$$

$$I_j = [x_{j-1}, x_j], h_j = x_j - x_{j-1}, j = 1, \cdots, N,$$

where the midpoint is defined by $x_{j-\frac{1}{2}} = (x_{j-1} + x_j)/2$.

Next, we set

$$I_0^* = [x_0, x_{\frac{1}{2}}], I_N^* = [x_{N-\frac{1}{2}}, x_N],$$

$$I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], (i = 1, 2, \cdots, N-1),$$

then all $I_i^*(i = 0, 1, 2, \cdots, N)$ compose the dual partition T_h^* of T_h. All $I_i^*, (i = 0, 1, 2, \cdots, N)$ are called control volumes.

Denote

$$\sigma = \{I_j : 1 \le j \le N\},$$

$$S_\sigma^{(k)(I)} = \{v \in L_2(I) : v|_{I_j} \in P_k, j = 1, 2, \cdots, N-1\},$$

where P_k denotes the set of polynomial of degree less than or equal to k.

$$S^{(k)}(I) = \bigcup_\sigma S_\sigma^{(k)}(I).$$

Let

$$S_{\sigma,E}^{(k)}(I) = \{v \in S_\sigma^{(k)}(I) : v(a^+) = v(b^-) = 0\},$$

$$S_E^{(k)}(I) = \bigcup_\sigma S_{\sigma,E}^{(k)}(I).$$

Let U_h be the trial space defined on T_h ([3], [9]),

$$U_h = \{v \in C(I) : v|_{I_i} \text{ is linear and } v(a) = v(b) = 0\},$$

and V_h be the test space defined on T_h^*,

$$V_h = \{v \in L^2(I) : v|_{I_j^*} \text{ is constant and } v(a) = v(b) = 0\}.$$

$$H_0^1(I) = \{v | v \in H^1(I), I = [a, b], v(a) = v(b) = 0\}.$$

III. DESCRIPTION OF THE ALGORITHM

A. Characteristics Finite Volume Element Method

Let

$$\psi(x) = [c^2(x) + b^2(x)]^{1/2}, \tag{3}$$

and let the characteristic direction associated with the operator $cu_t + bu_x$ be denoted by $\tau = \tau(x)$, where

$$\frac{\partial}{\partial \tau(x)} = \frac{c(x)}{\psi(x)}\frac{\partial}{\partial t} + \frac{b(x)}{\psi(x)}\frac{\partial}{\partial x}. \tag{4}$$

Then, equation (1a) – (1c) can be written in the equivalent form

$$\psi(x)\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x}(a(x)\frac{\partial u}{\partial x}) = f \;\; x \in I, \, t \in (0,T], \tag{5a}$$
$$u(x,t) = 0 \;\; x \in \partial I \;\; t \in (0,T], \tag{5b}$$
$$u(x,0) = u_0(x) \;\; x \in I. \tag{5c}$$

Multiplying the equation of (5a) by any $v \in V = S_E^{(k)}(I)$ and applying Green formula, the variational problem in accordance with (5a)–(5c) is: find $u \in H_0^1 \cap H^{r+1}$ such that

$$(\psi\frac{\partial u}{\partial \tau}, v) + A(u,v) = (f(u), v) \;\forall\; v \in V, \tag{6a}$$
$$u(x,0) = u_0(x), \tag{6b}$$

where

$$A(u,v) = \int_I a\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\,dx$$
$$= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} a\,u'\,v'\,dx + \sum_{i=1}^{N-1} u'(x_i)\,[v(x_i^+) - v(x_i^-)].$$

We consider a time step Δt and approximate the solution at times $t^n = n\Delta t, n = 1, 2, \cdots, M, \; \Delta t = \frac{T}{M}$. The characteristic derivative will be approximated basically in the following manner:

$$\psi(x)\frac{\partial u}{\partial \tau} \approx \psi(x)\frac{u(x,t^n) - u(\bar{x}, t^{n-1})}{[(x-\bar{x})^2 + (\Delta t)^2]^{1/2}}$$
$$= c(x)\frac{u(x,t^n) - u(\bar{x}, t^{n-1})}{\Delta t}, \tag{7}$$

where $\bar{x} = x - b(x)\Delta t/c(x)$.

The characteristics finite volume element method for (6a) and (6b) is defined as follows: for any $v \in V_h$, find $u_h^n \in U_h$ $(n = 1, 2, \cdots, M)$ such that

$$(c\frac{u_h^n - \bar{u}_h^{n-1}}{\Delta t}, v) + A(u_h^n, v) = (f(u_h^n), v), \tag{8a}$$
$$u_h^0 = u_{0h}, \tag{8b}$$

where

$$u_h^n = u_h(t^n), \quad \bar{u}_h^{n-1} = u_h(\bar{x}, t^{n-1}) = u_h(x - \frac{b(x)}{c(x)}\Delta t, t^{n-1}).$$

It is obvious that (8a) and (8b) determine $\{u_h^n\}$ uniquely in terms of that data u_0 and f.

B. Two-grid Technique Based on the Newton Method

To solve the nonlinear system (8a) and (8b), we use the Newton iterative method. The basic ingredient in our approach is the coarse grid $U_H(\subset U_h \subset H_0^1(I))$ defined on a coarser quasi-uniform partition (with mesh size $H > h$) of I. With the heuristic method of Taylor expansion, we have

$$\begin{cases} u_h^n \approx \widehat{u}_h^n = u_H^n + e_h^n + e_H^n, \\ f(u_h^n) \approx f'(u_H^n)e_h^n + \frac{1}{2}f''(u_H^n)(e_h^n)^2. \end{cases} \tag{9}$$

Then, the two-grid algorithm can be given as follows:
Algorithm : Find $\widehat{u}_h^n = u_H^n + e_h^n + e_H^n$ such that

$$1.\ u_H^n \in U_H, \begin{cases} (c\dfrac{u_H^n - \bar{u}_H^{n-1}}{\Delta t}, v) + A(u_H^n, v) \\ = (f(u_H^n), v) \ \ \forall v \in V_H, \\ u_H^0 = u_{0H}. \end{cases} \tag{10}$$

$$2.\ e_h^n \in U_h, \begin{cases} (c\dfrac{e_h^n - \bar{e}_h^{n-1}}{\Delta t}, v) + A(e_h^n, v) \\ -(f'(u_H^n)e_h^n, v) = (f(u_H^n), v) \\ -(c\dfrac{u_H^n - \bar{u}_H^{n-1}}{\Delta t}, v) \\ -A(u_H^n, v) \ \ \forall v \in V_h, \\ e_h^0 = e_{0h}. \end{cases} \tag{11}$$

$$3.\ e_H^n \in U_H, \begin{cases} (c\dfrac{e_H^n - \bar{e}_H^{n-1}}{\Delta t}, v) + A(e_H^n, v) \\ -(f'(u_H^n)e_h^n, v) \\ = \frac{1}{2}(f''(u_H^n)(e_h^n)^2, v) \ \forall v \in V_H, \\ e_H^0 = e_{0H}. \end{cases} \tag{12}$$

Nonlinear system (6a) and (6b) on the fine grid is reduced to solve a small nonlinear system on coarse-grid ΔH and a linear system on the fine-grid Δh. In step 3, a further coarse grid correction is performed, the linearized operator used in this step is based on the first coarse grid approximation U_H^n, such a correction indeed improves the accuracy of the approximation solution.

IV. Error Estimates

A. Error Estimates for the Characteristics Finite Volume Element Method

Define interpolating operator $\Pi_h : H_0^1 \to U_h$ by

$$\Pi_h u = \sum_{j=1}^{N-1} u(x_j)\phi_j(x),$$

and interpolating operator $\Pi_h^* : H_0^1 \to V_h$ by

$$\Pi_h^* u = \sum_{j=1}^{N-1} u(x_j)\omega_j(x),$$

where $\phi_j(x)$ *and* $\omega_j(x)$ is the basis of U_h *and* V_h, respectively.

The following approximation property is well known [3], [9].

$$\|v - \chi\|_{0,p} + h\|v - \chi\|_{1,p} \le c\|v\|_{2,p}h^2 \quad \forall v \in H^2 \cap H_0^1,\ 2 \le p < \infty. \tag{13}$$

Set $\omega_h: [0, T] \to U_h$, $\eta = u - \omega_h$, $\quad \xi = u_h - \omega_h$, *then* $u - u_h = \eta - \xi$. It is well known [10], for $p = 2$ or ∞ and $1 \le s \le r + 1$

$$\|\eta\|_{L^p(0,T;L^2(I))} + h\|\eta\|_{L^p(0,T;H^1(I))} \le C\|u\|_{L^p(0,T;H^s(I))}h^s, \tag{14}$$

for $p = 2$ and $1 \le s \le r + 1$

$$\|\frac{\partial \eta}{\partial t}\|_{L^2(0,T;H^{-1}(I))} + h\|\frac{\partial \eta}{\partial t}\|_{L^2(0,T;L^2(I))} \le C\|\frac{\partial u}{\partial t}\|_{L^2(0,T;H^{s-1}(I))}h^{s-1}. \tag{15}$$

Now we estimate the errors bounds for $u - u_h$. It is sufficient to estimate ξ for the relations of (14) and (15).

Theorem 4.1: Let u and u_h be the respective solution of (6a)–(6b) and (8a)–(8b), under assumptions (2a)–(2e), we have the error estimate

$$\max_{1\le n\le M} \|u^n - u_h^n\| \le C\Big(\Delta t\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^2(0,T;L^2)} + h^2\big(\|u\|_{L^\infty(0,T;H^2(I))} + \|\frac{\partial u}{\partial t}\|_{L^2(0,T;H^2(I))}\big)\Big) \le C\big(\Delta t + h^2\big). \tag{16}$$

Proof. At $t = t^n$, a calculation shows that for any $v \in V_h$, we have

$$\begin{aligned} &(c\frac{\xi^n - \xi^{n-1}}{\Delta t}, \Pi_h^*\xi^n) + A(\xi^n, \Pi_h^*\xi^n) \\ &= ((\psi\frac{\partial u}{\partial \tau})^n - c\frac{u^n - \bar{u}^{n-1}}{\Delta t}, \Pi_h^*\xi^n) \\ &+(c\frac{\eta^n - \eta^{n-1}}{\Delta t}, \Pi_h^*\xi^n) + (c\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, \Pi_h^*\xi^n) \\ &-(c\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, \Pi_h^*\xi^n) \\ &+(f(u^n) - f(u_h^n), \Pi_h^*\xi^n). \end{aligned} \tag{17}$$

By the Taylor expansion at any point to give

$$f(u^n(x)) - f(u_h^n(x)) = f'(\widetilde{u}_h^n(x))(u^n(x) - u_h^n(x)), \tag{18}$$

for some value $\widetilde{u}_h^n(x)$. Then we have

$$\begin{aligned} &(f(u^n(x)) - f(u_h^n(x)), \Pi_h^*\xi^n) \\ &= (f'(\widetilde{u}_h^n(x))(u^n(x) - u_h^n(x)), \Pi_h^*\xi^n) \\ &= (f'(\widetilde{u}_h^n(x))\eta^n, \Pi_h^*\xi^n) - (f'(\widetilde{u}_h^n(x))\xi^n, \Pi_h^*\xi^n). \end{aligned} \tag{19}$$

Combining (17)–(19), we know that

$$\begin{aligned} &(c\frac{\xi^n - \xi^{n-1}}{\Delta t}, \Pi_h^*\xi^n) + A(\xi^n, \Pi_h^*\xi^n) \\ &= ((\psi\frac{\partial u}{\partial \tau})^n - c\frac{u^n - \bar{u}^{n-1}}{\Delta t}, \Pi_h^*\xi^n) \\ &+(c\frac{\eta^n - \eta^{n-1}}{\Delta t}, \Pi_h^*\xi^n) + (c\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, \Pi_h^*\xi^n) \\ &-(c\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, \Pi_h^*\xi^n) + (f'(\widetilde{u}_h^n(x))\eta^n, \Pi_h^*\xi^n) \\ &-(f'(\widetilde{u}_h^n(x))\xi^n, \Pi_h^*\xi^n) \equiv \sum_{i=1}^{6} S_i. \end{aligned} \tag{20}$$

Let us introduce the discrete H_1 semi-norm and L_2 norm, for any $u_h \in U_h$

$$|u_h|_1 = \Big(\int_I (u_h')^2 dx\Big)^{1/2} = \Big(\sum_{j=1}^{N} (u_j - u_{j-1})^2/h_j\Big)^{1/2},$$

$$\|u_h\| = \left(\int_I (u_h)^2 dx\right)^{1/2} = \left(\sum_{j=1}^N (u_j)^2 h_j\right)^{1/2}.$$

We notice that the semi-norm $|.|_1$ and the norm $\|.\|_1$ are equivalent in the space H_0^1.

Lemma 4.1: The discrete bilinear form $A(\xi^n, \Pi_h^*\xi^n)$ is positive definite, i.e. there exists $\alpha_0 > 0$ independent of the space U_h such that

$$A(\xi^n, \Pi_h^*\xi^n) \geq \alpha_0\|\xi^n\|_1^2 \quad \forall\ \xi^n \in U_h. \tag{21}$$

Proof. First note that

$$\begin{aligned}
A(\xi^n, \Pi_h^*\xi^n) &= \sum_{j=1}^{N-1} \xi_j^n A(\xi_n, \omega_j)\\
&= \sum_{j=1}^{N-1} \xi_j^n \left[a_{j-\frac{1}{2}}\frac{(\xi_j^n - \xi_{j-1}^n)}{h_j} - a_{j+\frac{1}{2}}\frac{(\xi_{j+1}^n - \xi_j^n)}{h_{j+1}}\right]\\
&= \sum_{j=1}^{N-1} a_{j-\frac{1}{2}}\frac{(\xi_j^n)^2 - \xi_j^n\xi_{j-1}^n}{h_j}\\
&\quad -\sum_{j=2}^{N} a_{j-\frac{1}{2}}\frac{(\xi_j^n - \xi_{j-1}^n)\xi_{j-1}^n}{h_j}.
\end{aligned} \tag{22}$$

Next, taking the fact $\xi_0^n = \xi_N^n = 0$ in the above relation, we have

$$A(\xi^n, \Pi_h^*\xi^n) = \sum_{j=1}^{N} a_{j-\frac{1}{2}}\frac{(\xi_j^n - \xi_{j-1}^n)^2}{h_j}.$$

Then we complete the proof.

Lemma 4.2: Let Π_h^* be interpolating operator, then we get

$$(\xi^n, \Pi_h^*\xi^n) \geq \frac{1}{2}(\xi^n, \xi^n) \quad \forall\ \xi^n \in U_h. \tag{23}$$

We can estimate the left terms of (20) by Lemmas 4.1 and 4.2.

$$\begin{aligned}
&(c\frac{\xi^n - \xi^{n-1}}{\Delta t}, \Pi_h^*\xi^n) + A(\xi^n, \Pi_h^*\xi^n)\\
&\geq \frac{1}{4\Delta t}\left[(c\xi^n, \xi^n) - (c\xi^{n-1}, \xi^{n-1})\right] + a_0\|\xi^n\|_1^2.
\end{aligned} \tag{24}$$

and use the results provided in [10] to the right terms $S_1 - S_4$, we have

$$|S_1| \leq C\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^2(I\times[t^{n-1},\, t^n])}^2 \Delta t + \|\xi^n\|^2. \tag{25}$$

$$\begin{aligned}
|S_2| &\leq C\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} \|\frac{\partial \eta}{\partial t}\|_{-1} d\alpha \cdot \|\Pi_h^*\xi^n\|_1\\
&\leq \varepsilon\|\xi^n\|_1^2 + \frac{C}{\Delta t}\|\frac{\partial \eta}{\partial t}\|_{L^2(t^{n-1},t^n;H^{-1})}^2.
\end{aligned} \tag{26}$$

$$|S_3| \leq C\|\eta^{n-1}\|^2 + \frac{a_0}{4}\|\xi^n\|_1^2. \tag{27}$$

$$|S_4| \leq C\|\xi^{n-1}\|^2 + \frac{a_0}{4}\|\xi^n\|_1^2. \tag{28}$$

for any positive constant ε. Following the assumption (2b), we have

$$|S_5| \leq C\|\eta^n\|^2 + \|\xi^n\|^2. \tag{29}$$

$$|S_6| \leq C\|\xi^n\|^2. \tag{30}$$

Choosing proper ε, combing (25)–(30) to have the relation

$$\begin{aligned}
&\frac{1}{4\Delta t}\left\{(c\xi^n, \xi^n) - (c\xi^{n-1}, \xi^{n-1})\right\} + \frac{a_0}{4}\|\xi^n\|_1^2\\
&\leq C\left\{\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + \|\frac{\partial^2 u}{\partial \tau^2}\|_{L^2(t^{n-1},\, t^n;\, L^2)}^2 \Delta t\right.\\
&\left.+\frac{1}{\Delta t}\|\frac{\partial \eta}{\partial t}\|_{L^2(t^{n-1},t^n;H^{-1})}^2 + \|\eta^n\|^2 + \|\eta^{n-1}\|^2\right\}.
\end{aligned} \tag{31}$$

It is easy to see that $\xi^0 = 0$, multiplying (31) by $2\Delta t$, summing over n, and apply the discrete Gronwall Lemma, it follows that

$$\begin{aligned}
&\max_{1\leq n\leq M} \|\xi^n\|^2 + a_0\|\xi^n\|_1^2 \Delta t\\
&\leq C\left\{\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^2(0,\,T;\,L^2)}^2 (\Delta t)^2\right.\\
&\left.+\|\frac{\partial \eta}{\partial t}\|_{L^2(0,\,T;\,H^{-1})}^2 + \|\eta\|_{L^\infty(0,\,T;\,L^2)}^2\right\}\\
&\leq C\left\{\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^2(0,\,T;\,L^2)}^2 (\Delta t)^2 + \left(\|u\|_{L^\infty(0,\,T;\,H^2(I))}^2\right.\right.\\
&\left.\left.+\|\frac{\partial u}{\partial t}\|_{L^2(0,\,T;\,H^1(I))}^2\right) h^4\right\}.
\end{aligned} \tag{32}$$

Which together with $u^n - u_h^n = \eta^n - \xi^n$, (13) and (14) yield the desired result (16).

An optimal error estimate for $u^n - u_h^n$ in $H^1(I)$ can be derived in a similar fashion, starting from the test function $v = \Pi_h^*\frac{(\xi^n - \xi^{n-1})}{\Delta t}$ in (17). Then we have the following theorem.

Theorem 4.2: Let u and u_h be the respective solution of (6a)–(6b) and (8a)–(8b), under the assumptions (2a)–(2e). We have the error estimates

$$\begin{aligned}
&\max_{1\leq n\leq M} \|u^n - u_h^n\|_1 \leq C\left(\Delta t\|\frac{\partial^2 u}{\partial t^2}\|_{L^2(0,T;L^2)}\right.\\
&\left.+h(\|u\|_{L^\infty(0,T;H^2(I))} + \|\frac{\partial u}{\partial t}\|_{L^2(0,T;H^1(I))})\right)\\
&\leq C(\Delta t + h).
\end{aligned} \tag{33}$$

B. Error Estimates on Two-grid Technique

The two-grid technique is based on two different spaces defined respectively on one coarse grid with size H and one fine grid with size h, and $h \ll H$. This scheme is an economic and graceful method because of the nonlinear iteration only on the coarse grid, on the fine grid given a simpler, linear system. At last a further coarse grid correction is performed.

Lemma 4.3: For any $u_H \in H_0^1(\Omega)$, the following property is satisfied

1. $L_{u_H}\omega \equiv c\frac{\omega_H^n - \bar{\omega}_H^{n-1}}{\Delta t} - \frac{\partial}{\partial x}(a(x)\frac{\partial \omega}{\partial x}) - f'(u_H^n)\omega$: $H_0^1(\Omega) \cap H^2(\Omega) \mapsto L^2(\Omega)$, then we have

$$\|\omega\|_2 \leq C\|L_{u_H}\omega\| \quad \forall\ \omega \in H_0^1(\Omega) \cap H^2(\Omega).$$

2. If Δt is sufficient small, then there is $C(\Delta t)$,

$$\sup_{\chi\in V_h} \frac{B_H(\omega_h^n, \chi)}{\|\chi\|_1} \geq C(\Delta t)\|\omega_h\|_1, \quad \forall\ \omega_h \in V_h,$$

where $B_H(\omega_h^n,\psi) = (c\frac{\omega_h^n-\bar{\omega}_h^{n-1}}{\Delta t},\psi)+A(\omega_h^n,\psi)-(f'(u_H^n)\omega_h^n,\psi)$.

Theorem 4.3: Let u_h^n be the solution of (8a)–(8b), we have the error estimate relation

$$\|u_h^n-(u_H^n+e_h^n)\|_1\le C(\Delta t)H^4. \quad (34)$$

Proof.

$$\begin{aligned}&B_H(u_h^n,\chi)=(c\frac{u_h^n-\bar{u}_h^{n-1}}{\Delta t},\chi)\\&+A(u_h^n,\chi)-(f'(u_H^n)u_h^n,\chi)\\&=(f(u_h^n),\chi)-(f'(u_H^n)u_h^n,\chi).\end{aligned} \quad (35)$$

$$\begin{aligned}&B_H(u_H^n+e_h^n,\chi)=(c\frac{u_H^n-\bar{u}_H^{n-1}}{\Delta t},\chi)+A(u_H^n,\chi)\\&-(f'(u_H^n)u_H^n,\chi)+(f(u_H^n),\chi)\\&-(c\frac{u_H^n-\bar{u}_H^{n-1}}{\Delta t},\chi)+A(u_H^n,\psi)\\&=(f(u_H^n),\chi)-(f'(u_H^n)u_H^n,\chi).\end{aligned} \quad (36)$$

Combining (35) and (36), we get

$$\begin{aligned}&B_H(u_h^n-(u_H^n+e_h^n),\chi)\\&=(f(u_h^n)-f(u_H^n)-f'(u_H^n)(u_h^n-u_H^n),\chi)\\&=(b(u_h^n-u_H^n)^2,\chi),\end{aligned} \quad (37)$$

where $b=\int_0^1(1-t)f''(u_H^n+t(u_h^n-u_H^n))dt$.
Applying the Hölder inequality and embed theorem, we have

$$\begin{aligned}&(b(u_h^n-u_H^n)^2,\chi)\\&\le C\|u_h^n-u_H^n\|_{0,\frac{p}{2}}^2\|\chi\|_{0,\frac{p}{p-2}}\\&\le C\|u_h^n-u_H^n\|_{0,p}^2\|\chi\|_1.\end{aligned} \quad (38)$$

From the lemmas 4.1 and 4.3, it yields to

$$\|u_h^n-(u_H^n+e_h^n)\|_1\le C(\Delta t)H^4.$$

Lemma 4.4: For any $v\in V_h$ the following relation is satisfied

$$\begin{aligned}&B_H(\widehat{u}_h^n,v)\\&=(f(u_H^n)-f'(u_H^n)u_H^n+\frac{1}{2}f''(u_H^n)(e_h^n)^2,v)\\&-\frac{1}{2}(f''(u_H^n)(e_h^n)^2,v-\Pi_Hv).\end{aligned} \quad (39)$$

Proof. By the definition of Π_H and e_H^n , we can get

$$B_H(e_H^n,v)=B_H(e_H^n,\Pi_Hv)=\frac{1}{2}(f''(u_H^n)(e_h^n)^2,\Pi_Hv).$$

Then, the result is easily proved.

Lemma 4.5: we can get the following relation

$$\begin{aligned}&B_H(u_h^n-\widehat{u}_h^n,v)\\&=-\frac{1}{2}(f''(u_H^n)((e_h^n)^2-(u_h^n-u_H^n)^2),v)\\&+\frac{1}{2}(f''(u_H^n)(e_h^n)^2,v-\Pi_Hv)\\&+(O(u_h^n-u_H^n)^3,v).\end{aligned} \quad (40)$$

Proof. By the definition of $B_H(u_h^n,v)$ and $Taylor$ theorem, we have

$$\begin{aligned}&B_H(u_h^n,v)\\&=(f(u_H^n)-f'(u_H^n)u_H^n+\frac{1}{2}f''(u_H^n)(u_h^n-u_H^n)^2,v)\\&+(O(u_h^n-u_H^n)^3,v).\end{aligned}$$

From the lemma 4.3 for any $v\in V_h$, we can get

$$\begin{aligned}&B_H(u_h^n-\widehat{u}_h^n,v)=B_H(u_h^n,v)-B_H(\widehat{u}_h^n,v)\\&=-\frac{1}{2}(f''(u_H^n)((e_h^n)^2-(u_h^n-u_H^n)^2),v)\\&+\frac{1}{2}(f''(u_H^n)(e_h^n)^2,v-\Pi_Hv)+(O(u_h^n-u_H^n)^3,v).\end{aligned}$$

Theorem 4.4: Let u^n be the solution of (6a)–(6b) and $\widehat{u}_h^n=u_H^n+e_h^n+e_H^n$, under assumptions (2a)–(2e), we have the error estimate

$$\max_{1\le n\le M}\|u^n-\widehat{u}_h^n\|\le C(\Delta t)(\Delta t+h^2+H^6). \quad (41)$$

Proof. By the Hölder inequality and the embed theorem

$$\begin{aligned}&((e_h^n)^2-(u_h^n-u_H^n)^2,v)\\&\le\|(u_h^n-(u_H^n+e_h^n))(e_h^n+u_h^n-u_H^n)\|_{0,\frac{6}{5}}\|v\|_6\\&\le\|(u_h^n-(u_H^n+e_h^n)\|_{0,3}\|e_h^n+u_h^n-u_H^n\|_{0,2}\|v\|_{0,6}\\&\le\|(u_h^n-(u_H^n+e_h^n)\|_1\|e_h^n+u_h^n-u_H^n\|\|v\|_1.\end{aligned}$$

From the lemmas 4.1 and 4.3, we have

$$((e_h^n)^2-(u_h^n-u_H^n)^2,v)\le C(\Delta t)H^6\|v\|_1.$$

By the Cauchy-Schwarz inequality, (2b) and (13), we have

$$\begin{aligned}&(f''(u_H^n)(e_h^n)^2,v-\Pi_Hv)\\&\le\|e_h^n\|^2\|v-\Pi_Hv\|\\&\le CH^4\|v-\Pi_Hv\|.\end{aligned}$$

Applying the Hölder inequality and embed theorem, then we can get

$$\begin{aligned}&(O(u_h^n-u_H^n)^3,v)\\&\le C\|u_h^n-u_H^n\|_{0,\frac{6}{5}}^3\|v\|_{0,6}\\&\le C\|u_h^n-u_H^n\|_{0,4}^3\|v\|_1\\&\le CH^6\|v\|_1.\end{aligned}$$

The following relation is satisfied

$$B_H(u_h^n-\widehat{u}_h^n,v)\le C(\Delta t)H^6\|v\|_1+CH^4\|v-\Pi_Hv\|. \quad (42)$$

From the lemma 4.3, we can get

$$\|u_h^n-\widehat{u}_h^n\|\le C(\Delta t)H^5.$$

In order to get the estimation of L_2 norm, the dual technique can be applied. Find $\omega\in H_0^1(\Omega)\cap H^2(\Omega)$ such that

$$c\frac{\omega^n-\bar{\omega}^{n-1}}{\Delta t}-\frac{\partial}{\partial x}(a(x)\frac{\partial\omega}{\partial x})-(f'(u_H^n)\omega=u_h^n-\widehat{u}_h^n,$$

then

$$\begin{aligned}&\|u_h^n-\widehat{u}_h^n\|^2=B_H(u_h^n-\widehat{u}_h^n,\omega)\\&=B_H(u_h^n-\widehat{u}_h^n,\omega-\omega_h)+B_H(u_h^n-\widehat{u}_h^n,\omega_h).\end{aligned}$$

where ω_h is the interpolating function of ω. Noting

$$\begin{aligned}&B_H(u_h^n-\widehat{u}_h^n,\omega-\omega_h)\le C\|u_h^n-\widehat{u}_h^n\|_1\|\omega-\omega_h\|_1\\&\le C(\Delta t)H^5h\|\omega\|_2\le C(\Delta t)H^6\|u_h^n-\widehat{u}_h^n\|.\end{aligned}$$

From the expression of (42), we can get

$$\begin{aligned} &B_H(u_h^n - \widehat{u}_h^n, \omega_h) \\ &\le C(\Delta t)H^6\|\omega_h\|_1 + H^4\|(I-\Pi_H)\omega_h\| \\ &\le C(\Delta t)H^6\|\omega_h\|_2 + H^4(\|(I-\Pi_H)\omega\| \\ &+H\|\omega - \omega_h\|_1) \\ &\le C(\Delta t)H^6\|\omega\|_2 \\ &\le C(\Delta t)H^6\|u_h^n - \widehat{u}_h^n\|. \end{aligned} \quad (43)$$

From Lemma 4.3, we have

$$\|u_h^n - \widehat{u}_h^n\| \le C(\Delta t)H^6.$$

Combing the Theorem 4.1, we can get the Theorem 4.4.

Furthermore, the error estimate for $u_h^n - \widehat{u}_h^n$ in H^1 *norm* can be derived easily.

Theorem 4.5: Let u^n be the solution of (6a)–(6b) and $\widehat{u}_h^n = u_H^n + e_h^n + e_H^n$, under assumptions (2a)–(2e). we have the error estimate relation

$$\max_{1\le n\le M} \|u^n - \widehat{u}_h^n\|_1 \le C(\Delta t)(\Delta t + h + H^5). \quad (44)$$

V. Numerical Example

To illustrate the effectiveness of the two-grid linearization, we examine the following simple test problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) = -u^2 + G(x,t), \quad (45a)$$

$$u(0,t) = 1, \ u(1,t) = 0, \quad (45b)$$

$$u(x,0) = 1 - x, \quad (45c)$$

where $I = [0,1]$, $T = 0.2$ and $a(x) = 0.001$. The function $G(x,t)$ is determined by the exact solution $u(x,t) = (1-x)e^{xt}$.

We solve (45a)-(45c) with the two-grid scheme presented in this paper. The parameters are chosen as follows: $\Delta t = 0.125 \times 10^{-5}$, $H = 2^{-3}$ and $h = 2^{-9}$.

In order to check up the numerical accuracy of the proposed method, we obtain the nonlinear iteration solution u_h^n on the fine-grid U_h according to (6a)-(6b). The numerical results of the proposed method, the nonlinear iteration method and the exact solution are presented in Fig 1.

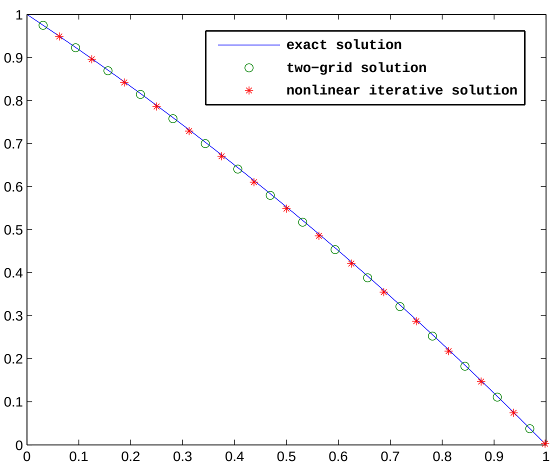

Fig 1 $\Delta t = 0.025$, T= 0.20

From Fig 1 we can conclude that the two-grid method solution $\widehat{u}_h^n$ has the same accuracy as that of nonlinear iteration solution u_h^n. Furthermore, we comprise the L_2 error and the costing CPU time. Tables I and II give the L_2 error and CPU time of two-grid characteristic finite volume element method and the nonlinear iterative method, respectively.

From Tables I and II, we can get the two-grid solution $\widehat{u}_h^n$ and nonlinear iterative solution u_h^n has the same precision, but the CPU time of two-grid method is much smaller than the nonlinear iterative method. Then the two-grid method is very effective for the nonlinear convection-dominated diffusion equation.

Table I. The L_2 error and costing time of Two-grid method

h	H	$\frac{\|\widehat{u}_h^n - u\|}{\|u\|}$	Time(s)
2^{-9}	2^{-3}	6.2×10^{-3}	$68''$

Table II. The L_2 error and costing time of nonlinear iterative method

h	$\frac{\|u_h^n - u\|}{\|u\|}$	Time(s)
2^{-9}	7.1×10^{-3}	$166''$

VI. Conclusion

In this paper, we use the characteristics finite volume element method to discrete the nonlinear convection-dominated diffusion equation. The solution of a nonlinear system on the fine space is much expensive, so we employ two-grid approach in our discretization schemes. The solution of a nonlinear system on the fine space is reduced to the solution of a nonlinear system on the coarse space and a linear system on the fine space. Theoretic analysis and numerical experiments are confirmed the effectiveness of the two-grid method for characteristics finite volume element of nonlinear convection-dominated diffusion equations. Combing the present method with some stabilization techniques solving Navier-Stokes equations with large Reynolds number and the possible extension of the method to other nonlinear coupling problems will be our further work.

Acknowledgment

The authors would like to thank the editor and anonymous reviewers for their valuable comments and helpful suggestions on the quality improvement of our present paper.

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