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Transcomplex Numbers: Properties, Topology and Functions

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***Abstract*—We derive the properties of transcomplex arithmetic from the usual definition of transcomplex numbers as a fraction of complex numbers, whose denominator may be zero. This is equivalent to giving an axiomatisation.**

In particular we characterise the partial associativity of transcomplex addition and the partial distributivity. We describe specifically how the transcomplex numbers depart from field structure and relate this to earlier work on transfields.

We review the transcomplex elementary functions and the topology of transcomplex numbers. Thus armed we extend several functional properties of the complex numbers to the transcomplex numbers.

***Index Terms*—Transcomplex number, transcomplex topology, transcomplex function, non-finite angle.**

I. Introduction

Transmathematics is the mathematics which arises from the transnumbers – numbers which extend the usual numbers and allow division by zero. The first, and so far most developed of the transnumbers, is the set of transreal numbers. Other sets are also studied in transmathematics such as, for example, the transnaturals and transcomplexes. The latter being the focus of this work. The set of transreals, denominated by $\mathbb{R}^T$, and their arithmetic, is an extension of real numbers and real arithmetic.

Transreal numbers were introduced by James Anderson at the turn of the millennium [3]. Anderson's motivation was to enable division by zero and to apply these new numbers to computer programming. The absence of exceptions is extremely powerful in computing. It makes it possible to construct computational systems where all syntactically correct expressions are semantically correct. This means that infinitely many exceptional states are removed from mathematics and from computer programs. This is of practical importance because it makes it possible to guarantee that if a program compiles then it does not terminate due to a logical exception. This is of very wide utility. In particular, meta-programs, such as genetic algorithms, can combine subprograms arbitrarily in the search for optimal solutions. The application of transarithmetic in both novel and conventional computer hardware and software is discussed in, among other places, [5] [1].

In $\mathbb{R}^T$, the four basic, arithmetical operations (addition, subtraction, multiplication and division) are closed, that is, the result of any of these operations between transreal numbers is a transreal number. In particular, division by zero is allowed. The set $\mathbb{R}^T$ is formed by all real numbers and three new elements, $-\infty$, ∞ and Φ, respectively denominated by minus infinity, infinity and nullity. Thus, $\mathbb{R}^T = \mathbb{R} \cup \{\infty, \infty, \Phi\}$. By definition: $\frac{-k}{0} = \frac{-1}{0} := -\infty$ and $\frac{k}{0} = \frac{1}{0} := \infty$ for all $k \in \mathbb{R}$ and $\frac{0}{0} := \Phi$ [7].

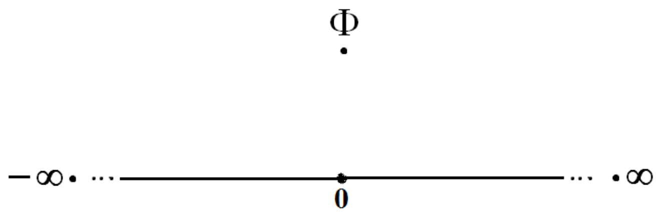

Fig. 1. Transreal line.

In [10] we introduce the set of transcomplex numbers, $\mathbb{C}^T$, and proved that transcomplex arithmetic is consistent. We construct this new set from the real numbers by means of equivalence classes of ordered pairs. The set of transcomplex numbers contains the ordinary set of complex numbers and the set of transreal numbers as proper subsets. Transcomplex numbers are a new system of numbers which is total, with respect to the four elementary, arithmetical operations. In particular, division by zero is allowed. In [8] we set up a topology for the set of transcomplex numbers.

In [6] [11] [12] we introduce transreal calculus with a transreal topology that extends real topology. In the same way we extend complex topology to the transcomplex plane and establish some results about limits and continuity of transcomplex functions, analogous to complex functions [9]. In [13] we extend every real, elementary function to the transreal domain and in [8] we extend every complex, elementary function to the transcomplex domain. This covers a lot of ground. An elementary function is defined so that every polynomial, root, exponential, logarithm, trigonometric and inverse trigonometric function is an elementary function; any finite composition of elementary functions is an elementary function; and any finite combination, using the four, elementary arithmetical operations, between elementary functions is an elementary function.

In the present paper we derive transcomplex arithmetic from the definition of transcomplex numbers as fractions of complex numbers, whose denominator may be zero. In particular we characterise the partial associativity and partial distributivity of transcomplex arithmetic. This identifies all of the assertions that would form an axiomatisation, thereby isolating the axioms that might be made the subject of a machine proof of consistency. The characterisation of the partialities also sharply defines how transcomplex arithmetic departs from field structure, though transcomplex arithmetic is already known to be a transfield [9].

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II. Transcomplex Arithmetic

In [10] we define fractions so that they allow a denominator of zero and we prove that transcomplex arithmetic is consistent. The set of transcomplex numbers is given by

$$\left\{\frac{x}{y};\ x, y \in \mathbb{C}\right\}.$$

Since any usual fraction $\frac{x}{y}$ (where x and y are complex numbers with $y \neq 0$) can be represented by the equivalent fraction $\frac{\frac{x}{y}}{1}$, every transcomplex number can be written as a fraction, $\frac{x}{y}$, where x is an ordinary complex number and y is either one or zero.

Definition 1: Let $\mathbb{C}^T$ denote the set of transcomplex numbers, which is defined by

$$\mathbb{C}^T := \left\{\frac{x}{y};\ x \in \mathbb{C} \text{ and } y \in \{0,1\}\right\}.$$

Next we establish an equivalence rule between fractions of this new kind that allow a denominator of zero.

Definition 2: Given arbitrary $\frac{x}{y}, \frac{w}{z} \in \mathbb{C}^T$, that is, $x, w \in \mathbb{C}$ and $y, z \in \{0,1\}$, we say that $\frac{x}{y} = \frac{w}{z}$ if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$ and $y = \alpha z$.

Next we define the arithmetical operations between transcomplex numbers.

Definition 3: Given arbitrary $\frac{x}{y}, \frac{w}{z} \in \mathbb{C}^T$, that is, $x, w \in \mathbb{C}$ and $y, z \in \{0,1\}$, it follows that:

a) (Adition) If $y = z = 0$, $x \neq 0$ and $w \neq 0$ then $\frac{x}{y} + \frac{w}{z} = \frac{\frac{x}{|x|} + \frac{w}{|w|}}{0}$ otherwise, $\frac{x}{y} + \frac{w}{z} = \frac{xz+wy}{yz}$.

b) (Opposite) $-\frac{x}{y} = \frac{-x}{y}$

c) (Subtraction) $\frac{x}{y} - \frac{w}{z} = \frac{x}{y} + \left(-\frac{w}{z}\right)$.

d) (Multiplication) $\frac{x}{y} \times \frac{w}{z} = \frac{xw}{yz}$.

e) (Reciprocal) If $x \neq 0$ then $\left(\frac{x}{y}\right)^{-1} = \frac{\frac{y}{x}}{1}$, otherwise $\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$.

f) (Division) $\frac{x}{y} \div \frac{w}{z} = \frac{x}{y} \times \left(\frac{w}{z}\right)^{-1}$.

Notice that with these definitions, when operations are performed between the usual complex numbers (fractions with denominator 1), the results obtained are exactly the same as the usual arithmetical results. This means that arithmetic in this new set of transcomplex numbers respects the arithmetic of the old set of complex numbers. Note, also, that the above rules are analogous to the usual rules.

As a consequence of the equivalence rule, given in Definition 2, it follows that

$$\mathbb{C}^T = \mathbb{C} \cup \left\{\frac{x}{0};\ x \in \mathbb{C},\ |x| = 1\right\} \cup \left\{\frac{0}{0}\right\}.$$

Indeed, if $\frac{x}{y} \in \mathbb{C}^T$ then $y = 1$ or $y = 0$. If $y = 1$ then $\frac{x}{y} \in \mathbb{C}$. On the other hand, if $y = 0$ then either $x \neq 0$ or else $x = 0$. In the first case, $x \neq 0$ implies $\alpha = |x|$ is a positive real number such that $x = \alpha\frac{x}{|x|}$, whence $\frac{x}{y} = \frac{\frac{x}{|x|}}{0}$. Further $\left|\frac{x}{|x|}\right| = 1$, whence $\frac{x}{y} = \frac{\frac{x}{|x|}}{0} \in \left\{\frac{x}{0};\ x \in \mathbb{C},\ |x| = 1\right\}$. In the second case, $x = 0$, whence $\frac{x}{y} = \frac{0}{0}$. Note that for each $x, z \in \mathbb{C}$, where $|z| = 1$, the elements x, $\frac{z}{0}$ and $\frac{0}{0}$ are pairwise distinct.

We name two special, transcomplex numbers: *infinity*, $\infty := \frac{1}{0}$, and *nullity,* $\Phi := \frac{0}{0}$.

The transcomplex plane is shown in figure 2. The usual complex plane is shown as a grey disk. It has no real bound but, after a gap, it is surrounded by a circle at infinity. The point at nullity, Φ, lies off the plane containing the complex plane and the circle at infinity. The transreal number line is shown as the x-axis, together with the point at nullity, Φ. Thus the transcomplex plane is obtained by a revolution of the transreal line.

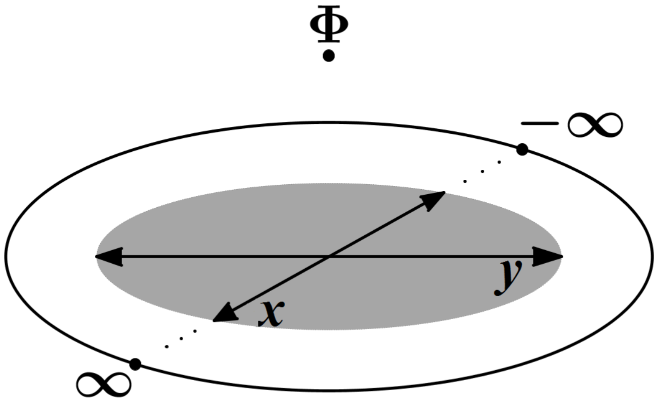

Fig. 2. The transreal numbers are shown as the extended x-axis, together with the point at nullity, Φ, as a subset of the transcomplex numbers.

Any complex number can be represented, in polar form, by an ordered pair (r, θ), where $r \in [0, \infty)$ and $\theta \in (-\pi, \pi]$. Note that zero does not have a unique description because $(0, \theta)$ describes zero for all $\theta \in (-\pi, \pi]$. Now we describe Φ by the ordered pair (Φ, θ), where θ is arbitrary in $(-\pi, \pi]$. We represent all transcomplex numbers in the form $\frac{u}{0}$ where $u \neq 0$, by the ordered pair (∞, θ), where $\theta = \text{Arg}(u)$. In this way all transcomplex numbers can be represented by an ordered pair, in the form (r, θ), where $r \in [0, \infty] \cup \{\Phi\}$ and $\theta \in (-\pi, \pi]$, observing that $(0, \theta)$ represents zero for all $\theta \in (-\pi, \pi]$ and (Φ, θ) represents Φ for all $\theta \in (-\pi, \pi]$. Thus we can write

$$\mathbb{C}^T = \mathbb{C} \cup \big\{(\infty, \theta);\ \theta \in (-\pi, \pi]\big\} \cup \{\Phi\}.$$

Though, later, we shall incorporate the non-finite angles $\theta \in \{-\infty, \infty, \Phi\}$.

Figure 3 shows that any point in the complex plane and the circle at infinity can be described in polar co-ordinates. The system of polar co-ordinates also describes the point at nullity which lies at nullity distance and nullity angle. Thus every point in the transcomplex plane, including the point at nullity, is described by polar co-ordinates.

Let us refer to the elements of $\mathbb{C}$ as *finite transcomplex numbers*, to the elements of $\big\{(\infty, \theta);\ \theta \in (-\pi, \pi]\big\} \bigcup \{\Phi\}$ as *non-finite transcomplex numbers* and, particularly, to the elements of $\big\{(\infty, \theta);\ \theta \in (-\pi, \pi]\big\}$ as *infinite transcomplex* *numbers*, then the elements of $\{(\infty, \theta);\ \theta \in (-\pi, \pi]\} \bigcup \{\Phi\}$ are *strictly transcomplex numbers*.

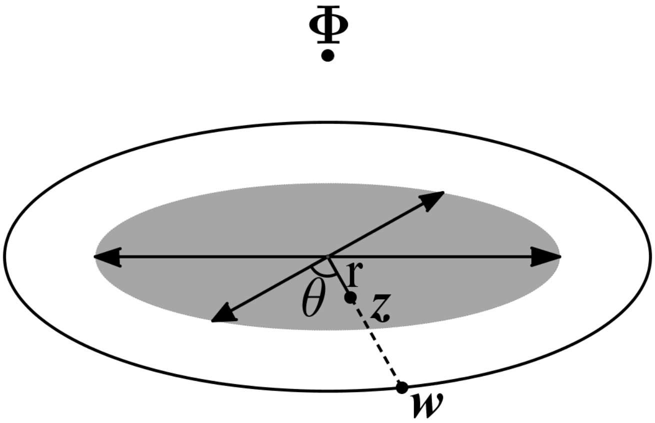

Fig. 3. Entire transcomplex plane described by polar co-ordinates: $z = (r, \theta)$ and $w = (\infty, \theta)$ and $\Phi = (\Phi, \Phi)$ with r, θ finite.

Remark 4: Let us denote the set of infinite transcomplex numbers by $\mathbb{C}^T_\infty$. That is,

$$\mathbb{C}^T_\infty := \left\{\frac{x}{0};\ x \in \mathbb{C},\ |x| = 1\right\}.$$

The reader should note that no restriction has been imposed on division. In this way, any transcomplex number, including complex numbers, can be divided by zero. Note that zero divided by zero results in nullity. Indeed,

$$0 \div 0 = \frac{0}{1} \div \frac{0}{1} = \frac{0}{1} \times \left(\frac{0}{1}\right)^{-1} = \frac{0}{1} \times \frac{1}{0} = \frac{0 \times 1}{1 \times 0} = \frac{0}{0} = \Phi.$$

And any non-zero, complex number, z, divided by zero, results in the transcomplex number of infinite radius, which has the same argument (angle) as z. Indeed,

$$z \div 0 = \frac{z}{1} \div \frac{0}{1} = \frac{z}{1} \times \frac{1}{0} = \frac{z \times 1}{1 \times 0} = \frac{z}{0} = (\infty, \mathrm{Arg}(z)).$$

Transcomplex arithmetic can be understood geometrically [4]. Multiplication and division are a generalisation of the usual rotation and dilatation, where dilatation of a finite, non-zero radius by ∞ is ∞, dilatation of a zero radius by ∞ is Φ, and dilatation of any radius by Φ is Φ. Addition is performed using a generalisation of the usual parallelogram rule, such that addition of an infinite number and a finite number involves a parallelogram whose one side has infinite length and whose other side has finite length, such that the diagonal has infinite length and lies at the same angle as the infinite side. The sum of two, non-opposite, infinite numbers involves a parallelogram with sides of equal and infinite length, such that the sum is the infinitely long diagonal. The sum of two, opposite, infinite numbers is Φ. The sum of any number with Φ is a diagonal of length Φ. The sum of finite numbers is given by the ordinary parallelogram rule.

III. Properties of Transcomplex Arithmetic

In this section we develop some elementary results of transcomplex arithmetic.

Proposition 5:

a) The sum of nullity with any transcomplex is nullity: $\Phi + z = z + \Phi = \Phi$ for all $z \in \mathbb{C}^T$.

b) The sum of any non-opposite, infinite transcomplexes is an infinite transcomplex: If $z, w \in \mathbb{C}^T_\infty$ and $z \neq -w$ then $z + w, w + z \in \mathbb{C}^T_\infty$.

c) The sum of opposite, infinite transcomplexes is nullity: If $z, w \in \mathbb{C}^T_\infty$ and $z = -w$ then $z + w = w + z = \Phi$.

d) The sum of an infinite transcomplex with a finite transcomplex is the infinite transcomplex: If $z \in \mathbb{C}^T_\infty$ and $w \in \mathbb{C}$ then $z + w = w + z = z$.

e) The opposite of nullity is nullity: $-\Phi = \Phi$.

f) Subtraction of a non-finite transcomplex from itself is nullity: If $z \in \mathbb{C}^T \setminus \mathbb{C}$ then $z - z = \Phi$.

g) The product of nullity with any transcomplex is nullity: $\Phi \times z = z \times \Phi = \Phi$ for all $z \in \mathbb{C}^T$.

h) The product of any infinite transcomplexes is an infinite transcomplex: If $z, w \in \mathbb{C}^T_\infty$ then $z \times w, w \times z \in \mathbb{C}^T_\infty$.

i) The product of an infinite transcomplex with a non-zero, finite transcomplex is an infinite transcomplex: If $z \in \mathbb{C}^T_\infty$ and $w \in \mathbb{C} \setminus \{0\}$ then $z \times w, w \times z \in \mathbb{C}^T_\infty$.

j) The product of an infinite transcomplex with zero is nullity: If $z \in \mathbb{C}^T_\infty$ then $z \times 0 = 0 \times z = \Phi$.

k) The reciprocal of nullity is nullity: $\Phi^{-1} = \Phi$.

l) The reciprocal of zero is infinity: $0^{-1} = \infty$.

m) The reciprocal of any infinite transcomplex is zero: If $z \in \mathbb{C}^T_\infty$ then $z^{-1} = 0$.

n) Division of non-finite transcomplexes is nullity: If $z, w \cup \mathbb{C}^T \setminus \mathbb{C}$ then $z \div w = \Phi$.

o) Zero divided by zero is nullity: $0 \div 0 = \Phi$.

Proof: Let $z, w \in \mathbb{C}^T$ and denote $z = \frac{x}{y}$ and $w = \frac{u}{t}$ where $x, u \in \mathbb{C}$ and $y, t \in \{0, 1\}$.

a) $\Phi + z = \frac{0}{0} + \frac{x}{y} = \frac{0 \times y + x \times 0}{0 \times y} = \frac{0}{0} = \Phi$.

b) If $z, w \in \mathbb{C}^T_\infty$ and $z \neq -w$ then $x \neq 0$, $u \neq 0$, $y = 0$, $t = 0$ and $\frac{x}{|x|} \neq -\frac{u}{|u|}$ whence $z + w = \frac{x}{0} + \frac{u}{0} = \frac{\frac{x}{|x|} + \frac{u}{|u|}}{0} \in \mathbb{C}^T_\infty$.

c) If $z, w \in \mathbb{C}^T_\infty$ and $z = -w$ then $x \neq 0$, $u \neq 0$, $y = 0$, $t = 0$ and $\frac{x}{|x|} = \neq -\frac{u}{|u|}$ whence $z + w = \frac{x}{0} + \frac{u}{0} = \frac{\frac{x}{|x|} + \frac{u}{|u|}}{0} = \frac{0}{0} = \Phi$.

d) If $z \in \mathbb{C}^T_\infty$ and $w \in \mathbb{C}$ then $x \neq 0$, $y = 0$, $t \neq 0$ whence $z + w = \frac{x}{0} + \frac{u}{1} = \frac{x \times 1 + u \times 0}{0 \times 1} = \frac{x}{0} = z$.

e) $-\Phi = -\frac{0}{0} = \frac{-0}{0} = \frac{0}{0} = \Phi$.

f) The result follows from items (a), (c) and (e).

g) $\Phi \times z = \frac{0}{0} \times \frac{x}{y} = \frac{0 \times x}{0 \times y} = \frac{0}{0} = \Phi$.

h) If $z, w \in \mathbb{C}^T_\infty$ then $x \neq 0$, $u \neq 0$, $y = 0$, $t = 0$ whence $z \times w = \frac{x}{0} \times \frac{u}{0} = \frac{xu}{0 \times 0} = \frac{xu}{0} \in \mathbb{C}^T_\infty$.

i) If $z \in \mathbb{C}^T_\infty$ and $w \in \mathbb{C} \setminus \{0\}$ then $x \neq 0$, $y = 0$, $u \neq 0$, $t \neq 0$ whence $z \times w = \frac{x}{0} \times \frac{u}{t} = \frac{x \times u}{0 \times t} = \frac{xu}{0} \in \mathbb{C}^T_\infty$.

j) If $z \in \mathbb{C}^T_\infty$ then $y = 0$ whence $z \times 0 = \frac{x}{0} \times \frac{0}{1} = \frac{x \times 0}{0 \times 1} = \frac{0}{0} = \Phi$.

k) $\Phi^{-1} = \left(\frac{0}{0}\right)^{-1} = \frac{0}{0} = \Phi$.

l) $0^{-1} = \left(\frac{0}{1}\right)^{-1} = \frac{1}{0} = \infty$.

m) If $z \in \mathbb{C}^T_\infty$ then $z^{-1} = \left(\frac{x}{0}\right)^{-1} = \frac{\frac{0}{x}}{1} = \frac{0}{1} = 0$.

n) The result follows from items (g), (j), (k) and (l).

o) The result follows from items (j) and (l).

The commutativity of addition and multiplication is proved at Proposition 21, below.

■

In what follows we establish, in $\mathbb{C}^T$, some definitions and properties that are similar in $\mathbb{C}$.

Definition 6: Given $z \in \mathbb{C}^T$ take $x \in \mathbb{C}$ and $y \in \{0, 1\}$ such that $z = \frac{x}{y}$ and define $\overline{z} := \frac{\overline{x}}{y}$. We call $\overline{z}$ the *conjugate* of the transcomplex number z.

We are abusing notation when we reuse the symbol for the conjugate of complex numbers to define conjugate in $\mathbb{C}^T$. However, this is not a problem because the context distinguishes the set to which the symbols refer. When we say that $\overline{z} = \frac{\overline{x}}{y}$ it is clear that the symbol " $\bar{\cdot}$ " on the left hand side of the equality refers to conjugate in $\mathbb{C}^T$ while the symbols " $\bar{\cdot}$ " on the right hand side of the equality refer to conjugate in $\mathbb{C}$. Moreover, when the operation of taking the conjugate in $\mathbb{C}^T$ is restricted to $\mathbb{C}$ it coincides with the usual conjugate in $\mathbb{C}$.

Proposition 7: The conjugate of a transcomplex number is well defined. That is, the conjugate is independent of the choice of the fraction which represents the transcomplex number. In other words, if $x, w \in \mathbb{C}$ and $y, t \in \{0, 1\}$ and $\frac{x}{y} = \frac{w}{t}$ then $\overline{\left(\frac{x}{y}\right)} = \overline{\left(\frac{w}{t}\right)}$.

Proof: Let $x, w \in \mathbb{C}$ and $y, t \in \{0, 1\}$ such that $\frac{x}{y} = \frac{w}{t}$. If $y = 1$ then $t = 1$ whence $x = w$ and the result is immediate. If $y = 0$ and $x = 0$ then $t = 0$ and $w = 0$ whence the result is also immediate. If $y = 0$ and $x \neq 0$ then $t = 0$ and $w \neq 0$ and $\frac{x}{|x|} = \frac{w}{|w|} \in \mathbb{C}$ whence $\frac{\overline{x}}{|\overline{x}|} = \frac{\overline{w}}{|\overline{w}|}$. Thus, $\overline{\left(\frac{x}{y}\right)} = \overline{\left(\frac{x}{0}\right)} = \frac{\overline{x}}{0} = \frac{\frac{\overline{x}}{|\overline{x}|}}{0} = \frac{\frac{\overline{w}}{|\overline{w}|}}{0} = \frac{\overline{w}}{0} = \overline{\left(\frac{w}{0}\right)} = \overline{\left(\frac{w}{t}\right)}$. ■

Of course, when $z \in \mathbb{C}^T$, $x \in \mathbb{C}$ and $y \in \{0, 1\}$ such that $z = \frac{x}{y}$, it follows that $\overline{z} = \frac{\overline{x}}{y}$.

Proposition 8: Given arbitrary $z, w \in \mathbb{C}^T$ it follows that:

a) $\overline{\overline{z}} = z$
b) $\overline{z + w} = \overline{z} + \overline{w}$
c) $\overline{-w} = -\overline{w}$
d) $\overline{z - w} = \overline{z} - \overline{w}$
e) $\overline{zw} = \overline{z}\,\overline{w}$
f) $\overline{w^{-1}} = \overline{w}^{-1}$
g) $\overline{z \div w} = \overline{z} \div \overline{w}$

Proof: Consider arbitrary $z, w \in \mathbb{C}^T$. Suppose $z = \frac{x}{y}$ and $w = \frac{u}{t}$ where $x, u \in \mathbb{C}$ and $y, t \in \{0, 1\}$.

a) $\overline{\overline{z}} = \overline{\overline{\left(\frac{x}{y}\right)}} = \overline{\left(\frac{\overline{x}}{y}\right)} = \frac{\overline{\overline{x}}}{y} = \frac{x}{y} = z$.

b) If $y = t = 0$, $x \neq 0$ and $u \neq 0$ then $\overline{z + w} = \overline{\frac{x}{0} + \frac{u}{0}} = \overline{\left(\frac{\frac{x}{|x|} + \frac{u}{|u|}}{0}\right)} = \frac{\overline{\frac{x}{|x|} + \frac{u}{|u|}}}{0} = \frac{\overline{\left(\frac{x}{|x|}\right)} + \overline{\left(\frac{u}{|u|}\right)}}{0} = \frac{\frac{\overline{x}}{|x|} + \frac{\overline{u}}{|u|}}{0} = \frac{\frac{\overline{x}}{|\overline{x}|} + \frac{\overline{u}}{|\overline{u}|}}{0} = \frac{\overline{x}}{0} + \frac{\overline{u}}{0} = \overline{\left(\frac{x}{0}\right)} + \overline{\left(\frac{u}{0}\right)} = \overline{z} + \overline{w}$. Otherwise $\overline{z + w} = \overline{\frac{x}{y} + \frac{u}{t}} = \overline{\left(\frac{xt + uy}{yt}\right)} = \frac{\overline{xt + uy}}{yt} = \frac{\overline{x}\,\overline{t} + \overline{u}\,\overline{y}}{yt} = \frac{\overline{x}t + \overline{u}y}{yt} = \frac{\overline{x}}{y} + \frac{\overline{u}}{t} = \overline{\left(\frac{x}{y}\right)} + \overline{\left(\frac{u}{t}\right)} = \overline{z} + \overline{w}$.

c) $\overline{-w} = \overline{\left(-\frac{u}{t}\right)} = \overline{\left(\frac{-u}{t}\right)} = \frac{\overline{-u}}{t} = \frac{-\overline{u}}{t} = -\frac{\overline{u}}{t} = -\overline{\left(\frac{u}{t}\right)} = -\overline{w}$.

d) This case follows from i(b) and (c).

e) $\overline{zw} = \overline{\frac{x}{y}\frac{u}{t}} = \overline{\left(\frac{xu}{yt}\right)} = \frac{\overline{xu}}{yt} = \frac{\overline{x}\,\overline{u}}{yt} = \frac{\overline{x}}{y}\frac{\overline{u}}{t} = \overline{\left(\frac{x}{y}\right)}\,\overline{\left(\frac{u}{t}\right)} = \overline{z}\,\overline{w}$

f) If $u \neq 0$ then $\overline{w^{-1}} = \overline{\left(\frac{u}{t}\right)^{-1}} = \overline{\left(\frac{\frac{t}{u}}{1}\right)} = \frac{\overline{\left(\frac{t}{u}\right)}}{1} = \frac{\frac{\overline{t}}{\overline{u}}}{1} = \left(\frac{\frac{\overline{t}}{\overline{u}}}{1}\right) = \left(\frac{\overline{u}}{\overline{t}}\right)^{-1} = \overline{\left(\frac{u}{t}\right)}^{-1} = \overline{w}^{-1}$. Otherwise $\overline{w^{-1}} = \overline{\left(\frac{u}{t}\right)^{-1}} = \overline{\left(\frac{t}{u}\right)} = \frac{\overline{t}}{\overline{u}} = \frac{t}{u} = \left(\frac{u}{t}\right)^{-1} = \left(\frac{\overline{u}}{\overline{t}}\right)^{-1} = \overline{\left(\frac{u}{t}\right)}^{-1} = \overline{w}^{-1}$.

g) This case follows from items (e) and (f).

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Definition 9: Given $z \in \mathbb{C}^T$, take $x \in \mathbb{C}$ and $y \in \{0, 1\}$, such that $z = \frac{x}{y}$, and define $|z| := \frac{|x|}{|y|}$. We call $|z|$ the *modulus* or *absolute value* of the transcomplex number z.

Once more we are abusing notation when we reuse the symbol for modulus. However, again, when we say that $|z| = \frac{|x|}{|y|}$ it is clear that the symbol "$|\cdot|$" on the left hand side of the equality refers to modulus in $\mathbb{C}^T$, while the symbol "$|\cdot|$", on the right hand side of the equality, refers to modulus in $\mathbb{C}$. Thus when the operation of taking the modulus in $\mathbb{C}^T$ is restricted to $\mathbb{C}$ it coincides with the usual modulus on $\mathbb{C}$.

Proposition 10: The modulus of a transcomplex number is well defined. That is, the modulus is independent of the choice of the fraction which represents the transcomplex number. In other words, if $x, w \in \mathbb{C}$ and $y, t \in \{0, 1\}$ and $\frac{x}{y} = \frac{w}{t}$ then $|\frac{x}{y}| = |\frac{w}{t}|$.

Proof: Let $x, w \in \mathbb{C}$ and $y, t \in \{0, 1\}$ such that $\frac{x}{y} = \frac{w}{t}$. If $y = 1$ then $t = 1$, whence $x = w$ and the result is immediate. If $y = 0$ and $x = 0$ then $t = 0$ and $w = 0$, whence the result is also immediate. If $y = 0$ and $x \neq 0$ then $t = 0$ and $w \neq 0$, whence $\left|\frac{x}{y}\right| = \infty = \left|\frac{w}{t}\right|$. ∎

Again, let $z \in \mathbb{C}^T$, $x \in \mathbb{C}$ and $y \in \{0, 1\}$, such that $z = \frac{x}{y}$. It follows that $|z| = \frac{|x|}{y}$.

Proposition 11: Given arbitrary $z, w \in \mathbb{C}^T$ it follows that:

a) $|z|^2 = z\overline{z}$
b) $|\overline{z}| = |z|$
c) $|zw| = |z||w|$
d) $|w^{-1}| = |w|^{-1}$
e) $|z \div w| = |z| \div |w|$
f) $|z + w| \not> |z| + |w|$

Proof: Consider arbitrary $z, w \in \mathbb{C}^T$. Suppose $z = \frac{x}{y}$ and $w = \frac{u}{t}$ where $x, u \in \mathbb{C}$ and $y, t \in \{0, 1\}$.

a) $z\overline{z} = \frac{x}{y}\overline{\left(\frac{x}{y}\right)} = \frac{x}{y}\frac{\overline{x}}{y} = \frac{x\overline{x}}{y^2} = \frac{x\overline{x}}{y^2} = \frac{|x|^2}{y^2} = \frac{|x|}{y}\frac{|x|}{y} = |z||z| = |z|^2$.

b) $|\overline{z}| = \left|\overline{\left(\frac{x}{y}\right)}\right| = \left|\frac{\overline{x}}{y}\right| = \frac{|\overline{x}|}{y} = \frac{|x|}{y} = |z|$.

c) $|zw| = \left|\frac{x}{y}\frac{u}{t}\right| = \left|\frac{xu}{yt}\right| = \frac{|xu|}{yt}$
$= \frac{|x||u|}{yt} = \frac{|x|}{y}\frac{|u|}{t} = |z||w|$

d) If $u \neq 0$ then $|w^{-1}| = \left|\left(\frac{u}{t}\right)^{-1}\right| = \left|\frac{\frac{t}{u}}{1}\right| = \frac{\left|\frac{t}{u}\right|}{1} = \frac{\frac{|t|}{|u|}}{1} = \left(\frac{|u|}{|t|}\right)^{-1} = \left|\frac{u}{t}\right|^{-1} = |w|^{-1}$. Otherwise, $|w^{-1}| = \left|\left(\frac{u}{t}\right)^{-1}\right| = \left|\frac{t}{u}\right| = \frac{|t|}{|u|} = \left(\frac{|u|}{|t|}\right)^{-1} = \left|\frac{u}{t}\right|^{-1} = |w|^{-1}$.

e) This case follows from (c) and (d).

f) (I) If $z = \Phi$ or $w = \Phi$, say $z = \Phi$, then $|z + w| = |\Phi + w| = |\Phi| = \Phi \not> \Phi = \Phi + |w| = |\Phi| + |w| = |z| + |w|$.

(II) If $z \in \mathbb{C}$ and $w \in \mathbb{C}$ the result follows from the ordinary Triangular Inequality of complex numbers.

(III) If either $z \in \mathbb{C}$ and $w \in \left\{\frac{x}{0};\ |x| = 1\right\}$ or $z \in \left\{\frac{x}{0};\ |x| = 1\right\}$ and $w \in \mathbb{C}$, say $z \in \mathbb{C}$ and $w \in \left\{\frac{x}{0};\ |x| = 1\right\}$, then $z+w \in \left\{\frac{x}{0};\ |x| = 1\right\}$, whence $|z + w| = \infty \not> \infty = |z| + \infty = |z| + |w|$.
(IV) If $z, w \in \left\{\frac{x}{0};\ |x| = 1\right\}$ and $z \neq -w$ then $z + w \in \left\{\frac{x}{0};\ |x| = 1\right\}$ whence $|z + w| = \infty \not> \infty = \infty + \infty = |z| + |w|$. (V) If $z, w \in \left\{\frac{x}{0};\ |x| = 1\right\}$ and $z = -w$ then $z + w = \Phi$ whence $|z + w| = |\Phi| = \Phi \not> \infty = \infty + \infty = |z| + |w|$.

∎

Definition 12: Given $z \in \mathbb{C}^T$, take $x \in \mathbb{C}$ and $y \in \{0, 1\}$, such that $z = \frac{x}{y}$ and define $\text{Arg}(z) := \text{Arg}(x)$. We call $\text{Arg}(z)$ the *principal argument* of the transcomplex number z.

Remark 13: We adopt the convention $\text{Arg}(0) := 0$.

Again we are abusing notation but, at this point, the reader can identify the correct use of the argument.

Proposition 14: The argument of a transcomplex number is well defined. That is, the argument is independent of the choice of the fraction which represents the transcomplex number. In other words, if $x, w \in \mathbb{C}$ and $y, t \in \{0, 1\}$ and $\frac{x}{y} = \frac{w}{t}$ then $\text{Arg}\left(\frac{x}{y}\right) = \text{Arg}\left(\frac{w}{t}\right)$.

Proof: Let $x, w \in \mathbb{C}$ and $y, t \in \{0, 1\}$, such that $\frac{x}{y} = \frac{w}{t}$. There is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$, whence $\text{Arg}\left(\frac{x}{y}\right) = \text{Arg}(x) = \text{Arg}(\alpha w) = \text{Arg}(w) = \text{Arg}\left(\frac{w}{t}\right)$. ∎

Now let $\frac{x}{y} \in \mathbb{C}^T$. When $y \neq 0$ we have $\frac{x}{y} \in \mathbb{C}$, whence $\frac{x}{y} = re^{i\theta}$ for some $r \in [0, \infty)$ and $\theta \in \mathbb{R}$. When $y = 0$, $\frac{x}{y} = \frac{x}{0}$. If $x = 0$ then $\frac{x}{y} = \frac{0}{0} = \Phi = \Phi \times e^{i\theta}$ for any $\theta \in \mathbb{R}$. If $x \neq 0$ then $\frac{x}{y} = \frac{x}{0} = \frac{x/|x|}{0} = \frac{1}{0} \times \frac{x/|x|}{1} = \infty \times \frac{x}{|x|} = \infty \times e^{i\theta}$ for some $\theta \in \mathbb{R}$. From now on we write $\Phi e^{i\theta} := \frac{0}{0}$, where $\theta \in \mathbb{R}$ is arbitrary, and $\infty e^{i\theta} := \frac{x}{0}$ where $\theta = \text{Arg}(x)$. Hence every transcomplex number z can be written as

$$z = |z|e^{i\text{Arg}(z)}.$$

Since, for each $\alpha \in \mathbb{R}$, $e^{i\alpha} = e^{i(\alpha + k2\pi)}$ for all $k \in \mathbb{Z}$, it follows that every transcomplex number z can be written as $z = |z|e^{i(\text{Arg}(z)+k2\pi)}$ for all $k \in \mathbb{Z}$. Thus $\mathbb{C}^T \subset \left\{re^{i\theta};\ r \in [0, \infty] \cup \{\Phi\},\ \theta \in \mathbb{R}\right\}$. Furthermore, clearly, $\left\{re^{i\theta};\ r \in [0, \infty] \cup \{\Phi\},\ \theta \in \mathbb{R}\right\} \subset \mathbb{C}^T$. Therefore

$$\mathbb{C}^T = \left\{re^{i\theta};\ r \in [0, \infty] \cup \{\Phi\},\ \theta \in \mathbb{R}\right\}.$$

Since for every $\alpha \in \mathbb{R}$ there is $\theta \in (-\pi, \pi]$ and $k \in \mathbb{Z}$ such that $\alpha = \theta + k2\pi$ whence $e^{i\alpha} = e^{i\theta}$, we can write

$$\mathbb{C}^T = \left\{re^{i\theta};\ r \in [0, \infty] \cup \{\Phi\},\ \theta \in (-\pi, \pi]\right\}.$$

Remark 15: Let $re^{i\theta}, se^{i\alpha} \in \mathbb{C}^T$. Notice that $re^{i\theta} = se^{i\alpha}$ if and only if $r = s$ and $\alpha = \theta + k2\pi$ for some $k \in \mathbb{Z}$.

Proposition 16: If $z, w \in \mathbb{C}^T$ then it is the case that $zw = |z||w|e^{i(\text{Arg}(z)+\text{Arg}(w))}$.

Proof: Let $z, w \in \mathbb{C}^T$ and $x, u \in \mathbb{C}$ and $y, t \in \{0, 1\}$, such that $z = \frac{x}{y}$ and $w = \frac{u}{t}$. It follows that, for all $k \in \mathbb{Z}$, $zw = \frac{x}{y}\frac{u}{t} = \frac{xu}{yt} = \left|\frac{xu}{yt}\right| e^{i\text{Arg}\left(\frac{xu}{yt}\right)} = \frac{|xu|}{yt}e^{i\text{Arg}(xu)} = \frac{|x||u|}{yt}e^{i(\text{Arg}(xu)+k2\pi)} = \frac{|x|}{y}\frac{|u|}{t}e^{i(\text{Arg}(x)+\text{Arg}(u))} = |z||w|e^{i(\text{Arg}(z)+\text{Arg}(w))}$. ∎

Corollary 17: If $z \in \mathbb{C}^T$ then $z^n = |z|^n e^{in\text{Arg}(z)}$.

Proposition 18: If $z, w \in \mathbb{C}^T$ and $n \in \mathbb{N}$ then $(zw)^n = z^n w^n$.

Proof: Let $z, w \in \mathbb{C}^T$ and $n \in \mathbb{N}$. It follows that $(zw)^n = \underbrace{(zw) \times (zw) \times \cdots \times (zw) \times (zw)}_{n \text{ times}}$.

Since transcomplex multiplication is associative and commutative (as proved in the Proposition 21), it follows that $(zw)^n = \underbrace{(zw) \times (zw) \times \cdots \times (zw) \times (zw)}_{n \text{ times}} = \underbrace{z \times z \times \cdots \times z \times z}_{n \text{ times}} \times \underbrace{w \times w \times \cdots \times w \times w}_{n \text{ times}} = z^n w^n$. ■

Definition 19: Given $z \in \mathbb{C}^T$ and $n \in \mathbb{N}$, we say that $w \in \mathbb{C}^T$ is an nth root of z if and only if $w^n = z$.

We define the transreal, non-negative, nth root of a non-negative, transreal number, a, in the following way: $\sqrt[n]{\Phi} := \Phi$; $\sqrt[n]{\infty} := \infty$, and, for all $a \in \mathbb{R}$, $\sqrt[n]{a}$ is the ordinary real positive nth root of a.

Notice that, for all $n \in \mathbb{N}$, zero is the only nth root of zero and nullity is the only nth root of nullity.

Proposition 20: Given $z \in \mathbb{C}^T \setminus \{0, \Phi\}$ and $n \in \mathbb{N}$, z has exactly n different nth roots, namely,

$$\sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+k2\pi}{n}\right)}$$

for each $k \in \{0, \ldots, n-1\}$ where $\sqrt[n]{|z|}$ denotes the transreal nth root of $|z|$.

Proof: Let $z \in \mathbb{C}^T \setminus \{0, \Phi\}$ and $n \in \mathbb{N}$. An $re^{i\theta} \in \mathbb{C}^T$ is the nth root of z if and only if $z = \left(re^{i\theta}\right)^n$, whence $|z|e^{i\text{Arg}(z)} = z = \left(re^{i\theta}\right)^n = r^n \left(e^{i\theta}\right)^n = r^n e^{in\theta}$. Hence $|z| = r^n$ and $\text{Arg}(z) = n\theta + k2\pi$ where $k \in \mathbb{Z}$, whence $r = \sqrt[n]{|z|}$ and $\theta = \frac{\text{Arg}(z)+k2\pi}{n}$ where $k \in \mathbb{Z}$. Thus the nth roots of z are the transcomplex numbers of the form $\sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+k2\pi}{n}\right)}$ where $k \in \mathbb{Z}$. Now notice that, reciprocally, for each $k \in \mathbb{Z}$, $\sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+k2\pi}{n}\right)}$ is an nth root of z. Notice also if $k, l \in \{0, \ldots, n-1\}$ and $k \neq l$ then $\sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+k2\pi}{n}\right)} \neq \sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+l2\pi}{n}\right)}$ and, furthermore, for all $l \in \mathbb{Z}$ there is $k \in \{0, \ldots, n-1\}$ such that $\sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+l2\pi}{n}\right)} = \sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+k2\pi}{n}\right)}$. Therefore, for all $k \in \{0, \ldots, n-1\}$, $\sqrt[n]{|z|}e^{i\left(\frac{\text{Arg}(z)+k2\pi}{n}\right)}$ are the n different nth roots of z. ■

Since $\Phi + z = \Phi$ for every $z \in \mathbb{C}^T$, there is no $z \in \mathbb{C}^T$ such that $\Phi + z = 0$. The fact that Φ does not have an additive inverse is sufficient to show that $\mathbb{C}^T$ is not a field, though it is a transfield [9]. In the next theorem we establish which field properties do hold in $\mathbb{C}^T$ and for all field properties that do not hold, we indicate the necessary restrictions. Thus we sharply delimit the field properties of $\mathbb{C}^T$ and obtain statements that could be taken as axioms in an axiomatic development of the transcomplex numbers.

Proposition 21: Let $a, b, c \in \mathbb{C}^T$. It follows that:

a) (Additive Commutativity) $a + b = b + a$.

b) (Additive Associativity) $(a+b)+c = a+(b+c)$ if and only if one of the five following conditions holds:
(I) either $a \notin \mathbb{C}^T_\infty$ or $b \notin \mathbb{C}^T_\infty$ or $c \notin \mathbb{C}^T_\infty$ or
(II) $a = c$ or
(III) $a = -c$ and $b = a$ or
(IV) $a = -c$ and $b = c$ or
(V) there is $z \in \mathbb{C}^T_\infty$ such that a, b and c are all cube roots of z.

c) (Additive Identity) $a + 0 = 0 + a = a$.

d) (Additive Inverse) $a - a = 0$ if and only if $a \notin \mathbb{C}^T \setminus \mathbb{C}$.

e) (Multiplicative Commutativity) $a \times b = b \times a$.

f) (Multiplicative Associativity) $(a \times b) \times c = a \times (b \times c)$.

g) (Multiplicative Identity) $a \times 1 = 1 \times a = a$.

h) (Multiplicative Inverse) $a \div a = 1$ if and only if $a \notin \{0\} \cup (\mathbb{C}^T \setminus \mathbb{C})$.

i) (Distributivity) $a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a)$ if and only if one of five following conditions holds:
(I) either $a = \Phi$ or $b = \Phi$ or $c = \Phi$ or
(II) $a \notin \mathbb{C}^T_\infty$ or
(III) $a, b, c \in \mathbb{C}^T \setminus \mathbb{C}$ or
(IV) $|b| = |c|$ or
(V) $\text{Arg}(b) = \text{Arg}(c)$ and $bc \neq 0$.

The proof of this Proposition 21 is in the Appendix.

In Proposition 21 we looked at what arithmetical properties of complex numbers are valid in the transcomplexes. As we have said, transcomplex numbers make a transfield. A transfield is a generalisation of a field – not in the sense that every transfield is a field, but in the sense that a field is a system of axioms that establishes properties of addition and multiplication which are extended by the special axioms of a transfield so that addition, multiplication, subtraction and division are total operations. Thus a transfield contains a field and supplies total operations.

We define a transfield so that the smallest number of axioms are used to admit the maximum possible structure of real arithmetic, subject to the constraint that a transfield describes both the transreals and the transcomplexes. A transfield is a set, T, provided with two binary operations, $+$ and $\times$, and two unary operations, $-$ and $^{-1}$, such that: T is closed for $+$, $\times$, $-$ and $^{-1}$; $+$ and $\times$ are commutative, each has an identity element and $\times$ is associative; there is $F \subset T$ such that F is a field, with respect to $+$ and $\times$, and F and T have common additive and multiplicative identities; and for each $x \in F$, $-x$ coincides with the additive inverse of x in F and, for each $x \in F$, different from the additive identity, x^{-1} coincides with the multiplicative inverse of x in F. The reader can find more about transfields in [9].

IV. Topology, Limits and Continuity

Henceforth we take $\theta \in (-\pi, \pi]$ in every $re^{i\theta} \in \mathbb{C}^T$.

Let $D := \{z \in \mathbb{C};\ |z| < 1\}$, $\overline{D} := \{z \in \mathbb{C};\ |z| \leq 1\}$ and

$$\begin{array}{rcl} \varphi : \mathbb{C}^T \setminus \{\Phi\} & \to & \overline{D} \subset \mathbb{C}^T \\ re^{i\theta} & \mapsto & \frac{1}{1+\frac{1}{r}} e^{i\theta} \end{array} . \tag{1}$$

Note that $\varphi_{|\mathbb{C}}$ is an homeomorphism between $\mathbb{C}$ and D with respect to the usual topology on $\mathbb{C}$.

Proposition 22: Define $d : \mathbb{C}^T \times \mathbb{C}^T \to \mathbb{R}$ where

$$d(z,w) = \begin{cases} 0, & \text{if } z = w = \Phi \\ 2, & \text{if } z = \Phi \text{ or else } w = \Phi. \\ |\varphi(z) - \varphi(w)|, & \text{otherwise} \end{cases}$$

We have that d is a metric on $\mathbb{C}^T$ and, therefore, $\mathbb{C}^T$ is a metric space.

Proof: Clearly, for all $z, w \in \mathbb{C}^T$, $d(z,w) = 0$ if and only if $z = w$, $d(z,w) = d(w,z)$ and $d(z,w) \geq 0$. If $z, w, u \in \mathbb{C}^T \setminus \{\Phi\}$ then $d(z,u) = |\varphi(z) - \varphi(u)| = |\varphi(z) - \varphi(w) + \varphi(w) - \varphi(u)| \leq |\varphi(z) - \varphi(w)| + |\varphi(w) - \varphi(u)| = d(z,w) + d(w,u)$. The reader can verify that the triangular inequality is also true when $z, w, u \in \mathbb{C}^T \setminus \{\Phi\}$ does not hold. ∎

Proposition 23: The topology on $\mathbb{C}$, induced by the topology of $\mathbb{C}^T$, is the usual topology of $\mathbb{C}$. That is, if $U \subset \mathbb{C}^T$ is open on $\mathbb{C}^T$ then $U \cap \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$ and if $U \subset \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$ then U is open on $\mathbb{C}^T$.

Proof: Let us denote the ball of centre z and radius ρ on $\mathbb{C}^T$ as $B_{\mathbb{C}^T}(z,\rho)$, that is, $B_{\mathbb{C}^T}(z,\rho) = \{w \in \mathbb{C}^T;\ |\varphi(z) - \varphi(w)| < \rho\}$, and denote the ball of centre z and radius ρ on $\mathbb{C}$ as $B_{\mathbb{C}}(z,\rho)$, that is, $B_{\mathbb{C}}(z,\rho) = \{w \in \mathbb{C};\ |z - w| < \rho\}$.

Let $U \subset \mathbb{C}^T$ be open on $\mathbb{C}^T$ and let $z \in U \cap \mathbb{C}$. As U is open on $\mathbb{C}^T$, there is a positive $\varepsilon \in \mathbb{R}$ such that $B_{\mathbb{C}^T}(z,\varepsilon) \subset U$. As $\varphi_{|\mathbb{C}}$ is continuous, there is a positive $\delta \in \mathbb{R}$ such that if $w \in \mathbb{C}^T \setminus \{\Phi\}$ and $|z - w| < \delta$ then $|\varphi(z) - \varphi(w)| < \varepsilon$. Thus $B_{\mathbb{C}}(z,\delta) \subset B_{\mathbb{C}^T}(z,\varepsilon) \cap \mathbb{C} \subset U \cap \mathbb{C}$, whence $U \cap \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$.

Now, let $U \subset \mathbb{C}$ be open (in the usual sense) on $\mathbb{C}$ and let $z \in U$. Notice that $z = re^{i\theta}$ for some $r \in [0,\infty)$ and some $\theta \in (-\pi, \pi]$. As U is open (in the usual sense) on $\mathbb{C}$, there is a positive $\varepsilon \in \mathbb{R}$ such that $B_{\mathbb{C}}(z,\varepsilon) \subset U$. As $\varphi^{-1}_{|D}$ is continuous, there is a positive $\delta \in \mathbb{R}$ such that $\delta < |\varphi(z) - e^{i\theta}|$ and if $\varphi(w) \in D$ and $|\varphi(z) - \varphi(w)| < \delta$ then $w \in \mathbb{C}$ and $|z - w| < \varepsilon$. Thus $B_{\mathbb{C}^T}(z,\delta) \subset B_{\mathbb{C}}(z,\varepsilon) \subset U$, whence U is open on $\mathbb{C}^T$. ∎

Corollary 24: If $A \subset \mathbb{C}^T$ is closed on $\mathbb{C}^T$ then $A \cap \mathbb{C}$ is closed (in the usual sense) on $\mathbb{C}$.

The next Remark 25 gives more details about our observations in Remark 4 of [8].

Remark 25: Note that φ is an homeomorphism. Indeed, let us show that φ is continuous. The proof that φ^{-1} is continuous is analogous. Let $z \in \mathbb{C}^T \setminus \{\Phi\}$ be arbitrary. If $z \in \mathbb{C}$ then the result holds from the Proposition 23 and the fact of $\varphi_{|\mathbb{C}}$ being continuous. If $z \in \mathbb{C}^T_\infty$ let $\varepsilon \in \mathbb{R}$ be positive arbitrary. Let $\theta_z \in (-\pi, \pi]$ such that $z = \infty e^{i\theta_z}$. As the function $[0,\infty] \ni r \mapsto \frac{1}{1+\frac{1}{r}} \in [0,1]$ is continuous, there is a positive $M \in \mathbb{R}$ such that $\frac{1}{2} - \frac{1}{1+\frac{1}{\frac{1}{1+\frac{1}{r}}}} < \frac{\varepsilon}{2}$ whenever $r \in (M, \infty]$. As the function $\mathbb{R} \ni \theta \mapsto e^{i\theta} \in \mathbb{C}$ is continuous, there is a positive $\delta \in \mathbb{R}$ such that $|e^{i\theta} - e^{i\theta_z}| < \varepsilon$ whenever $\theta \in \mathbb{R}$ and $|\theta - \theta_z| < \delta$. Thus if $re^{i\theta} \in \mathbb{C}^T$ and $r \in (M, \infty]$ and $|\theta - \theta_z| < \delta$ then $d(\varphi(re^{i\theta}), \varphi(\infty e^{i\theta_z})) \leq d(\varphi(re^{i\theta}), \varphi(re^{i\theta_z})) + d(\varphi(re^{i\theta_z}), \varphi(\infty e^{i\theta_z})) = |\varphi(\varphi(re^{i\theta})) - \varphi(\varphi(re^{i\theta_z}))| + |\varphi(\varphi(re^{i\theta_z})) - \varphi(\varphi(\infty e^{i\theta_z}))| = \frac{1}{1+\frac{1}{\frac{1}{1+\frac{1}{r}}}}|e^{i\theta} - e^{i\theta_z}| + \left(\frac{1}{2} - \frac{1}{1+\frac{1}{\frac{1}{1+\frac{1}{r}}}}\right) < \frac{1}{2}\varepsilon + \frac{\varepsilon}{2} < \varepsilon$. Therefore φ is continuous.

Remark 26: Because of Proposition 23:

i) Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and let $L \in \mathbb{C}$, it follows that $\lim_{n\to\infty} x_n = L$ on $\mathbb{C}^T$ if and only if $\lim_{n\to\infty} x_n = L$, in the usual, sense on $\mathbb{C}$.
ii) Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$, $x \in A'$ and $L \in \mathbb{C}$, it follows that $\lim_{x\to x} f(z) = L$ on $\mathbb{C}^T$ if and only if $\lim_{x\to x} f(z) = L$, in the usual sense, on $\mathbb{C}$.
iii) Given $x \in A$, it follows that f is continuous in x on $\mathbb{C}^T$ if and only if f is continuous in x, in the usual sense, on $\mathbb{C}$.

Proposition 27: $\mathbb{C}^T$ is disconnected.

Proof: $\mathbb{C}^T = \left\{re^{i\theta};\ r \in [0,\infty],\ \theta \in (-\pi,\pi]\right\} \cup \{\Phi\}$ and the sets $\left\{re^{i\theta};\ r \in [0,\infty],\ \theta \in (-\pi,\pi]\right\}$ and $\{\Phi\}$ are open. ∎

Notice that Φ is the unique isolated point of $\mathbb{C}^T$.

Remark 28:

i) Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}^T$. Notice that $\lim_{n\to\infty} x_n = \Phi$ if and only if there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$.
ii) Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}^T$ and $x \in A'$, it follows that $\lim_{x\to x} f(z) = \Phi$ if and only if there is a neighbourhood U of x such that $f(z) = \Phi$ for all $x \in U \setminus \{x\}$.
iii) If $\Phi \in A$ then f is continuous in Φ.

Proposition 29: $\mathbb{C}^T$ is a separable space.

Proof: $(\mathbb{Q} + \mathbb{Q}i) \cup \{\Phi\}$ is countable and dense in $\mathbb{C}^T$. ∎

Proposition 30: Every sequence of transcomplex numbers has a convergent subsequence.

Proof: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}^T$. If $\{n;\ x_n \neq \Phi\}$ is a finite set then clearly $\lim_{n\to\infty} x_n = \Phi$. If $\{n;\ x_n \neq \Phi\}$ is an infinite set then denote, by $(y_k)_{k\in\mathbb{N}}$, the subsequence of $(x_n)_{n\in\mathbb{N}}$ of all elements of $(x_n)_{n\in\mathbb{N}}$ that are distinct from Φ. Note that $(\varphi(y_k))_{k\in\mathbb{N}}$ (φ defined in (1)) is a bounded sequence of complex numbers, whence it has a convergent subsequence, denoted $(\varphi(y_{k_m}))_{m\in\mathbb{N}}$. As φ is an homeomorphism, $(y_{k_m})_{m\in\mathbb{N}}$ is convergent. ∎

Proposition 31: $\mathbb{C}^T$ is compact.

Proof: As $\mathbb{C}^T$ is a metric space and every sequence from $\mathbb{C}^T$ has a convergent subsequence, $\mathbb{C}^T$ is compact. ∎

Corollary 32: Let $A \subset \mathbb{C}^T$. It follows that A is compact if and only if A is closed.

Proposition 33: $\mathbb{C}^T$ is complete.

Proof: Every compact, metric space is complete and $\mathbb{C}^T$ is compact and metric. ∎

V. ELEMENTARY FUNCTIONS

A. Polynomial Functions

A function, f, is a complex, polynomial function if and only if there is $n \in \mathbb{N}$ and $a, \dots, a_n \in \mathbb{C}$ such that $f(z) = a_n x^n + \cdots + a_1 x + a$ for all $x \in \mathbb{C}$. As every arithmetical operation is well-defined in transcomplex numbers, we extend the function f to $\mathbb{C}^T$ naturally. In the complex domain, $0 \times x^k = 0$ for all complex x but $0 \times x^k = 0$ does not hold for all transcomplex x. In order to avoid this problem we adopt the following definition.

Definition 34: A function, f, is a transcomplex, polynomial function if and only if there is $n, k \in \mathbb{N}$; $n_1, \dots, n_k \in \{1, \dots, n-1\}$ and $a, a_{n_1} \dots, a_{n_k}, a_n \in \mathbb{C}$ such that $a_{n_1} \dots, a_{n_k}, a_n$ are different from zero and

$$\begin{array}{rcl} f : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & a_n x^n + a_{n_k} x^{n_k} + \cdots + a_{n_1} x^{n_1} + a \end{array}.$$

Remark 35: For every non-constant, transcomplex, polynomial function, f, we have that $f(\Phi) = \Phi$.

B. Exponential Functions

In [13] we defined the transreal, exponential function. We have that $e^{-\infty} = 0$, $e^{\infty} = \infty$ and $e^{\Phi} = \Phi$.

For every ordinary, complex number, $z = re^{i\theta}$, we have

$$\begin{array}{rcl} \exp(z) & = & \exp\left(re^{i\theta}\right) \\ & = & \exp\left(r\cos(\theta) + ir\sin(\theta)\right) \\ & = & e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta))). \end{array}$$

In particular, when $\theta \in \{0, \pi\}$, we have that $\sin(\theta) = 0$ whence

$$\begin{array}{rcl} \exp\left(re^{i\theta}\right) & = & e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta))) \\ & = & e^{r\cos(\theta)}(\cos(0) + i\sin(0)) \\ & = & e^{r\cos(\theta)}. \end{array}$$

Motivated by this, we extend the exponential function to the transcomplex domain in the following way.

Definition 36: A function, f, is a transcomplex, natural, exponential function if and only if

$$\begin{array}{rcl} f : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ re^{i\theta} & \longmapsto & \exp\left(re^{i\theta}\right) \end{array}$$

where $\exp\left(re^{i\theta}\right) = e^{r\cos(\theta)}$ if $\theta \in \{0, \pi\}$ and $\exp\left(re^{i\theta}\right) = e^{r\cos(\theta)}(\cos(\infty\sin(\theta)) + i\sin(\infty\sin(\theta)))$ if $\theta \notin \{0, \pi\}$.

Notice that $\exp(\infty) = \exp\left(\infty e^{i0}\right) = e^{\infty\cos(0)} = e^{\infty}$ and $\exp(-\infty) = \exp\left(\infty e^{i\pi}\right) = e^{\infty\cos(\pi)} = e^{-\infty}$. Furthermore, if $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$ then

$$\begin{array}{rcl} \exp\left(\infty e^{i\theta}\right) & = & e^{\infty\cos(\theta)}(\cos(\infty\sin(\theta)) + i\sin(\infty\sin(\theta))) \\ & = & e^{\infty\cos(\theta)}(\Phi + i\Phi) \\ & = & e^{\infty\cos(\theta)}\Phi \\ & = & \Phi. \end{array}$$

Therefore:

i) $\exp(z) = e^z$ for every $z \in \mathbb{C}$.
ii) $\exp(-\infty) = 0$.
iii) $\exp(\infty) = \infty$.
iv) $\exp\left(\infty e^{i\theta}\right) = \Phi$ for all $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$.
v) $\exp(\Phi) = \Phi$.

Remark 37: Notice that $\exp$ is discontinuous in all infinities.

Remark 38: Unfortunately, the property $\exp(z+w) = \exp(z)\exp(w)$ does not hold for all $z, w \in \mathbb{C}^T$. For example, let $z = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{0}$ and $w = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}{0}$. We have that $z + w = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{0} + \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}{0} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}{0} = \frac{\sqrt{2}}{0} = \frac{1}{0} = \infty$ whence $\exp(z+w) = e^{\infty} = \infty$. But $z = \infty e^{\frac{\pi}{4}i}$ and $w = \infty e^{-\frac{\pi}{4}i}$ whence $\exp(z) = \exp(\infty e^{\frac{\pi}{4}i}) = \Phi$ and $\exp(w) = \exp(\infty e^{\frac{-\pi}{4}i}) = \Phi$. Thus $\exp(z)\exp(w) = \Phi \times \Phi = \Phi$. Therefore $\exp(z+w) \neq \exp(z)\exp(w)$. Another example is when $z = \infty$ and $w = i$. We have that $z + w = \infty + i = \infty$ whence $\exp(z+w) = \exp(\infty) = \infty$ but $\exp(z) = e^{\infty} = \infty$ and $\exp(w) = e^i$ whence $\exp(z)\exp(w) = \infty e^i$. Thus $\exp(z+w) \neq \exp(z)\exp(w)$.

C. Logarithmic Functions

In [13] we defined the transreal, logarithmic function. We have that $\ln(0) = -\infty$, $\ln(\infty) = \infty$ and $\ln(\Phi) = \Phi$.

A function, f, is the complex, logarithmic function if and only if $f(re^{i\theta}) = \ln(r) + i\theta$ for all $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$. Motivated by this, we extend the logarithmic function to the transcomplex domain in the following way.

Definition 39: A function, f, is a transcomplex, natural, logarithmic function if and only if

$$\begin{array}{rcl} f : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ re^{i\theta} & \longmapsto & \ln(r) + i\theta \end{array}.$$

Remark 40: Notice that, for every $\theta \in (-\pi, \pi]$, we have $\ln(\infty e^{i\theta}) = \ln(\infty) + i\theta = \infty + i\theta = \infty$. So $\ln(z) = \infty$ for every transcomplex infinity z.

Remark 41: The property $\ln(\exp(z)) = z$ does not hold for all $z \in \mathbb{C}^T$. If $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$ then $\ln(\exp(\infty e^{i\theta})) = \ln(\Phi) = \Phi \neq \infty e^{i\theta}$. But $\ln(\exp(z)) = z$ holds in the other cases. Indeed:

- if $z = a + bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$ and $b \in (-\pi, \pi]$, then we already know that $\ln(\exp(z)) = z$,
- if $z = \Phi$ then $\ln(\exp(z)) = \ln(\exp(\Phi)) = \ln(\Phi) = \Phi = z$,
- if $z = -\infty$ then $\ln(\exp(z)) = \ln(\exp(-\infty)) = \ln(0) = -\infty = z$, and
- if $z = \infty$ then $\ln(\exp(z)) = \ln(\exp(\infty)) = \ln(\infty) = \infty = z$.

In the same way $\exp(\ln(z)) = z$ does not hold for all $z \in \mathbb{C}^T$. If $\theta \in (-\pi, \pi] \setminus \{0\}$ then $\exp(\ln(\infty e^{i\theta})) = \exp(\infty) = \infty \neq \infty e^{i\theta}$. But $\exp(\ln(z)) = z$ holds in the other cases. Indeed:

- if $z \in \mathbb{C} \setminus \{0\}$ then we already know that $\exp(\ln(z)) = z$,
- if $z = 0$ then $\exp(\ln(z)) = \exp(\ln(0)) = \exp(-\infty) = 0 = z$,
- if $z = \Phi$ then $\exp(\ln(z)) = \exp(\ln(\Phi)) = \exp(\Phi) = \Phi = z$ and
- if $z = \infty$ then $\exp(\ln(z)) = \exp(\ln(\infty)) = \exp(\infty) = \infty = z$.

Remark 42: We know that, for all $z, w \in \mathbb{C} \setminus \{0\}$, $\ln(zw) = \ln(z) + \ln(w) + ki2\pi$ for some $k \in \mathbb{Z}$. Fortunately this property also holds in transcomplex domain. That is, for all $z, w \in \mathbb{C}^T$, $\ln(zw) = \ln(z) + \ln(w) + ki2\pi$ for some $k \in \mathbb{Z}$. In particular if the two conditions $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C} \setminus \{0\}$ do not hold simultaneously then $\ln(zw) = \ln(z) + \ln(w)$. The reader can prove this with simple calculations.

Remark 43: Definition 39 can give us powers of every transcomplex base so we define $z^w := \exp(w \ln(z))$ for all $z, w \in \mathbb{C}^T$.

D. Trigonometric Functions

In [13] we defined the transreal trigonometric functions. We have that $\sin(-\infty) = \cos(-\infty) = \tan(-\infty) = \csc(-\infty) = \sec(-\infty) = \cot(-\infty) = \sin(\infty) = \cos(\infty) = \tan(\infty) = \csc(\infty) = \sec(\infty) = \cot(\infty) = \sin(\Phi) = \cos(\Phi) = \tan(\Phi) = \csc(\Phi) = \sec(\Phi) = \cot(\Phi) = \Phi$.

A function, f, is the complex, sine function if and only $f(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$ for all $z \in \mathbb{C}$ and f is the complex, cosine function if and only $f(z) = \frac{\exp(iz) + \exp(-iz)}{2}$ for all $z \in \mathbb{C}$. Furthermore, for all $k \in \mathbb{Z}$, it is the case that $\frac{\sin(z)}{\cos(z)}$, $\frac{1}{\sin(z)}$, $\frac{1}{\cos(z)}$ and $\frac{\cos(z)}{\sin(z)}$ are lexically well-defined at $\frac{\pi}{2} + k\pi$ and $k\pi$ in the transcomplex domain. Because of this we extend the trigonometric functions to $\mathbb{C}^T$ in the following way.

Definition 44: A function is a transcomplex, trigonometric function if and only if it is one of:

a) $\begin{array}{rcl} \sin : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i} \end{array}$,

b) $\begin{array}{rcl} \cos : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \end{array}$,

c) $\begin{array}{rcl} \tan : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \tan(z) = \frac{\sin(z)}{\cos(z)} \end{array}$,

d) $\begin{array}{rcl} \csc : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \csc(z) = \frac{1}{\sin(z)} \end{array}$,

e) $\begin{array}{rcl} \sec : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \sec(z) = \frac{1}{\cos(z)} \end{array}$ and

f) $\begin{array}{rcl} \cot : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \cot(z) = \frac{\cos(z)}{\sin(z)} \end{array}$.

Remark 45: In [13] we show that $\sin^2(x) + \cos^2(x) = 1^x$ for all $x \in \mathbb{R}^T$. Unfortunately this property does not hold for all transcomplex numbers. We have that $\sin^2(z) + \cos^2(z) = 1^z$ if and only if $z \in \mathbb{C}^T \setminus \{-i\infty, i\infty\}$. Note that, by Remark 43, $1^z = \Phi$ if $z \in \mathbb{C}^T \setminus \mathbb{C}$.

VI. TOTALISATION

A. Recursive Totalisation

The work, above, develops the elementary, transcomplex functions as functions of transcomplex numbers, which numbers are expressible as tuples, $\langle r, \theta \rangle$, of a transreal radius, r, and a transreal angle, θ. This is adequate from a mathematical point of view but it is not sufficient for computer science where total functions are wanted whose domain can be recursively decomposed into the entire domain of transreal numbers so that, here, r and θ could be any transreals. As usual when a negative r occurs, we map r to its modulus and increment the angle by π so that all transreal radii are admitted. We observe that for all non-finite angles, $\theta \in \{-\infty, \infty, \Phi\}$, it is the case that $re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = r(\Phi + \Phi) = r\Phi = \Phi$ so that the exponential, logarithmic and trigonometric functions admit all transreal angles. The totalisation of the remaining elementary functions is immediate.

It is well known that the trigonometric functions can be defined, equivalently, by power series or by geometrical constructions. The totalisation of angle, just given, relies on power series. We now give a geometrical construction of the transreal angles.

B. Geometrical Construction of the Transreal Angles

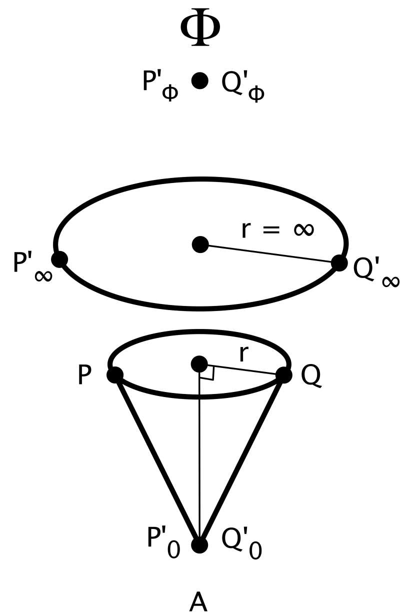

Fig. 4. Transreal cone

Let us explore both finite and non-finite angles in a geometrical construction before settling on a definition of transreal angle.

Consider a transreal cone with apex A, as shown in Figure 4. A right cross-section of the cone is a circle on which a radius, r, may be drawn. On the circle at unit radius, $r = 1$, mark off, not necessarily distinct, points P and Q. Project the lines AP and AQ, taking a point P' anywhere on AP, including the point P'_0 at A, the point P'_∞ on the circle at infinity and the point P'_Φ at the point at nullity, shown as Φ in the figure. Similarly take Q' on AQ.

At $r = 1$ the angle from P to Q is defined to be the arc length $\widehat{PQ}$ taken zero, positive or negative according to the usual sign convention. It is then shown that identical plane rotations arise for all non-negative, finite radii, $0 < r < \infty$, when the angle is given by $\widehat{PQ} = \widehat{P'Q'}/r$ when P' and Q' lie on the circle with radius r. We now consider the cases $r \in \{0, \infty, \Phi\}$. The reader is free to construct negative radii in a double cone.

At $r = \Phi$ we have $\widehat{P'_\Phi Q'_\Phi}/r_\Phi = \widehat{\Phi\Phi}/0 = 0/0 = \Phi$, which is to say that the angle nullity occurs at $r = \Phi$. Now $re^{i\Phi} = r(\cos(\Phi) + i\sin(\Phi)) = r(\Phi + \Phi) = r\Phi = \Phi$ for all transreal r. Thus the nullity rotation, by angle nullity, $\theta = \Phi$, maps the whole of its domain onto the point at nullity, Φ.

At $r = 0$ we have $\widehat{P_0'Q_0'}/r_0 = \widehat{AA}/0 = 0/0 = \Phi$, which is to say that the angle nullity also occurs at $r = 0$. This is a redundancy which we shall presently resolve.

At $r = \infty$ the zero angle, $\theta = 0$, arises when P'_∞ and Q'_∞ are co-punctal but when P'_∞ and Q'_∞ are distinct we have $\theta = \widehat{P'_\infty Q'_\infty}/r_\infty = \infty/\infty = \Phi$ so that only the angles zero and nullity can arise, from this geometrical construction, in the circle at infinity. This computed angle of nullity is degenerate in the sense that it hides the true value of any non-zero, finite angle in the circle at infinity. That is it hides all points $\infty e^{i\theta}$ with $\theta \neq 0$. Information hiding is discussed in [2]. We shall presently avoid this degeneracy.

We now construct the infinite angle via a winding on the ordinary, unit cone.

By definition P and Q lie in the unit circle, separated by an angle $\theta = \widehat{PQ}$. When P and Q are distinct we take an arc length $\alpha = \widehat{PQ}$ and when P and Q are co-punctal we take $\alpha = 2\pi$. We now take the arc at a smaller radius and wind it from P', once fully round the cone, and continue exactly to Q'. This winding marks off the angle $\theta_1 = \theta + 2\pi$. We continue in this way, recursively winding at smaller radii, to produce the family of angles $\theta_k = \theta + k2\pi$. We suppose that the winding process is continuous to that at $r = 0$ we produce the winding $\theta_\infty = \theta + 2\infty\pi = \infty$. But this rotation is identically the rotation at $r = 0$ so the infinite angle is equivalent to the nullity angle. This agrees with the result obtained from power series.

Notice that all transreal angles are given uniquely when we define a zero angle at a fixed point, Z, on the base of the cone. Let us take $Z = Q$ in Figure 4. Now all angles, θ, in the principle range $-\pi < \theta \leq \pi$ are given uniquely by a point in the unit circle with radius $r = 1$. All finite angles, $-\infty < \theta < \infty$, are given uniquely by windings on the cone at all positive radii, $0 < r \leq 1$. And the equivalence class of all non-finite angles is given uniquely by the apex of the cone at $r = 0$. Hence the apex of the unit cone uniquely defines the non-finite angle and the unit cone, with the apex punctured and a zero point identified, defines each finite angle uniquely. Thus there is an injection from transreal angles to points on the unit cone, with the zero point identified. This being the case we accept the winding construction in the unit cone as our definition of transreal angle and simply note the extraneous behaviour in the circle at infinity and the point at nullity.

In the definition we have just adopted, we assume continuity of the winding process to obtain the infinite angle equal to the nullity angle, which is what we wanted to achieve. This commits us to continuity of winding everywhere, including all applications of winding in topology, in complex analysis and in mathematical physics.

VII. Discussion

We have established several properties on transcomplex numbers which are similar to the properties in $\mathbb{C}$. The finite, transcomplex numbers are identical to the complex numbers but that their field structure is generalised to a transfield to account for the non-finite transcomplexes. The properties of the complement, modulus, exponential, logarithm and n'th root generalise in a natural way. In particular, we have proved that:

- Transcomplex addition is commutative, is partially associative, has an identity element and every finite, non-zero, transcomplex number has an inverse element; and transcomplex multiplication is commutative, is associative, is partially distributive with respect to transcomplex addition, has identity element and every finite, non zero, transcomplex number has an inverse element.
- $\overline{\overline{z}} = z$; $\overline{z+w} = \overline{z} + \overline{w}$; $\overline{-w} = -\overline{w}$; $\overline{z-w} = \overline{z} - \overline{w}$; $\overline{zw} = \overline{z}\overline{w}$; $\overline{w^{-1}} = \overline{w}^{-1}$ and $\overline{z \div w} = \overline{z} \div \overline{w}$; for all $z, w \in \mathbb{C}^T$.
- $|z|^2 = z\overline{z}$; $|\overline{z}| = |z|$; $|zw| = |z||w|$; $|w^{-1}| = |w|^{-1}$; $|z \div w| = |z| \div |w|$ and $|z + w| \not> |z| + |w|$; for all $z, w \in \mathbb{C}^T$.
- $zw = |z||w|e^{i(\mathrm{Arg}(z)+\mathrm{Arg}(w))}$ and $z^n = |z|^n e^{in\mathrm{Arg}(z)}$ for all $z, w \in \mathbb{C}^T$ and given $z \in \mathbb{C}^T \setminus \{0, \Phi\}$ and $n \in \mathbb{N}$, z has exactly n different nth roots, namely, $\sqrt[n]{|z|}e^{i\left(\frac{\mathrm{Arg}(z)+k2\pi}{n}\right)}$ for each $k \in \{0, \ldots, n-1\}$.

We have equipped the set of transcomplex numbers with a topology, given by a metric, that contains the usual topology of the complex numbers. This preserves many properties of complex numbers and leads to consistent generalisations of them. In particular it extends the usual geometrical constructions of the real trigonometric functions to their transreal counterparts, from which we obtain the transcomplex, trigonometric functions. Thus the topology provides a firm foundation for our work. As $\mathbb{C}^T$ is a metric space, all usual results of that space follow. For example: $\mathbb{C}^T$ is a Hausdorff space; the limit of a sequence, when it exists, is unique; when $A \subset \mathbb{C}^T$, $f : A \to \mathbb{C}^T$, $x \in A'$ and $L \in \mathbb{C}^T$, we have that $\lim_{x\to x} f(z) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for all $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$; and when $A \subset \mathbb{C}^T$, $f : A \to \mathbb{C}^T$ and $x \in A$, we have that f is continuous in x if and only if $\lim_{n\to\infty} f(x_n) = f(z)$ for all $(x_n)_{n\in\mathbb{N}} \subset A$ and $\lim_{n\to\infty} x_n = x$.

In [12] we adopted the following procedure to extend an elementary function from the real to the transreal domain. If the usual expression of the function is lexically well-defined, at a transreal number, then we define the function by simply applying its expression at that transreal number. If the function, f, is not lexically well-defined at a transreal number, x, but there is a limit, $\lim_{x\to x} f(z)$, then we choose to define the function at x by $\lim_{x\to x} f(z)$. Otherwise we choose to define the function by way of its power series if it converges. And if, nevertheless, its power series does not converge, we keep the function undefined. But the transcomplex space is more complicated than transreal space. Transreal space has only two infinite numbers and there is only one path, one direction, to each one of these infinities but there are several (infinite) paths and directions to each infinite transcomplex number. Hence many limits do not exist at infinite transcomplexes.

Now let us address some remarks to why we did not adopt other ways to define the exponential function on the transcomplex plane.

i) We cannot define the transcomplex, exponential function by a lexical expression because the exponential is not defined by finitely many arithmetical operations. In particular we cannot take the usual algebraic definition that if $z = a + bi$ then $\exp(z) = e^a(\cos(b) + i\sin(b))$ because the infinite transcomplex numbers do not have any algebraic representation.

ii) We cannot define the transcomplex, exponential function by limits because, for every $r \in [0, \infty)$, $\exp\left(re^{i\theta}\right) = \exp\left(r\cos(\theta) + ir\sin(\theta)\right) = e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta)))$, whence there is no $\lim_{r\to\infty}\exp\left(re^{i\theta}\right)$ for every $\theta \in (-\pi, \pi]\setminus\{0, \pi\}$.

iii) We cannot define the transcomplex, exponential function by power series because $1 + \sum_{n=1}^{\infty}\frac{\left(\infty e^{i\theta}\right)^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{\infty e^{in\theta}}{n!} = 1 + \sum_{n=1}^{\infty}\infty e^{in\theta}$ diverges for every $\theta \in (-\pi, \pi]\setminus\{0, \pi\}$.

iv) We could think about the homeomorphism φ. Notice that the circle at infinity, $\{\infty e^{i\theta};\ \theta \in (-\pi, \pi]\}$, is a homeomorphic copy, by the function φ, of the unitary circle, $\partial D := \{e^{i\theta};\ \theta \in (-\pi, \pi]\}$. So, in order to define $\exp$ at an infinite transcomplex number $\infty e^{i\theta}$, we could transform $\infty e^{i\theta}$ to $\varphi(\infty e^{i\theta}) = e^{i\theta}$, then we would take $\exp\left(e^{i\theta}\right) = \exp\left(\cos(\theta) + i\sin(\theta)\right) = e^{\cos(\theta)}(\cos(\sin(\theta)) + i\sin(\sin(\theta)))$, after that we would transform $e^{\cos(\theta)}(\cos(\sin(\theta)) + i\sin(\sin(\theta)))$ to $\frac{e^{\cos(\theta)}(\cos(\sin(\theta))+i\sin(\sin(\theta)))}{|e^{\cos(\theta)}(\cos(\sin(\theta))+i\sin(\sin(\theta)))|} = \cos(\sin(\theta)) + i\sin(\sin(\theta))$ and, finally, we would transform $\cos(\sin(\theta)) + i\sin(\sin(\theta))$ to $\varphi^{-1}\left(\cos(\sin(\theta)) + i\sin(\sin(\theta))\right)$. In this way, denoting the function $\mathbb{C}\setminus\{0\} \ni z \mapsto \frac{z}{|z|} \in \partial D$ by h, we would define $\exp\left(\infty e^{i\theta}\right) := \left(\varphi^{-1}\circ h \circ \exp\circ\varphi\right)\left(\infty e^{i\theta}\right)$. This would define the exponential of all transcomplex infinities but the transcomplex exponential, $\exp_{\mathbb{C}^T}$, would not be an extension of the transreal exponential, $\exp_{\mathbb{R}^T}$. In fact $\exp_{\mathbb{C}^T}(-\infty) = \exp_{\mathbb{C}^T}\left(\infty e^{i\pi}\right) = \left(\varphi^{-1}\circ h\circ\exp\circ\varphi\right)\left(\infty e^{i\pi}\right) = \left(\varphi^{-1}\circ h\circ\exp\right)\left(\varphi\left(\infty e^{i\pi}\right)\right) = \left(\varphi^{-1}\circ h\circ\exp\right)\left(e^{i\pi}\right) = \left(\varphi^{-1}\circ h\circ\exp\right)(-1) = (\varphi^{-1}\circ h)(\exp(-1)) = (\varphi^{-1}\circ h)(e^{-1}) = \varphi^{-1}(h(e^{-1})) = \varphi^{-1}\left(\frac{e^{-1}}{|e^{-1}|}\right) = \varphi^{-1}(1) = \infty \neq 0 = \exp_{\mathbb{R}^T}(-\infty)$.

In future we intend to extend the differential and integral calculi from the complex to the transcomplex domain, opening up the way to extend our generalisation of Newtonian Physics [2] to both relativistic and quantum physics.

VIII. Conclusion

The transcomplex numbers, introduced elsewhere, contain the complex, transreal and real numbers and support division by zero, consistently, in all of their arithmetics.

Here we show that the transcomplex complement and modulus are well defined and we supply the set of transcomplex numbers with a topology that contains the usual topology of both the complex and real numbers. It is easy to see that our transcomplex topology also contains transreal topology. Thus we maintain all of these topologies within a single number system.

We extend the exponential from the complex to the transcomplex domain so that it contains the complex, transreal and real exponentials. Hence we obtain the transcomplex logarithm and the transcomplex, trigonometric functions and all transcomplex, elementary functions, such that they contain their complex, transreal and real counterparts. This gives us the n'th roots of a transcomplex number.

We give a geometrical construction of transreal angle, including the non-finite angles. Thus the equivalence of geometrical and power series definitions of the trigonometric functions is maintained. We stress that our geometrical construction of angle assumes continuity of the winding process so we are committed to this continuity wherever winding occurs, for example in topology, in complex analysis and in mathematical physics.

All of the transarithmetics are total. This removes infinitely many exceptions from mathematics and from computer programs. Thus our corpus of work continues to offer both theoretical and practical advantages.

Appendix A
Proof of Proposition 21

Proof: Let $a, b, c \in \mathbb{C}^T$. Let us denote $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$ and $c = \frac{c_1}{c_2}$ where $a_1, b_1, c_1 \in \mathbb{C}$ and $a_2, b_2, c_2 \in \{0, 1\}$.

a) If $a_2 = b_2 = 0$, $a_1 \neq 0$ and $b_1 \neq 0$ then $a + b = \frac{a_1}{a_2} + \frac{b_1}{b_2} = \frac{\frac{a_1}{|a_1|}+\frac{b_1}{|b_1|}}{0} = \frac{\frac{b_1}{|b_1|}+\frac{a_1}{|a_1|}}{0} = \frac{b_1}{b_2} + \frac{a_1}{a_2} = b + a$. Otherwise $a + b = \frac{a_1}{a_2} + \frac{b_1}{b_2} = \frac{a_1b_2+b_1a_2}{a_2b_2} = \frac{b_1a_2+a_1b_2}{b_2a_2} = \frac{b_1}{b_2} + \frac{a_1}{a_2} = b + a$.

b) (I) Suppose $a \notin \mathbb{C}^T_\infty$ (the cases $b \notin \mathbb{C}^T_\infty$ and $c \notin \mathbb{C}^T_\infty$ are analogous). Having $a \notin \mathbb{C}^T_\infty$, we have two possibilities:
- $b, c \in \mathbb{C}^T_\infty$. In this case: If $a_2 = 0$ then $a = \Phi$ whence $(a + b) + c = (\Phi + b) + c = \Phi + c = \Phi = \Phi + (b + c) = a + (b + c)$; if $a_2 \neq 0$ then $(a + b) + c = \left(\frac{a_1}{1} + \frac{b_1}{0}\right) + \frac{c_1}{0} = \frac{a_1\times 0+b_1\times 1}{1\times 0} + \frac{c_1}{0} = \frac{0+b_1}{0} + \frac{c_1}{0} = \frac{b_1}{0} + \frac{c_1}{0} = \frac{\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}}{0} = \frac{0+\left(\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}\right)}{0} = \frac{a_1\times 0+\left(\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}\right)\times 1}{1\times 0} = \frac{a_1}{1} + \frac{\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}}{0} = \frac{a_1}{1} + \left(\frac{b_1}{0} + \frac{c_1}{0}\right) = a + (b + c)$.
- $b, c \in \mathbb{C}^T_\infty$ does not hold. In this case, $(a + b) + c = \left(\frac{a_1}{a_2} + \frac{b_1}{b_2}\right) + \frac{c_1}{c_2} = \frac{a_1b_2+b_1a_2}{a_2b_2} + \frac{c_1}{c_2} = \frac{(a_1b_2+b_1a_2)c_2+c_1a_2b_2}{a_2b_2c_2} = \frac{a_1b_2c_2+b_1a_2c_2+c_1a_2b_2}{a_2b_2c_2} = \frac{a_1b_2c_2+b_1c_2a_2+c_1b_2a_2}{a_2b_2c_2} = \frac{a_1b_2c_2+(b_1c_2+c_1b_2)a_2}{a_2b_2c_2} = \frac{a_1}{a_2} + \frac{b_1c_2+c_1b_2}{b_2c_2} = \frac{a_1}{a_2} + \left(\frac{b_1}{b_2} + \frac{c_1}{c_2}\right) = a + (b + c)$.

(II) If $a = c$ then $(a+b)+c = (a+b)+a = a+(a+b) = a + (b + a) = a + (b + c)$.

(III) Suppose $a = -c$ and $b = a$. If either $a \notin \mathbb{C}^T_\infty$ or $b \notin \mathbb{C}^T_\infty$ or $c \notin \mathbb{C}^T_\infty$ then the result follows from item (I); otherwise, $(a+b)+c = (a+a)-a = 2a-a = a-a = \Phi = a+\Phi = a+(-c+c) = a+(a+c) = a+(b+c)$.

(IV) Suppose $a = -c$ and $b = c$. This case is analogous to item (III).

(V) If there is $z \in \mathbb{C}^T_\infty$ such that a, b and c are all cube roots of z then $(a + b) + c = -c + c = \Phi = a - a = a + (b + c)$.

Now suppose that $(a + b) + c = a + (b + c)$. If either $a \notin \mathbb{C}^T_\infty$ or $b \notin \mathbb{C}^T_\infty$ or $c \notin \mathbb{C}^T_\infty$ then there is nothing to show; otherwise let us show that either $a = c$ or $a = -c$ and $b = a$ or $a = -c$ and $b = c$ or there is $z \in \mathbb{C}^T_\infty$ such that a, b and c are all cube roots of z. Indeed, $a = \frac{a_1}{0}$, $b = \frac{b_1}{0}$ and $c = \frac{c_1}{0}$. Without loss of generality, suppose $|a_1| = |b_1| = |c_1| = 1$. Hence $(a + b) + c = \left(\frac{a_1}{0} + \frac{b_1}{0}\right) + \frac{c_1}{0} = \frac{a_1+b_1}{0} + \frac{c_1}{0}$ and $a + (b + c) = \frac{a_1}{0} + \left(\frac{b_1}{0} + \frac{c_1}{0}\right) = \frac{a_1}{0} + \frac{b_1+c_1}{0}$. Therefore, if $(a+b)+c = a+(b+c)$ then we have five possibilities:

- $b_1 = -a_1$ and $b_1 = -c_1$. In this case, $a_1 = c_1$ whence $a = c$.
- $b_1 = -a_1$ and $b_1 \neq -c_1$ and $a_1|b_1+c_1|+b_1+_1c = 0$. In this case, $-b_1|b_1+c_1|+b_1+c_1 = 0$ whence $c_1 = b_1(|b_1+c_1|-1)$. Notice that, since $|b_1| = 1$ and $|c_1| = 1$, $1 = |c_1| = |b_1(|b_1+c_1|-1)| = |b_1|||b_1+c_1|-1| = ||b_1+c_1|-1|$ and, since $|b_1+c_1|-1 \in \mathbb{R}$, it follows that either $||b_1+c_1|-1| = -1$ or $||b_1+c_1|-1| = 1$. Hence, since $b_1 \neq -c_1$, $||b_1+c_1|-1| = 1$ whence $c_1 = b_1$ whence $c = b$.
- $b_1 \neq -a_1$ and $a_1+b_1+c_1|a_1+b_1| = 0$ and $b_1 = -c_1$. In this case, $a_1+b_1-b_1|a_1+b_1| = 0$ whence $a_1 = b_1(|a_1+b_1|-1)$. Notice that, since $|a_1| = 1$ and $|b_1| = 1$, $1 = |a_1| = |b_1(|a_1+b_1|-1)| = |b_1|||a_1+b_1|-1| = ||a_1+b_1|-1|$ and, since $|a_1+b_1|-1 \in \mathbb{R}$, it follows that either $||a_1+b_1|-1| = -1$ or $||a_1+b_1|-1| = 1$. Hence, since $b_1 \neq -a_1$, $||a_1+b_1|-1| = 1$ whence $a_1 = b_1$ whence $a = b$.
- $b_1 \neq -a_1$ and $a_1+b_1+c_1|a_1+b_1| = 0$ and $b_1 \neq -c_1$ and $a_1|b_1+c_1|+b_1+c_1 = 0$. In this case, $a_1+b_1+c_1|a_1+b_1| = 0 = a_1|b_1+c_1|+b_1+c_1$ whence $a_1(|b_1+c_1|-1) = c_1(|a_1+b_1|-1)$. If $|b_1+c_1|-1 = 0$ then $|a_1+b_1|-1 = 0$ whence $|b_1+c_1| = 1$ and $|a_1+b_1| = 1$ whence there is $z \in \mathbb{C}$ such that a_1, b_1 and c_1 are all cube roots of z.
  If $|b_1+c_1|-1 \neq 0$ then $a_1 = \frac{|a_1+b_1|-1}{|b_1+c_1|-1}c_1$ whence either $a_1 = -c_1$ or $a_1 = c_1$. If $a_1 = -c_1$ then $a_1+b_1-a_1|a+_1b_1| = 0$ whence $b_1 = a_1(|a_1+b_1|-1)$ whence $b_1 = a_1$, furthermore $a_1|b_1+c_1|+b_1+c_1 = 0$ whence $b_1 = c_1(|b_1+c_1|-1)$ whence $b_1 = c_1$, hence $a = c$, which is absurd. If $a_1 = c_1$ then $c_1|b+_1c_1|+b_1+c_1 = 0$ whence $c_1(|b_1+c_1|+1) = -b_1$, hence $c_1 = -b_1$, which is an absurd.
- $b_1 \neq -a_1$ and $a_1+b_1+c_1|a_1+b_1| \neq 0$ and $b_1 \neq -c_1$ and $a_1|b_1+c_1|+b_1+c_1 \neq 0$. In this case, $\frac{a_1+b_1+c_1|a_1+b_1|}{0} = \frac{a_1|b_1+c_1|+b_1+c_1}{0}$ whence $\frac{\frac{a_1+b_1}{|a_1+b_1|}+c_1}{0} = \frac{a_1+\frac{b_1+c_1}{|b_1+c_1|}}{0}$. Let us show that $a_1 = c_1$. Denote $\alpha := \mathrm{Arg}(a_1)$, $\beta := \mathrm{Arg}(b_1)$ and $\gamma := \mathrm{Arg}(c_1)$. Suppose $\alpha, \beta, \gamma \in [0, \pi]$. The other cases are analogous. Firstly notice that $\mathrm{Arg}(a_1+b_1) = \frac{\alpha+\beta}{2}$. Indeed, since $a_1+b_1 = \cos(\alpha)+\cos(\beta)+(\sin(\alpha)+\sin(\beta))i$, it is sufficient to show that $\cos(\alpha)+\cos(\beta) = |a_1+b_1|\cos\left(\frac{\alpha+\beta}{2}\right)$ and $\sin(\alpha)+\sin(\beta) = |a_1+b_1|\sin\left(\frac{\alpha+\beta}{2}\right)$. It follows that $|a_1+b_1| = \sqrt{(\cos(\alpha)+\cos(\beta))^2+(\sin(\alpha)+\sin(\beta))^2} = \sqrt{2}\sqrt{1+\cos(\alpha-\beta)}$. Further, we know that $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) = 2\sqrt{1-\sin^2\left(\frac{\alpha-\beta}{2}\right)}\cos\left(\frac{\alpha+\beta}{2}\right) = 2\sqrt{1-\left(1-\frac{\cos(\alpha-\beta)}{2}\right)}\cos\left(\frac{\alpha+\beta}{2}\right) = \sqrt{2}\sqrt{1+\cos(\alpha-\beta)}\cos\left(\frac{\alpha+\beta}{2}\right) = |a_1+b_1|\cos\left(\frac{\alpha+\beta}{2}\right)$. In similar way we see that $\sin(\alpha)+\sin(\beta) = |a_1+b_1|\sin\left(\frac{\alpha+\beta}{2}\right)$. This concludes that $\mathrm{Arg}(a_1+b_1) = \frac{\alpha+\beta}{2}$. Analogously it follows that $\mathrm{Arg}\left(\frac{a_1+b_1}{|a_1+b_1|}+c_1\right) = \frac{\frac{\alpha+\beta}{2}+\gamma}{2}$ and $\mathrm{Arg}(b_1+c_1) = \frac{\beta+\gamma}{2}$ and $\mathrm{Arg}\left(a_1+\frac{b_1+c_1}{|b_1+c_1|}\right) = \frac{\alpha+\frac{\beta+\gamma}{2}}{2}$. Hence $\mathrm{Arg}\left(\frac{a_1+b_1}{|a_1+b_1|}+c_1\right) = \frac{\alpha+\beta+2\gamma}{4}$ and $\mathrm{Arg}\left(a_1+\frac{b_1+c_1}{|b_1+c_1|}\right) = \frac{2\alpha+\beta+\gamma}{4}$. Since $\frac{\frac{a_1+b_1}{|a_1+b_1|}+c_1}{0} = \frac{a_1+\frac{b_1+c_1}{|b_1+c_1|}}{0}$, it follows that $\mathrm{Arg}\left(\frac{a_1+b_1}{|a_1+b_1|}+c_1\right) = \mathrm{Arg}\left(a_1+\frac{b_1+c_1}{|b_1+c_1|}\right)$ whence $\frac{\alpha+\beta+2\gamma}{4} = \frac{2\alpha+\beta+\gamma}{4}$. Thus $\alpha = \gamma$. Therefore $a_1 = c_1$ whence $a = c$.

c) $a+0 = \frac{a_1}{a_2}+\frac{0}{1} = \frac{a_1\times 1+0\times a_2}{a_2\times 1} = \frac{a_1+0}{a_2} = \frac{a_1}{a_2} = a$.

d) Suppose $a \notin \mathbb{C}^T \setminus \mathbb{C}$. We have that $a-a = \frac{a_1}{1}-\frac{a_1}{1} = \frac{a_1}{1}+\frac{-a_1}{1} = \frac{a_1\times 1+(-a_1)\times 1}{1\times 1} = \frac{a_1-a_1}{1} = \frac{0}{1} = 0$.
Suppose $a \in \mathbb{C}^T \setminus \mathbb{C}$. We have that $a-a = \Phi \neq 0$.

e) $a\times b = \frac{a_1}{a_2}\times\frac{b_1}{b_2} = \frac{a_1b_1}{a_2b_2} = \frac{b_1a_1}{b_2a_2} = \frac{b_1}{b_2}\times\frac{a_1}{a_2} = b\times a$.

f) $(a\times b)\times c = \left(\frac{a_1}{a_2}\times\frac{b_1}{b_2}\right)\times\frac{c_1}{c_2} = \frac{a_1b_1}{a_2b_2}\times\frac{c_1}{c_2} = \frac{(a_1b_1)c_1}{(a_2b_2)c_2} = \frac{a_1(b_1c_1)}{a_2(b_2c_2)} = \frac{a_1}{a_2}\times\frac{b_1c_1}{b_2c_2} = \frac{a_1}{a_2}\times\left(\frac{b_1}{b_2}\times\frac{c_1}{c_2}\right) = a\times(b\times c)$.

g) $a\times 1 = \frac{a_1}{a_2}\times\frac{1}{2} = \frac{a_1\times 1}{a_2\times 1} = \frac{a_1}{a_2} = a$.

h) Suppose $a \notin \{0\}\cup\mathbb{C}^T\setminus\mathbb{C}$. We have that $a = \frac{a_1}{1}$ with $a_1 \neq 0$, whence $a\div a = \frac{a_1}{1}\div\frac{a_1}{1} = \frac{a_1}{1}\times\left(\frac{a_1}{1}\right)^{-1} = \frac{a_1}{1}\times\frac{\frac{1}{a_1}}{1} = \frac{a_1\times\frac{1}{a_1}}{1\times 1} = \frac{1}{1} = 1$.
Suppose $a \in \{0\}\cup\mathbb{C}^T\setminus\mathbb{C}$. We have that $a\div a = \Phi$.

i) (I) Suppose either $a = \Phi$ or $b = \Phi$ or $c = \Phi$. In this case, $a\times(b+c) = \Phi = (a\times b)+(a\times c)$.
(II) Suppose $a \notin \mathbb{C}^T_\infty$. We have three possibilities:
   - $a = \Phi$. In this case, $a\times(b+c) = \Phi\times(b+c) = \Phi = \Phi+\Phi = (\Phi\times b)+(\Phi\times c) = (a\times b)+(a\times c)$.
   - $a = 0$. In this case, $a = \frac{0}{1}$. If $b, c \in \mathbb{C}^T_\infty$ then $a\times(b+c) = \frac{0}{1}\times\left(\frac{b_1}{0}+\frac{c_1}{0}\right) = \frac{0}{1}\times\frac{\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}}{0} = \frac{0\times\left(\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}\right)}{1\times 0} = \frac{0}{0} = \Phi = \Phi+\Phi = \frac{0}{0}+\frac{0}{0} = \frac{0\times b_1}{1\times 0}+\frac{0\times c_1}{1\times 0} = \left(\frac{0}{1}\times\frac{b_1}{0}\right)+\left(\frac{0}{1}\times\frac{c_1}{0}\right) = (a\times b)+(a\times c)$; otherwise $a\times(b+c) = \frac{0}{1}\times\left(\frac{b_1}{b_2}+\frac{c_1}{c_2}\right) = \frac{0}{1}\times\frac{b_1c_2+c_1b_2}{b_2c_2} = \frac{0\times(b_1c_2+c_1b_2)}{1\times b_2c_2} = \frac{0}{b_2c_2} = \frac{0+0}{b_2c_2} = \frac{0\times c_2+0\times b_2}{b_2c_2} = \frac{0}{b_2}+\frac{0}{c_2} = \frac{0\times b_1}{1\times b_2}+\frac{0\times c_1}{1\times c_2} = \left(\frac{0}{1}\times\frac{b_1}{b_2}\right)+\left(\frac{0}{1}\times\frac{c_1}{c_2}\right) = (a\times b)+(a\times c)$.
   - $a \in \mathbb{C}\setminus\{0\}$. In this case, $a = \frac{a_1}{1}$ with $a_1 \neq 0$. If $b, c \in \mathbb{C}^T_\infty$ then $a\times(b+c) = \frac{a_1}{1}\times\left(\frac{b_1}{0}+\frac{c_1}{0}\right) = \frac{a_1}{1}\times\frac{\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}}{0} = \frac{a_1\times\left(\frac{b_1}{|b_1|}+\frac{c_1}{|c_1|}\right)}{1\times 0} = \frac{a_1\frac{b_1}{|b_1|}+a_1\frac{c_1}{|c_1|}}{0} = \frac{\frac{a_1b_1}{|b_1|}+\frac{a_1c_1}{|c_1|}}{0} = \frac{\frac{1}{|a_1|}\left(\frac{a_1b_1}{|b_1|}+\frac{a_1c_1}{|c_1|}\right)}{0} = \frac{\frac{1}{|a_1|}\frac{a_1b_1}{|b_1|}+\frac{1}{|a_1|}\frac{a_1c_1}{|c_1|}}{0} = \frac{\frac{a_1b_1}{|a_1||b_1|}+\frac{a_1c_1}{|a_1||c_1|}}{0} = \frac{\frac{a_1b_1}{|a_1b_1|}+\frac{a_1c_1}{|a_1c_1|}}{0} = \frac{a_1b_1}{0}+\frac{a_1c_1}{0} = \frac{a_1\times b_1}{1\times 0}+\frac{a_1\times c_1}{1\times 0} = \left(\frac{a_1}{1}\times\frac{b_1}{0}\right)+\left(\frac{a_1}{1}\times\frac{c_1}{0}\right) = (a\times b)+(a\times c)$; otherwise $a\times(b+c) = \frac{a_1}{1}\times\left(\frac{b_1}{b_2}+\frac{c_1}{c_2}\right) = \frac{a_1}{1}\times\frac{b_1c_2+c_1b_2}{b_2c_2} = \frac{a_1(b_1c_2+c_1b_2)}{1\times b_2c_2} = \frac{a_1b_1c_2+a_1c_1b_2}{b_2c_2} = \frac{(a_1b_1)c_2+(a_1c_1)b_2}{b_2c_2} = \frac{a_1b_1}{b_2}+\frac{a_1c_1}{c_2} = \frac{a_1b_1}{1\times b_2}+\frac{a_1c_1}{1\times c_2} = \left(\frac{a_1}{1}\times\frac{b_1}{b_2}\right)+\left(\frac{a_1}{1}\times\frac{c_1}{c_2}\right) = (a\times b)+(a\times c)$.

(III) Suppose $a, b, c \in \mathbb{C}^T \setminus \mathbb{C}$. If either $a = \Phi$ or $b = \Phi$ or $c = \Phi$ then $a \times (b + c) = \Phi = (a \times b) + (a \times c)$; otherwise $a, b, c \in \mathbb{C}^T_\infty$ whence $a \times (b + c) = \frac{a_1}{0} \times \left(\frac{b_1}{0} + \frac{c_1}{0}\right) = \frac{a_1}{0} \times \frac{\frac{b_1}{|b_1|} + \frac{c_1}{|c_1|}}{0} = \frac{a_1\left(\frac{b_1}{|b_1|} + \frac{c_1}{|c_1|}\right)}{0\times 0} = \frac{a_1\left(\frac{b_1}{|b_1|} + \frac{c_1}{|c_1|}\right)}{0} = \frac{a_1\frac{b_1}{|b_1|} + a_1\frac{c_1}{|c_1|}}{0} = \frac{\frac{a_1 b_1}{|b_1|} + \frac{a_1 c_1}{|c_1|}}{0} = \frac{\frac{1}{|a_1|}\left(\frac{a_1 b_1}{|b_1|} + \frac{a_1 c_1}{|c_1|}\right)}{0} = \frac{\frac{1}{|a_1|}\frac{a_1 b_1}{|b_1|} + \frac{1}{|a_1|}\frac{a_1 c_1}{|c_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1||b_1|} + \frac{a_1 c_1}{|a_1||c_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1 b_1|} + \frac{a_1 c_1}{|a_1 c_1|}}{0} = \frac{a_1 b_1}{0} + \frac{a_1 c_1}{0} = \frac{a_1 \times b_1}{0\times 0} + \frac{a_1 \times c_1}{0\times 0} = \left(\frac{a_1}{0} \times \frac{b_1}{0}\right) + \left(\frac{a_1}{0} \times \frac{c_1}{0}\right) = (a \times b) + (a \times c)$.

(IV) Suppose $|b| = |c|$. We have five possibilities:

- $a \notin \mathbb{C}^T_\infty$. This case is proved in item (II).
- $a \in \mathbb{C}^T_\infty$ and $b = c = \Phi$. This case is proved in item (I).
- $a \in \mathbb{C}^T_\infty$ and $b, c\mathbb{C}^T_\infty$. This case is proved in (III).
- $a \in \mathbb{C}^T_\infty$ and $b, c \in \mathbb{C}$ and $b = c = 0$. In this case $a \times (b + c) = a \times (0 + 0) = a \times 0 = \Phi = \Phi + \Phi = a \times 0 + a \times 0 = a \times b + a \times c$
- $a \in \mathbb{C}^T_\infty$ and $b, c \in \mathbb{C}$ and $bc \neq 0$. In this case $a \times (b+c) = \frac{a_1}{0} \times \left(\frac{b_1}{1} + \frac{c_1}{1}\right) = \frac{a_1}{0} \times \frac{b_1 \times 1 + c_1 \times 1}{1\times 1} = \frac{a_1}{0} \times \frac{b_1 + c_1}{1} = \frac{a_1(b_1+c_1)}{0\times 1} = \frac{a_1(b_1+c_1)}{0} = \frac{a_1 b_1 + a_1 c_1}{0} = \frac{\frac{a_1 b_1 + a_1 c_1}{|a_1 b_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1 b_1|} + \frac{a_1 c_1}{|a_1 b_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1||b_1|} + \frac{a_1 c_1}{|a_1||b_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1||b_1|} + \frac{a_1 c_1}{|a_1||c_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1 b_1|} + \frac{a_1 c_1}{|a_1 c_1|}}{0} = \frac{a_1 b_1}{0} + \frac{a_1 c_1}{0} = \frac{a_1 b_1}{0\times 1} + \frac{a_1 c_1}{0\times 1} = \left(\frac{a_1}{0} \times \frac{b_1}{1}\right) + \left(\frac{a_1}{0} \times \frac{c_1}{1}\right) = (a \times b) + (a \times c)$.

(V) Suppose $\mathrm{Arg}(b) = \mathrm{Arg}(c)$ and $bc \neq 0$. We have four possibilities:

- $a \notin \mathbb{C}^T_\infty$. This case is proved in item (II).
- $a \in \mathbb{C}^T_\infty$ and $b = c = \Phi$. This case is proved in item (I).
- $a \in \mathbb{C}^T_\infty$ and $b, c \in \mathbb{C}^T_\infty$. This case is proved in item (III).
- $a \in \mathbb{C}^T_\infty$ and $b, c \in \mathbb{C}$. In this case, since $\mathrm{Arg}(b) = \mathrm{Arg}(c)$ and $bc \neq 0$, it follows that $b = \alpha c$ for some positive $\alpha \in \mathbb{R}$, whence $b_1 = \alpha c_1$, and $b_1 \neq 0$ and $c_1 \neq 0$. Thus $a \times (b + c) = \frac{a_1}{0} \times \left(\frac{b_1}{1} + \frac{c_1}{1}\right) = \frac{a_1}{0} \times \frac{b_1 \times 1 + c_1 \times 1}{1\times 1} = \frac{a_1}{0} \times \frac{b_1 + c_1}{1} = \frac{a_1(b_1+c_1)}{0\times 1} = \frac{a_1(b_1+c_1)}{0} = \frac{a_1(\alpha c_1 + c_1)}{0} = \frac{a_1(\alpha+1)c_1}{0} = \frac{a_1 c_1}{0} = \frac{2\frac{a_1 c_1}{|a_1 c_1|}}{0} = \frac{\frac{a_1 c_1}{|a_1 c_1|} + \frac{a_1 c_1}{|a_1 c_1|}}{0} = \frac{a_1 c_1}{0} + \frac{a_1 c_1}{0} = \frac{a_1 \alpha c_1}{0} + \frac{a_1 c_1}{0} = \frac{a_1 b_1}{0\times 1} + \frac{a_1 c_1}{0\times 1} = \left(\frac{a_1}{0} \times \frac{b_1}{1}\right) + \left(\frac{a_1}{0} \times \frac{c_1}{1}\right) = (a \times b) + (a \times c)$.

Now suppose that $\mathrm{Arg}(b) = \mathrm{Arg}(c)$ and $bc \neq 0$ do not hold simultaneously and $|a| + |b| + |c| = \Phi$ does not hold and $a \notin \mathbb{C}^T_\infty$ does not hold and $a, b, c \in \mathbb{C}^T \setminus \mathbb{C}$ does not hold and $|b| = |c|$ does not hold. Let us show that $a \times (b + c) \neq (a \times b) + (a \times c)$. We have five possibilities:

- $b = 0$ and $c \neq 0$ and $c \neq \Phi$ and $a \in \mathbb{C}^T_\infty$. In this case, $a \times (b + c) = a \times (0 + c) = a \times c = ac \in \mathbb{C}^T_\infty$ and $(a \times b) + (a \times c) = (a \times 0) + (a \times c) = \Phi + ac = \Phi$ whence $a \times (b + c) \neq (a \times b) + (a \times c)$.
- $b \neq 0$ and $c = 0$ and $b \neq \Phi$ and $a \in \mathbb{C}^T_\infty$. This case is analogous to the previous one.
- $\mathrm{Arg}(b) \neq \mathrm{Arg}(c)$ and $bc \neq 0$ and $a \in \mathbb{C}^T_\infty$ and $b, c \in \mathbb{C}$ and $|b| \neq |c|$. In this case, $a \times (b + c) = \frac{a_1}{0} \times \left(\frac{b_1}{1} + \frac{c_1}{1}\right) = \frac{a_1}{0} \times \frac{b_1 \times 1 + c_1 \times 1}{1\times 1} = \frac{a_1}{0} \times \frac{b_1 + c_1}{1} = \frac{a_1(b_1+c_1)}{0\times 1} = \frac{a_1(b_1+c_1)}{0}$ and $(a \times b) + (a \times c) = \left(\frac{a_1}{0} \times \frac{b_1}{1}\right) + \left(\frac{a_1}{0} \times \frac{c_1}{1}\right) = \frac{a_1 b_1}{0\times 1} + \frac{a_1 c_1}{0\times 1} = \frac{a_1 b_1}{0} + \frac{a_1 c_1}{0} = \frac{\frac{a_1 b_1}{|a_1 b_1|} + \frac{a_1 c_1}{|a_1 c_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1||b_1|} + \frac{a_1 c_1}{|a_1||c_1|}}{0} = \frac{\frac{1}{|a_1|}\left(\frac{a_1 b_1}{|b_1|} + \frac{a_1 c_1}{|c_1|}\right)}{0} = \frac{\frac{a_1 b_1}{|b_1|} + \frac{a_1 c_1}{|c_1|}}{0} = \frac{\frac{a_1 b_1 |c_1| + a_1 c_1 |b_1|}{|b_1||c_1|}}{0} = \frac{a_1 b_1 |c_1| + a_1 c_1 |b_1|}{0}$. If $a \times (b + c) = (a \times b) + (a \times c)$ then it would follow that $\frac{a_1(b_1+c_1)}{0} = \frac{a_1 b_1 |c_1| + a_1 c_1 |b_1|}{0}$. Hence there would be a positive $\alpha \in \mathbb{R}$ such that $a_1(b_1 + c_1)\alpha = a_1 b_1 |c_1| + a_1 c_1 |b_1|$ whence $(b_1 + c_1)\alpha = b_1|c_1| + c_1|b_1|$. If $\alpha = |c_1|$ then $(b_1 + c_1)\alpha = b_1|c_1| + c_1|b_1|$ whence $(b_1 + c_1)\alpha = b_1\alpha + c_1|b_1|$ whence $b_1\alpha + c_1\alpha = b_1\alpha + c_1|b_1|$ whence $c_1\alpha = c_1|b_1|$ whence $\alpha = |b_1|$ whence $|c_1| = \alpha = |b_1|$ whence $|c| = |b|$, which is an absurd. If $\alpha \neq |c_1|$ then $1 - \frac{|c_1|}{\alpha} \neq 0$ whence $(b_1 + c_1)\alpha = b_1|c_1| + c_1|b_1|$ whence $b_1 + c_1 = b_1\frac{|c_1|}{\alpha} + c_1\frac{|b_1|}{\alpha}$ whence $b_1 - b_1\frac{|c_1|}{\alpha} = c_1\frac{|b_1|}{\alpha} - c_1$ whence $b_1\left(1 - \frac{|c_1|}{\alpha}\right) = c_1\left(\frac{|b_1|}{\alpha} - 1\right)$ whence $b_1 = c_1\frac{\frac{|b_1|}{\alpha} - 1}{1 - \frac{|c_1|}{\alpha}}$. If $\frac{\frac{|b_1|}{\alpha} - 1}{1 - \frac{|c_1|}{\alpha}} = 0$ then $b_1 = 0$ whence $b = 0$, which is an absurd. If $\frac{\frac{|b_1|}{\alpha} - 1}{1 - \frac{|c_1|}{\alpha}} > 0$ then $\mathrm{Arg}(b_1) = \mathrm{Arg}(c_1)$ whence $\mathrm{Arg}(b) = \mathrm{Arg}(c)$, which is an absurd. If $\frac{\frac{|b_1|}{\alpha} - 1}{1 - \frac{|c_1|}{\alpha}} < 0$ then, calling $\frac{\frac{|b_1|}{\alpha} - 1}{1 - \frac{|c_1|}{\alpha}} = -\lambda$ with $\lambda > 0$, $b_1 = -\lambda c_1$ whence $(b_1 + c_1)\alpha = b_1|c_1| + c_1|b_1|$ whence $(-\lambda c_1 + c_1)\alpha = -\lambda c_1|c_1| + c_1| - \lambda c_1|$ whence $(1 - \lambda)c_1\alpha = -\lambda c_1|c_1| + c_1\lambda|c_1|$ whence $(1 - \lambda)c_1\alpha = -\lambda c_1|c_1| + \lambda c_1|c_1|$ whence $(1 - \lambda)c_1\alpha = 0$ whence $\lambda = 1$ whence $b_1 = -c_1$ whence $|b_1| = |c_1|$ whence $|b| = |c|$, which is an absurd. Anyway the equality $a \times (b + c) = (a \times b) + (a \times c)$ produces an absurd. Therefore $a \times (b + c) \neq (a \times b) + (a \times c)$.
- $\mathrm{Arg}(b) \neq \mathrm{Arg}(c)$ and $bc \neq 0$ and $a \in \mathbb{C}^T_\infty$ and $b \in \mathbb{C}$ and $c \in \mathbb{C}^T_\infty$. In this case, $a \times (b + c) = a \times c = \frac{a_1}{0} \times \frac{c_1}{0} = \frac{a_1 c_1}{0}$ and $(a \times b) + (a \times c) = \left(\frac{a_1}{0} \times \frac{b_1}{1}\right) + \left(\frac{a_1}{0} \times \frac{c_1}{0}\right) = \frac{a_1 b_1}{0\times 1} + \frac{a_1 c_1}{0\times 0} = \frac{a_1 b_1}{0} + \frac{a_1 c_1}{0} = \frac{\frac{a_1 b_1}{|a_1 b_1|} + \frac{a_1 c_1}{|a_1 c_1|}}{0} = \frac{\frac{a_1 b_1}{|a_1||b_1|} + \frac{a_1 c_1}{|a_1||c_1|}}{0} = \frac{\frac{1}{|a_1|}\left(\frac{a_1 b_1}{|b_1|} + \frac{a_1 c_1}{|c_1|}\right)}{0} = \frac{\frac{a_1 b_1}{|b_1|} + \frac{a_1 c_1}{|c_1|}}{0} = \frac{\frac{a_1 b_1 |c_1| + a_1 c_1 |b_1|}{|b_1||c_1|}}{0} = \frac{a_1 b_1 |c_1| + a_1 c_1 |b_1|}{0}$. If $a \times (b + c) = (a \times b) + (a \times c)$ then it would follow that $\frac{a_1(b_1+c_1)}{0} = \frac{a_1 b_1 |c_1| + a_1 c_1 |b_1|}{0}$. Hence there would be a positive $\alpha \in \mathbb{R}$ such that $a_1 c_1 \alpha = a_1 b_1 |c_1| + a_1 c_1 |b_1|$ whence $c_1\alpha = b_1|c_1| + c_1|b_1|$ whence $c_1\alpha - c_1|b_1| = b_1|c_1|$ whence $c_1(\alpha - |b_1|) = b_1|c_1|$. If $\alpha - |b_1| = 0$ then either $b_1 = 0$ or $c_1 = 0$, which is an absurd. If $\alpha - |b_1| \neq 0$ then $c_1 = b_1\frac{|c_1|}{\alpha - |b_1|}$. If $\alpha - |b_1| > 0$ then $\mathrm{Arg}(c_1) = \mathrm{Arg}(b_1)$ whence $\mathrm{Arg}(c) = \mathrm{Arg}(b)$, which is an absurd. If $\alpha - |b_1| < 0$ then, call $\frac{|c_1|}{\alpha - |b_1|} = -\lambda$ with $\lambda > 0$, $c_1 = -\lambda b_1$ whence $c_1\alpha = b_1|c_1| + c_1|b_1|$ whence $c_1\alpha = b_1| - \lambda b_1| - \lambda b_1|b_1|$ whence $c_1\alpha = \lambda b_1|b_1| - \lambda b_1|b_1|$ whence $c_1\alpha = 0$ whence

$c_1 = 0$ whence $c = 0$, which is an absurd. Anyway the equality $a \times (b+c) = (a \times b) + (a \times c)$ produces an absurd. Therefore $a \times (b+c) \neq (a \times b) + (a \times c)$.

- $\mathrm{Arg}(b) \neq \mathrm{Arg}(c)$ and $bc \neq 0$ and $a \in \mathbb{C}^T_\infty$ and $b \in \mathbb{C}^T_\infty$ and $c \in \mathbb{C}$. This case is analogous to the previous one.

■

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