Permanence and Extinction of Delayed Stage-Structured Predator-Prey System on Time Scales

Lili Wang, Pingli Xie

Abstract—**This paper is concerned with a delayed Leslie-Gower predator-prey system with stage-structure on time scales. By using the theory of calculus on time scales and some mathematical methods as well as some dynamic inequalities on time scales, sufficient conditions which guarantee the predator and the prey to be permanent are obtained. Moreover, under some suitable conditions, we show that the prey species will be driven to extinction. Finally, an example and its corresponding numerical simulations are presented to explain our theoretical results.**

Index Terms—**Permanence; Extinction; stage-structure; Dynamic inequality; Time scale.**

I. Introduction

PERMANENCE (or persistence) is an important property of dynamical systems and of the systems arising in ecology, epidemics etc, since permanence addresses the limits of growth for some or all components of a system, while persistence also deals with the long-term survival of some or all components of the system. As is well known, most prey species have a life history that includes multiple stages juvenile and adults or immature and mature. In particular, we have in mind mammalian populations and some amphibious animals. In the past few years, permanence (or persistence) and extinction of different types of continuous or discrete predator-prey systems with stage-structure have been studied wildly both in theories and applications; see, for example, [1-4] and the references therein.

Notice that, in the nature world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

A time scale is a nonempty closed subset of $\mathbb{R}$. The theory of calculus on time scales (see [5]) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 (see [6]). The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors powerfully, see [7-10]. However, to the best of the authors' knowledge, there are few papers considered permanence of predator-prey system with stage-structure on time scales.

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Motivated by the above statements, in this paper, we first establish a limitation theorem on time scales by using the theory of calculus on time scales and some mathematical methods, then, based on the theorem and some inequalities obtained in [7], as an application, we shall study the permanence and the extinction of the following delayed Leslie-Gower predator-prey system with stage-structure on time scales:

$$\begin{cases} x_1^{\Delta}(t) = \alpha x_2(t) - r_1 x_1(t) - \alpha e^{-r_1\tau} x_2(\delta_-(\tau,t)), \\ x_2^{\Delta}(t) = \alpha e^{-r_1\tau} x_2(\delta_-(\tau,t)) - r_2 x_2(t) - r_3 x_2^2(t) \\ \qquad - \dfrac{a_1 y_1(t) x_2(t)}{x_2(t)+k_1}, \\ y_1^{\Delta}(t) = \big(\beta_1 - \dfrac{a_2 y_1(t)}{x_2(t)+k_2}\big) y_1(t) \\ \qquad + D_1(y_2(t)-y_1(t)), \\ y_2^{\Delta}(t) = (\beta_2 - r_4 y_2(t)) y_2(t) + D_2(y_1(t)-y_2(t)), \end{cases} \tag{1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is a time scale. $x_1(t)$ and $x_2(t)$ represent the densities of immature and mature individual prey in patch 1 at time t, $y_i(t)$ denote the density of predator species in patch i, $i = 1, 2$ at time t. The prey only lives in patch 1. For immature prey, α is birth rate, r_1 is death rate, and the term $\alpha e^{-r_1\tau} x_2(\delta_-(\tau,t))$ represents the number of immature prey that was born at time $\delta_-(\tau,t)$, which still survive at time t and are transferred from the immature stage to the mature stage at time t. For mature prey, r_2 is death rate, r_3 is the intraspecific competition rate of mature prey, a_1 is the maximum value of the per-capita reduction rate of x_2 due to y_1, and k_1 (resp., k_2) measures the extent to which environment provides protection to prey x_2 (resp., to the predator y_1). For the predator, β_i is the birth rate of predator in patch $i, i = 1, 2$, D_i is the dispersion rate of predator between two patches, r_4 is death rate of predator in patch 2, a_2 has a similar meaning to a_1. For the ecological justification of (1), one can refer to [11,12].

$\delta_-(\tau,t)$ is a delay function with $t \in \mathbb{T}$ and $\tau \in [0,+\infty)_{\mathbb{T}}$, where δ_- is a backward shift operator on the set $\mathbb{T}^*$, and $\mathbb{T}^*$ is a non-empty subset of the time scale $\mathbb{T}$. More details about backward shift operator, one may see [13].

The initial conditions of (1) are of the form

$$x_i(t) = \phi_i(t), y_i(t) = \psi_i(t), \\ \phi_i(0) > 0, \psi_i(0) > 0, i = 1, 2, \tag{2}$$

where $(\phi_1(t), \phi_2(t), \psi_1(t), \psi_2(t)) \in C([\delta_-(\tau,0),0]_{\mathbb{T}}, \mathbb{R}^4_{+0})$, the Banach space of continuous function mapping the interval $[\delta_-(\tau,0),0]_{\mathbb{T}}$ into $\mathbb{R}^4_{+0}$, where $\mathbb{R}^4_{+0} = \{(x_1,x_2,y_1,y_2) : x_i > 0, y_i > 0, i = 1, 2\}$.

The organization of this paper is as follows. In section 2, we introduce some notations and definitions and state some preliminary results needed in later sections; a limitation theorem on time scales is established, which plays an important role in this paper. Section 3 is devoted to discussing the permanence and the extinction of system (1). An Example together with its numerical simulations is provided in Section 4; and a conclusion is made in section 5.

II. PRELIMINARIES

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

For the basic theory of calculus on time scales, see [5].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \ \ominus p = -\frac{p}{1 + \mu p}, \ p \ominus q = p \oplus (\ominus q).$$

Lemma 1. (see [5]) *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $(e_{\ominus p}(t, s))^\Delta = (\ominus p)(t)e_{\ominus p}(t, s)$.

In order to discuss the permanence of system (1), we need the following lemmas.

Lemma 2. (see [7]) *Assume that $a > 0, b > 0$ and $-a \in \mathcal{R}^+$. Then*

$$x^\Delta(t) \geq (\leq) b - ax(t), \ x(t) > 0, \ t \in [t_0, \infty)_\mathbb{T}$$

implies

$$x(t) \geq (\leq) \frac{b}{a}[1 + (\frac{ax(t_0)}{b} - 1)e_{-a}(t, t_0)], \ t \in [t_0, \infty)_\mathbb{T}.$$

Lemma 3. *Consider the following equation:*

$$x^\Delta(t) = ax(\delta_-(\tau, t)) - bx(t) - cx^2(t),$$

where a, b, c, and τ are positive constants, $x(t) > 0$ for $t \in [\delta_-(\tau, 0), 0]_\mathbb{T}$. Then
(i) *if $a > b$, then $\lim_{t\to+\infty} x(t) = \frac{a-b}{c}$;*
(ii) *if $a < b$, then $\lim_{t\to+\infty} x(t) = 0$.*

Proof: **Case** (i) It is easy to show that $x(t)$ is positive and bounded for all $t \in [0, +\infty)_\mathbb{T}$. Clearly, $x^* = \frac{a-b}{c}$ is the unique positive equilibrium of the equation. Suppose that $x(t)$ is eventually monotonic, then $\lim_{t\to+\infty} x(t)$ exists. Denote $L = \lim_{t\to+\infty} x(t)$, we show that $L = x^*$, otherwise, if $L > x^*$, then

$$\lim_{t\to+\infty} x^\Delta(t) = aL - bL - cL^2 = cL(x^* - L) < 0,$$

which implies $\lim_{t\to+\infty} x(t) = -\infty$, a contradiction, therefore $L = x^*$.

Now, suppose that $x(t)$ is not eventually monotonic, since $x(t)$ is bounded, let $\eta = \limsup_{t\to+\infty} |x(t) - x^*|$, then η is bounded, we can show that $\eta = 0$, otherwise, if $\eta > 0$, then there exists a sequence $x(t_i)(t_i > t_{i-1}, \lim_{i\to+\infty} t_i = +\infty)$ such that $\lim_{i\to+\infty} x(t_i) = x^* + \eta$ or $\lim_{i\to+\infty} x(t_i) = x^* - \eta, (x^* \geq \eta)$. Without loss of generalization, we only consider the first case, then there exists an $\varepsilon\,(0 < \varepsilon \leq \frac{c(x^*+\eta)\eta}{a+b+2c(x^*+\eta)})$ such that

$$a(x^* + \eta + \varepsilon) - c(x^* + \eta - \varepsilon)^2 - b(x^* + \eta - \varepsilon) < 0. \quad (3)$$

For this ε, there exists a $T = T(\varepsilon) > \tau$ such that for $t_i \in (\delta_-(\tau, T), +\infty)_\mathbb{T}$, we have

$$x(t_i) < x^* + \eta + \varepsilon. \quad (4)$$

We also know that there exists a $\overline{t_i} \in (T, +\infty)_\mathbb{T}$ such that $x^\Delta(\overline{t_i}) = 0, x(\overline{t_i}) - x^* > \eta - \varepsilon$. This implies that

$$ax(\delta_-(\tau, \overline{t_i})) = cx^2(\overline{t_i}) + bx(\overline{t_i}).$$

Thus

$$ax(\delta_-(\tau, \overline{t_i})) > c(x^* + \eta - \varepsilon)^2 + b(x^* + \eta - \varepsilon).$$

By (3), we have

$$ax(\delta_-(\tau, \overline{t_i})) > a(x^* + \eta + \varepsilon).$$

Hence

$$x(\delta_-(\tau, \overline{t_i})) > x^* + \eta + \varepsilon,$$

a contradiction to (4), then $\eta = 0$, that is

$$\lim_{t\to+\infty} x(t) = x^*.$$

Case (ii) If $x(t)$ is eventually monotonic, then $\lim_{t\to+\infty} x(t)$ exists. Denote $L_1 = \lim_{t\to+\infty} x(t)$, then $L_1 \geq 0$, we show that $L_1 = 0$, otherwise, if $L_1 > 0$, then

$$\lim_{t\to+\infty} x^\Delta(t) = aL_1 - bL_1 - cL_1^2 < 0,$$

which implies $\lim_{t\to+\infty} x(t) = -\infty$, a contradiction, therefore $\lim_{t\to+\infty} x(t) = 0$.

Now, suppose that $x(t)$ is not eventually monotonic, we can show that $u = \limsup_{t\to+\infty} x(t) = 0$, otherwise, if $u > 0$, then there exists a sequence $x(t_i)(t_i > t_{i-1}, \lim_{i\to+\infty} t_i = +\infty)$ such that $\lim_{i\to+\infty} x(t_i) = u$. Since $au - bu - cu^2 < 0$, then there exists an $\varepsilon > 0$ such that

$$a(u+\varepsilon) - c(u-\varepsilon)^2 - b(u-\varepsilon) < 0. \tag{5}$$

For this ε, there exists a $T = T(\varepsilon) > \tau$ such that for $t_i \in (\delta_-(\tau, T), +\infty)_{\mathbb{T}}$, we have

$$x(t_i) < u + \varepsilon. \tag{6}$$

We also know that there exists a $\overline{t_i} \in (T, +\infty)_{\mathbb{T}}$ such that $x^{\Delta}(\overline{t_i}) = 0, x(\overline{t_i}) > u - \varepsilon$. This implies that

$$ax(\delta_-(\tau, \overline{t_i})) = cx^2(\overline{t_i}) + bx(\overline{t_i}).$$

Thus

$$ax(\delta_-(\tau, \overline{t_i})) > c(u-\varepsilon)^2 + b(u-\varepsilon).$$

By (5), we have

$$x(\delta_-(\tau, \overline{t_i})) > u + \varepsilon,$$

a contradiction to (6), then $u = 0$, that is

$$\lim_{t\to+\infty} x(t) = 0.$$

This completes the proof. ■

III. Permanence and extinction

As an application, based on the results obtained in section 2 and [7], we shall discuss the permanence and the extinction of system (1).

Definition 1. *System (1) is said to be permanent if there exist a compact region $\mathbb{D} \subseteq \mathrm{Int}\mathbb{R}^4_{+0}$, such that for any positive solution $(x_1(t), y_1(t), x_2(t), y_2(t))$ of system (1) with initial conditions (2) eventually enters and remains in region $\mathbb{D}$.*

Proposition 1. *Assume that $(x_1(t), x_2(t), y_1(t), y_2(t))$ be a positive solution of system (1) with initial conditions (2). If $-r_1 \in \mathcal{R}^+$, then there exists a $T_4 > 0$ such that*

$$x_i(t) \le M, y_i(t) \le M,\ i = 1, 2,\ t \in [T_4, +\infty)_{\mathbb{T}}.$$

where M is a constant and

$$\begin{aligned}
&M > \max\{M_1, M_2, M^*\},\\
&M_1 = \frac{\alpha M_2}{r_1}(1 - e^{-r_1\tau}) + \varepsilon,\\
&M_2 = \frac{\alpha e^{-r_1\tau}}{r_3} + \varepsilon,\\
&M^* = \frac{\alpha^2}{4Ar_3} + \frac{(A + D_2 + \beta_1)^2(M_2 + k_2)}{4Aa_2}\\
&\qquad + \frac{(A + D_1 + \beta_2)^2}{4Aa_1} + \varepsilon,\\
&A = \min\{r_1, r_2\}.
\end{aligned}$$

Proof: Assume that $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1) with initial conditions (2). It follows from the second equation of system (1) that for $t \in [\tau, +\infty)_{\mathbb{T}}$,

$$x_2^{\Delta}(t) \le \alpha e^{-r_1\tau} x_2(\delta_-(\tau, t)) - r_3 x_2^2(t).$$

Consider the following auxiliary equation:

$$u^{\Delta}(t) = \alpha e^{-r_1\tau} u(\delta_-(\tau, t)) - r_3 u^2(t),$$

by Lemma 3, we can get

$$\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-r_1\tau}}{r_3}.$$

According to the comparison principle, it follows that

$$\limsup_{t\to+\infty} x_2(t) \le \frac{\alpha e^{-r_1\tau}}{r_3}.$$

Therefore, for arbitrary small $\varepsilon > 0$, there exists a $T_1 > \tau$ such that

$$x_2(t) \le \frac{\alpha e^{-r_1\tau}}{r_3} + \varepsilon := M_2,\ t \in [T_1, +\infty)_{\mathbb{T}}. \tag{7}$$

Setting $T_2 = T_1 + \tau$, from the first equation of system (1) and (7), for $t \in [T_2, +\infty)_{\mathbb{T}}$,

$$x_1^{\Delta}(t) \le \alpha(1 - e^{-r_1\tau})M_2 - r_1 x_1(t),$$

by Lemma 2, for arbitrary small $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$x_1(t) \le \frac{\alpha M_2}{r_1}(1 - e^{-r_1\tau}) + \varepsilon := M_1,\ t \in [T_3, +\infty)_{\mathbb{T}}.$$

Define

$$v(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t),\ t \in [T_3, +\infty)_{\mathbb{T}},$$

calculating the derivative of $v(t)$ along the solutions of (1), we have

$$\begin{aligned}
v^{\Delta}(t) \quad \le \quad & -Av(t) + \frac{\alpha^2}{4Ar_3} + \frac{(A + D_2 + \beta_1)^2(M_2 + k_2)}{4Aa_2}\\
& + \frac{(A + D_1 + \beta_2)^2}{4Aa_1},
\end{aligned} \tag{8}$$

where $A = \min\{r_1, r_2\}$.

By Lemma 2, there exists a $T_4 > T_3$, it follows from (8) that, for $t \in [T_4, +\infty)_{\mathbb{T}}$,

$$\begin{aligned}
v(t) \quad \le \quad & \frac{\alpha^2}{4Ar_3} + \frac{(A + D_2 + \beta_1)^2(M_2 + k_2)}{4Aa_2}\\
& + \frac{(A + D_1 + \beta_2)^2}{4Aa_1} + \varepsilon := M^*.
\end{aligned} \tag{9}$$

Let $M > \max\{M_1, M_2, M^*\}$, then

$$x_i(t) \le M, y_i(t) \le M,\ i = 1, 2,\ t \in [T_4, +\infty)_{\mathbb{T}}.$$

This completes the proof. ■

Proposition 2. *Assume that $(x_1(t), x_2(t), y_1(t), y_2(t))$ be a positive solution of system (1) with initial conditions (2). If $\alpha e^{-r_1\tau} > r_2 + \frac{a_1 M}{k_1}$, where M is defined in Proposition 1, then there exists a $T_8 > 0$ such that*

$$x_i(t) \ge m, y_i(t) \ge m,\ i = 1, 2,\ t \in [T_8, +\infty)_{\mathbb{T}}.$$

where m is a constant and

$$\begin{aligned}
&0 < m < \min\{m_1, m_2, m_3\},\\
&m_1 = \frac{\alpha m_2}{r_1}(1 - e^{-r_1\tau}) - \varepsilon,\\
&m_2 = \frac{\alpha e^{-r_1\tau} - r_2 - a_1 M/k_1}{r_3} - \varepsilon,\\
&m_3 = m_3^* - \varepsilon,\\
&m_3^* = \min\left\{\frac{\beta_1(m_2 + k_2)}{a_2}, \frac{\beta_2}{r_4}\right\}.
\end{aligned}$$

Proof: Assume that $(x_1(t), y_1(t), x_2(t), y_2(t))$ be any positive solution of system (1) with initial conditions (2). It follows from the second equation of system (1) that for $t \in [T_4, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} x_2^{\Delta}(t) &\geq \alpha e^{-r_1\tau} x_2(\delta_-(\tau,t)) - \left(r_2 + \frac{a_1 M}{k_1}\right) x_2(t) \\ &\quad - r_3 x_2^2(t). \end{aligned}$$

Consider the following auxiliary equation:

$$\begin{aligned} u^{\Delta}(t) &= \alpha e^{-r_1\tau} u(\delta_-(\tau,t)) - \left(r_2 + \frac{a_1 M}{k_1}\right) u(t) \\ &\quad - r_3 u^2(t), \end{aligned}$$

by Lemma 3, we can get

$$\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-r_1\tau} - r_2 - a_1 M/k_1}{r_3}.$$

According to the comparison principle, it follows that

$$\liminf_{t\to+\infty} x_2(t) \geq \frac{\alpha e^{-r_1\tau} - r_2 - a_1 M/k_1}{r_3}.$$

Therefore, for arbitrary small $\varepsilon > 0$, there exists a $T_5 > T_4$ such that

$$x_2(t) \geq \frac{\alpha e^{-r_1\tau} - r_2 - a_1 M/k_1}{r_3} - \varepsilon := m_2, \qquad (10)$$

for $t \in [T_5, +\infty)_{\mathbb{T}}$.

By the third and forth equation of system (1), we have

$$\begin{cases} y_1^{\Delta}(t) \geq (\beta_1 - \frac{a_2 y_1(t)}{m_2+k_2}) y_1(t) + D_1(y_2(t) - y_1(t)), \\ y_2^{\Delta}(t) = (\beta_2 - r_4 y_2(t)) y_2(t) + D_2(y_1(t) - y_2(t)), \end{cases}$$

for $t \in [T_5, +\infty)_{\mathbb{T}}$.

Consider the following auxiliary equation:

$$\begin{cases} u_1^{\Delta}(t) = (\beta_1 - \frac{a_2 u_1(t)}{m_2+k_2}) u_1(t) + D_1(u_2(t) - u_1(t)), \\ u_2^{\Delta}(t) = (\beta_2 - r_4 u_2(t)) u_2(t) + D_2(u_1(t) - u_2(t)), \end{cases}$$

for $t \in [T_5, +\infty)_{\mathbb{T}}$.

Define

$$v(t) = \min\{u_1(t), u_2(t)\}.$$

Using a similar argument in the proof of [14, Lemma 2.1], we can obtain

$$\liminf_{t\to+\infty} v(t) \geq \min\left\{\frac{\beta_1(m_2+k_2)}{a_2}, \frac{\beta_2}{r_4}\right\} := m_3^*.$$

Therefore, for arbitrary small $\varepsilon > 0$, there exists a $T_6 > T_5$ such that

$$y_1(t) \geq m_3^* - \varepsilon := m_3,\ y_2(t) \geq m_3^* - \varepsilon := m_3.$$

Setting $T_7 = T_6 + \tau$, from the first equation of system (1) and (10), for $t \in [T_7, +\infty)_{\mathbb{T}}$,

$$x_1^{\Delta}(t) \geq \alpha(1 - e^{-r_1\tau}) m_2 - r_1 x_1(t),$$

by Lemma 2, for arbitrary small $\varepsilon > 0$, there exists a $T_8 > T_7$ such that

$$x_1(t) \geq \frac{\alpha m_2}{r_1}(1 - e^{-r_1\tau}) - \varepsilon := m_1,\ t \in [T_8, +\infty)_{\mathbb{T}}.$$

Let $0 < m < \min\{m_1, m_2, m_3\}$ then

$$x_i(t) \geq m, y_i(t) \geq m,\ i = 1, 2,\ t \in [T_8, +\infty)_{\mathbb{T}}.$$

This completes the proof. ■

Together with Propositions 1 and 2, we can obtain the following theorem.

Theorem 1. *If conditions in Propositions 1 and 2 hold, then system (1) is permanent.*

Theorem 2. *If $\alpha e^{-r_1\tau} < r_2$, then the mature and immature prey population in system (1) will go to extinction.*

Proof: Assume that $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1) with initial conditions (2). It follows from the second equation of system (1) that

$$x_2^{\Delta}(t) \leq \alpha e^{-r_1\tau} x_2(\delta_-(\tau,t)) - r_2 x_2(t) - r_3 x_2^2(t).$$

Consider the following auxiliary equation:

$$u^{\Delta}(t) = \alpha e^{-r_1\tau} u(\delta_-(\tau,t)) - r_2 u(t) - r_3 u^2(t),$$

by Lemma 3, we can get

$$\lim_{t\to+\infty} u(t) = 0.$$

A standard comparison argument shows that

$$\lim_{t\to+\infty} x_2(t) = 0.$$

Therefore, for arbitrary small $\varepsilon > 0$, there exists a $T_9 > \tau$ such that

$$0 < x_2(t) < \frac{r_1 \varepsilon}{2\alpha(1 - e^{-r_1\tau})},\ t \in [T_9, +\infty)_{\mathbb{T}}. \qquad (11)$$

Setting $T_{10} = T_9 + \tau$, from the first equation of system (1) and (11), for $t \in [T_{10}, +\infty)_{\mathbb{T}}$,

$$x_1^{\Delta}(t) \leq r_1 \varepsilon - r_1 x_1(t),$$

by Lemma 2, there exists a $T_{11} > T_{10}$ such that

$$x_1(t) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ t \in [T_{11}, +\infty)_{\mathbb{T}},$$

that is

$$\lim_{t\to+\infty} x_1(t) = 0.$$

This completes the proof. ■

IV. Numerical Example and Simulations

Consider the following system on time scales

$$\begin{cases} x_1^{\Delta}(t) = 5x_2(t) - 0.5x_1(t) \\ \qquad -5e^{-0.5\tau} x_2(\delta_-(\tau,t)), \\ x_2^{\Delta}(t) = 5e^{-0.5\tau} x_2(\delta_-(\tau,t)) - r_2 x_2(t) \\ \qquad -3x_2^2(t) - \frac{0.8 y_1(t) x_2(t)}{x_2(t)+8}, \\ y_1^{\Delta}(t) = (0.2 - \frac{1.5 y_1(t)}{x_2(t)+1.5}) y_1(t) \\ \qquad +0.5(y_2(t) - y_1(t)), \\ y_2^{\Delta}(t) = (1.5 - y_2(t)) y_2(t) \\ \qquad +0.5(y_1(t) - y_2(t)), \end{cases} \qquad (12)$$

that is $\alpha = 5, r_1 = 0.5, r_3 = 3, r_4 = 1, a_1 = 0.8, a_2 = 1.5, k_1 = 8, k_2 = 1.5, \beta_1 = 0.2, \beta_2 = 1.5, D_1 = 0.5, D_2 = 0.5$, τ is a positive constant.

Consider system (12) on $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ with $t_0 = 0$. Obviously, $-r_1 \in \mathcal{R}^+$. Let $\tau = 1$, then $\delta_-(\tau, t) = t - 1$.

Case 1. Let $r_2 = 1.2$ in (12), by a direct calculation, we can get $\alpha e^{-r_1\tau} = 3.0327$, $r_2 + \frac{a_1 M}{k_1} = 2.3622$. Therefore, all conditions in Propositions 1 and 2 hold, by Theorem 1, system (12) is permanent, see Figures 1 and 2.

Case 2. Let $r_2 = 4$ in (12), by a direct calculation, we can get $\alpha e^{-r_1\tau} = 3.0327 < 4 = r_2$. The condition in Theorem 2 holds, then the mature and immature prey population in system (12) will go to extinction, see Figures 3 and 4.

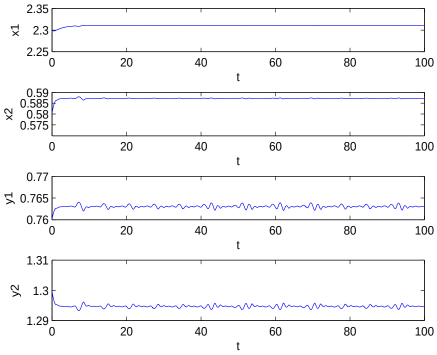

Fig. 1. $\mathbb{T} = \mathbb{R}$. Dynamics behavior of system (12) with initial conditions $x_1(0) = 2.3, x_2(0) = 0.58, y_1(0) = 0.76, y_2(0) = 1.3$.

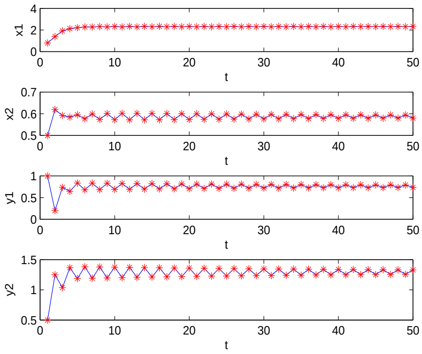

Fig. 2. $\mathbb{T} = \mathbb{Z}$. Dynamics behavior of system (12) with initial conditions $x_1(1) = 0.8, x_2(1) = 0.5, y_1(1) = 1, y_2(1) = 0.5$.

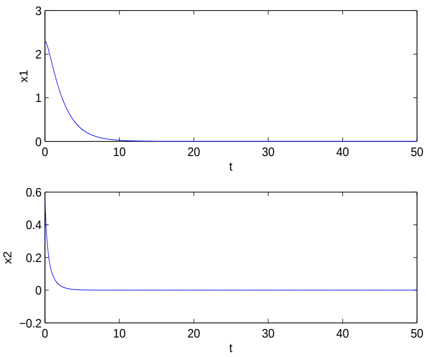

Fig. 3. $\mathbb{T} = \mathbb{R}$. Dynamics behavior of system (12) with initial conditions $x_1(0) = 2.3, x_2(0) = 0.58, y_1(0) = 0.76, y_2(0) = 1.3$.

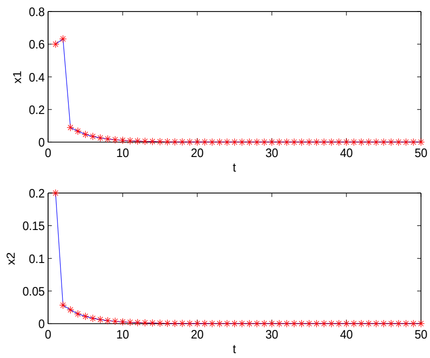

Fig. 4. $\mathbb{T} = \mathbb{Z}$. Dynamics behavior of system (12) with initial conditions $x_1(1) = 0.6, x_2(1) = 0.2, y_1(1) = 1, y_2(1) = 0.5$.

V. Conclusion

Two problems for a delayed Leslie-Gower predator-prey system with stage-structure on time scales have been studied, namely, permanence and extinction. It is important to notice that the methods used in this paper can be extended to other types of biological models [15-17]. Future work will include biological or epidemic dynamic systems modeling and analysis on time scales.

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