C^1 Rational Cubic/Linear Trigonometric Interpolation Spline with Positivity-preserving Property

Xiangbin Qin and Qingsong Xu

Abstract—**A class of C^1 rational cubic/linear trigonometric interpolation spline with two local parameters is proposed. Simple sufficient conditions for constructing positivity-preserving interpolation curves are developed. By using the boolean sum of quadratic trigonometric interpolating operators to blend together the proposed rational cubic/linear trigonometric interpolation splines as four boundary functions, a kind of C^1 blending rational cubic/linear interpolation surface with four families of local parameters is constructed. Simple sufficient data dependent conditions are also deduced for generating C^1 positivity-preserving interpolation surfaces on rectangular grids.**

Index Terms—**Data visualization, Trigonometric interpolation spline, Positivity-preserving, Local parameter.**

I. Introduction

Spline has a wide range of applications in Engineering, such as data fitting [1], principal components analysis [2], signal restoration [3] and so on. In visualizing scientific data, when the data are arising from some complex function or from some scientific phenomena, it becomes crucial that the resulting spline can preserve the shape features of the data. Positivity is one of the essential shape features of data. Many physical situations have entities that gain meaning only when their values are positive, such as a probability distribution function, monthly rainfall amounts, speed of winds at different intervals of time, and half-life of a radioactive substance and so on. For given positive data, ordinary interpolation spline methods such as the classical cubic interpolation spline usually ignore positive characteristic. Thus constructing positivity-preserving interpolation spline is an essential problem and many methods have been proposed, such as the cubic interpolation spline methods [4], the rational polynomial interpolation spline methods [5], [6], [7], [8], [9], [10], [11].

In the last years, for generating positivity-preserving interpolation curves, some trigonometric interpolation spline methods have been proposed, see for example [12], [13], [14], [15], [16], [17], and the references quoted therein. In [12], a kind of rational cubic/cubic trigonometric interpolation spline with four families of parameters was developed, and sufficient conditions were given for constructing positivity-preserving interpolation curves. In [13], a positivity-preserving interpolation spline was presented by using rational quadratic/quadratic trigonometric interpolation spline with two families of parameters. Latter, in [14], the rational quadratic/quadratic trigonometric interpolation spline with two families of parameters was further extended to four families of parameters. In [16], a new rational quadratic/quadratic trigonometric interpolation spline with four families of parameters was constructed for generating positivity-preserving interpolation curves. The smoothness of the resulting positivity-preserving interpolation curves by the above rational trigonometric interpolation spline methods attains C^1 continuity. Recently, in [15], [17], two kinds of quadratic trigonometric interpolation splines were proposed for generating GC^1 continuous positivity-preserving interpolation curves.

For constructing C^1 positivity-preserving interpolation surfaces, the well known Coons surface technique [18] has been widely used, see for example [19], [20], [21], [22], [23] and the references quoted therein. In [19], [20], by exchanging the cubic Hermite blending functions used for the classical bi-cubic Coons surface with two different kinds of rational cubic blending functions, two classes of C^1 rational bi-cubic were presented. And constrains concerning the local free parameters were given for visualizing 3D positive data on rectangular grids. Like the classical bi-cubic Coons surface technique, these schemes need to provide the twists on the grid lines for generating interpolation surfaces. In [21], [22], [23], based upon the boolean sum of cubic interpolating operators, by blending different rational cubic interpolation splines as the boundary functions, simpler schemes without making use of twists for constructing C^1 positive interpolation of gridded data were given. These rational bi-cubic partially blended interpolation spline methods are convenience since they are possible to control the shape of the interpolation surfaces by using the boundary functions, though they have to pay the price that the generated surfaces have zero twist vectors at the data points.

The sufficient conditions for generating positivity-preserving interpolation surfaces developed in [21], [22], [23] were based on the claim given in [24]: bi-cubic partially blended interpolation surface patch inherits all the properties of network of boundary curves. Thus, these methods have a common point that the positivity of the global interpolation surfaces are determined by the positivity of the four boundary curves of each local interpolation surface patch respectively. However, as it was pointed out in [20] that, these methods did not depict the positive surfaces due to the coon patches because they conserved the shape of data only on the boundaries of patch not inside the patch.

There are also some positivity-preserving interpolation surface schemes developed by using trigonometric interpolation spline methods. In [12], a kind of rational bi-

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cubic trigonometric interpolation spline was constructed for generating C^1 continuous positivity-preserving interpolation surfaces. In [13], a class of rational bi-quadratic trigonometric interpolation spline was developed for constructing C^1 continuous positivity-preserving interpolation surfaces. The disadvantages of these two methods lie in that the parameters do not have local control property for generating interpolation surfaces. Recently, in [25], a kind of bi-quadratic trigonometric interpolation spline was developed for generating GC^1 continuous positivity-preserving interpolation surfaces with local control parameters.

In this paper, we propose a kind of C^1 rational cubic/linear trigonometric interpolation spline with two local parameters. By using the boolean sum of quadratic trigonometric interpolating operators to blend together the proposed rational cubic/linear trigonometric interpolation splines as four boundary functions, a class of C^1 blending rational cubic/linear interpolation surface with four families of local parameters is constructed. Simple sufficient data dependent conditions are also deduced for generating C^1 positivity-preserving interpolation curves and surfaces. The developed schemes improve on the existing methods in some ways:
(1) The smoothness of bi-quadratic interpolant given in [8] is C^0 and the quadratic trigonometric interpolation splines given in [15], [17], [25] are GC^1 continuous while in this paper it is C^1.
(2) The rational trigonometric interpolation schemes developed in [12], [13] do not allow the designer to locally refine the positive surface as per consumers demand. Whereas, the given method is done by introducing local free parameters which they are used in the description of interpolation curves and surfaces.
(3) In [21], [22], [23], the authors claimed that the rational bi-cubic partially blended functions (coon patches) generated a positive surface but they conserved the shape of data only on the boundaries of patch not inside the patch and they did not provide proof that the conditions given [21], [22], [23] will be always sufficient to generate positivity-preserving interpolation surfaces everywhere in the domain. In contrast, we develop new constrain conditions on the boundary curves of each local interpolation surface patch and the given conditions are sufficient to generate positivity-preserving interpolation surfaces everywhere in the domain with theory proving.

The rest of this paper is organized as follows. Section II gives the construction of C^1 rational cubic/linear interpolation spline with two local free parameters. The sufficient conditions for generating positivity-preserving interpolation curves are discussed. In section III, a kind of C^1 blending rational cubic/linear interpolation surfaces with four families of local free parameters is described. Simple sufficient data dependent constraints are derived on the local free parameters to preserve the shape of 3D positive data on rectangular grids. Several numerical examples are also given to prove the worth of the new developed schemes. Conclusion is given in the section IV.

II. C^1 POSITIVITY-PRESERVING INTERPOLATION CURVES

In this section, we firstly construct a new kind of C^1 rational cubic/linear trigonometric interpolation spline with two families of free parameters, and then we developed simple sufficient conditions for generating positivity-preserving interpolation curves. Several numerical examples and comparisons are also given.

A. Rational cubic/linear trigonometric interpolation spline

Let $f_i \in R$, $i = 1, ..., n$, be data given at the distinct knots $x_i \in R$, $i = 1, ..., n$, with interval spacing $h_i = x_{i+1} - x_i > 0$, and let $d_i \in R$ denote the first derivative values defined at the knots. For $x \in [x_i, x_{i+1}]$, a piecewise rational cubic/linear trigonometric interpolation spline with two local parameters α_i and β_i is defined as follows

$$\begin{aligned} T(x) = {} & B_0(t;\alpha_i)f_i + B_1(t;\alpha_i)\left[f_i + \tfrac{2h_i}{\pi(1+\alpha_i)}d_i\right] \\ & + B_2(t;\beta_i)\left[f_{i+1} - \tfrac{2h_i}{\pi(1+\beta_i)}d_{i+1}\right] + B_3(t;\beta_i)f_{i+1}, \end{aligned} \tag{1}$$

where $t = \pi(x - x_i)/(2h_i)$, $\alpha_i, \beta_i \in [0, +\infty), i = 1, 2, ..., n-1$, and the four rational cubic/linear trigonometric basis functions $B_j(t;\alpha_i)$ and $B_{3-j}(t;\beta_i)$, $j = 0, 1$ are given by

$$\begin{cases} B_0(t;\alpha_i) = \frac{1-\sin t}{1+\alpha_i \sin t}, \\ B_1(t;\alpha_i) = \frac{\sin t(1-\sin t)(1+\alpha_i+\alpha_i \sin t)}{1+\alpha_i \sin t}, \\ B_2(t;\beta_i) = \frac{\cos t(1-\cos t)(1+\beta_i+\beta_i \cos t)}{1+\beta_i \cos t}, \\ B_3(t;\beta_i) = \frac{1-\cos t}{1+\beta_i \cos t}. \end{cases}$$

The spline given in (1) is a C^1 interpolation spline as it satisfies the following interpolation properties

$$\begin{cases} T(x_i) = f_i, \quad T(x_{i+1}) = f_{i+1}, \\ T'(x_i) = d_i, \quad T'(x_{i+1}) = d_{i+1}. \end{cases}$$

For any $t \in [0, \pi/2]$ and $\alpha_i, \beta_i \in [0, +\infty)$, it is easy to check that the four rational cubic/linear trigonometric basis functions possess the properties of partition of unity and nonnegativity, that is

$$\begin{cases} B_0(t;\alpha_i) + B_1(t;\alpha_i) + B_2(t;\beta_i) + B_3(t;\beta_i) = 1, \\ B_i(t;\alpha_i) \ge 0, \quad B_{3-i}(t;\beta_i) \ge 0, \quad i = 0, 1. \end{cases}$$

From the expression of the interpolation spline $T(x)$ given in (1), we have

$$\lim_{\alpha_i,\beta_i \to +\infty} T(x) = f_i \cos^2 t + f_{i+1} \sin^2 t$$

which implies that the parameters α_i, β_i serve as tension parameters.

In applications, the first derivative values d_i, $i = 1, 2, \ldots, n$ are not known and should be specified in advance. In this paper, they are computed by using the following Arithmetic mean method

$$\begin{cases} d_1 = \Delta_1 - \frac{h_1}{h_1+h_2}(\Delta_2 - \Delta_1), \\ d_i = \frac{\Delta_{i-1}+\Delta_i}{2}, i = 2, 3, \ldots, n-1, \\ d_n = \Delta_{n-1} + \frac{h_{n-1}}{h_{n-2}+h_{n-1}}(\Delta_{n-1} - \Delta_{n-2}), \end{cases}$$

where $\Delta_i = (f_{i+1} - f_i)/h_i$. This Arithmetic mean method is the three-point difference approximation based on arithmetic calculation, which is computationally economical and suitable for visualization of shaped data, see for example [21].

For convenience, in the following discussion, for $x \in [x_i, x_{i+1}]$, we will also denote the interpolation spline $T(x)$ given in (1) as $T(t; f_i, f_{i+1}; d_i, d_{i+1}; \alpha_i, \beta_i)$.

B. Positivity-preserving condition

For given positivity data set $\{(x_i, f_i), i = 1, 2, \ldots, n\}$, since the four rational cubic/linear trigonometric basis functions are nonnegative on $[0, \pi/2]$ and strict positive in $(0, \pi/2)$, it is obvious that the interpolation spline $T(x)$ given in (1) is positive on each subinterval $I_i = [x_i, x_{i+1}]$ if

$$\begin{cases} \alpha_i \geq 0, & \beta_i \geq 0, \\ f_i + \frac{2h_i d_i}{\pi(1+\alpha_i)} \geq 0, & f_{i+1} - \frac{2h_i d_{i+1}}{\pi(1+\beta_i)} \geq 0. \end{cases}$$

From these, we can immediately obtain the following sufficient conditions for $T(x)$ $(x \in [x_1, x_n])$ preserving positivity

$$\begin{cases} \alpha_i = \max\left\{-1 - \frac{2h_i d_i}{\pi f_i}, 0\right\} + a_i, & a_i \geq 0, \\ \beta_i = \max\left\{-1 + \frac{2h_i d_{i+1}}{\pi f_{i+1}}, 0\right\} + b_i, & b_i \geq 0, \end{cases} \tag{2}$$

where $i = 1, 2, \ldots, n-1$ and a_i, b_i serve as free parameters for the users to interactively adjust the shape of the obtained positivity-preserving interpolation curves.

C. Numerical examples and comparisons

TABLE I
POSITIVE DATA SET GIVEN IN [21].

i	1	2	3	4	5	6	7
x_i	0	2	4	10	28	30	32
f_i	20.8	8.8	4.2	0.5	3.9	6.2	9.6

TABLE II
THE POSITIVE DATA SET GIVEN IN [19].

i	1	2	3	4	5	6	7
x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	4	10

TABLE III
THE POSITIVE DATA SET GIVEN IN [14].

i	1	2	3	4	5	6	7	8
x_i	0	0.04	0.05	0.06	0.07	0.08	0.12	0.13
f_i	0.82	1.2	0.978	0.6	0.3	0.1	0.15	0.48

Fig. 1 shows the positivity-preserving interpolation curves generated by using the conditions (2) with different free parameters a_i and b_i for the positive data set given in Tab. I and the graphics of their first derivatives, respectively. From the results, we can see that the interpolation curves preserve the shape of the positive data set given in Tab. I nicely and they all achieve C^1 continuity.

Fig. 2 shows the positivity-preserving interpolation curves generated by using the conditions (2) with different free parameters a_i and b_i for the positive data set given in Tab. II and the graphics of their first derivatives, respectively. In can be seen from 2 that the interpolation curves clearly preserve the shape of the positive data set given in Tab. II and they all reach C^1 continuity.

Fig. 3 and Fig. 4 show the positivity-preserving interpolation curves generated by the methods given in [12], [13], [14], [15], [16] and our method with a set of appropriate parameters for the positive data sets given in Tab. III and Tab. IV, respectively. From the results, it can be seen that our piecewise rational cubic/linear trigonometric interpolation spline describes the positive data set more fairly than the methods given in [12], [13], [14], [15], [16].

III. C^1 POSITIVITY-PRESERVING INTERPOLATION SURFACES

In this section, by using the boolean sum of quadratic trigonometric interpolating operators to blend together the constructed rational cubic/linear trigonometric interpolation splines as the four boundary functions, we shall construct a class of C^1 blending rational cubic/linear trigonometric interpolation surface with four families of local parameters. By developing new constrains on the boundary functions, we will also theoretically deduce simple sufficient data dependent conditions on the local parameters to generate C^1 positivity-preserving interpolation surfaces for positive data on rectangular grids.

A. Blending rational cubic/linear trigonometric interpolation surfaces

Let $\{(x_i, y_i, F_{ij}), i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ be a given set of data points defined over the rectangular domain $D = [x_1, x_n] \times [y_1, y_m]$, where $\pi_x : x_1 < x_2 < \ldots < x_n$ is the partition of $[x_1, x_n]$ and $\pi_y : y_1 < y_2 < \ldots < y_m$ is the partition of $[y_1, y_m]$. For $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, by using the boolean sum of quadratic trigonometric interpolating operators to blend together the rational cubic/linear trigonometric interpolation splines (1) as the boundary functions, a new blending rational cubic/linear trigonometric interpolation surface is given as follows

$$F(x, y) = -\begin{bmatrix} -1 & b_0(t) & b_1(t) \end{bmatrix} H \begin{bmatrix} -1 \\ b_0(s) \\ b_1(s) \end{bmatrix}, \tag{3}$$

where $h_i^x = x_{i+1} - x_i, h_j^y = y_{j+1} - y_j, t = \pi(x - x_i)/(2h_i^x), s = \pi(y - y_j)/(2h_j^y)$ and

$$H = \begin{bmatrix} 0 & F(x, y_j) & F(x, y_{j+1}) \\ F(x_i, y) & F_{i,j} & F_{i,j+1} \\ F(x_{i+1}, y) & F_{i+1,j} & F_{i+1,j+1}, \end{bmatrix}$$

and

$$\begin{aligned} &b_0(z) := \cos^2 z, \qquad b_1(z) := \sin^2 z, \\ &F(x, y_j) := T(t; F_{i,j}, F_{i+1,j}; D_{i,j}^x, D_{i+1,j}^x; \alpha_{i,j}^x, \beta_{i,j}^x), \\ &F(x_i, y) := T(s; F_{i,j}, F_{i,j+1}; D_{i,j}^y, D_{i,j+1}^y; \alpha_{i,j}^y, \beta_{i,j}^y). \end{aligned}$$

Here, $D_{i,j}^x, D_{i,j}^y$ are known as the first partial derivatives at the grid point (x_i, y_j) and $\left(\alpha_{i,j}^x\right)_{(n-1)\times m}$, $\left(\beta_{i,j}^x\right)_{(n-1)\times m}$, $\left(\alpha_{i,j}^y\right)_{n\times(m-1)}$, $\left(\beta_{i,j}^y\right)_{n\times(m-1)}$ are called four families of local parameters. From the interpolation surface $T(x, y)$ given in (1), we can see that the changes of a local parameter $\alpha_{i,j}^x$ will affect the shape of two neighboring patches $F(x, y)$ defined in the domain $(x, y) \in (x_i, x_{i+1}) \times (y_{j-1}, y_{j+1})$. And the changes of a local parameter $\alpha_{i,j}^y$ will affect the shape of two neighboring patches $F(x, y)$ defined in the domain $(x, y) \in (x_{i-1}, x_{i+1}) \times (y_j, y_{j+1})$. And the local parameters $\beta_{i,j}^x$ and $\beta_{i,j}^y$ have the same effect region on the shape of the generated interpolation surface $F(x, y)$ as that of the local parameters $\alpha_{i,j}^x$ and $\alpha_{i,j}^y$, respectively. Since the four boundary functions $F(x, y_j)$, $F(x, y_{j+1})$, $F(x_i, y)$ and

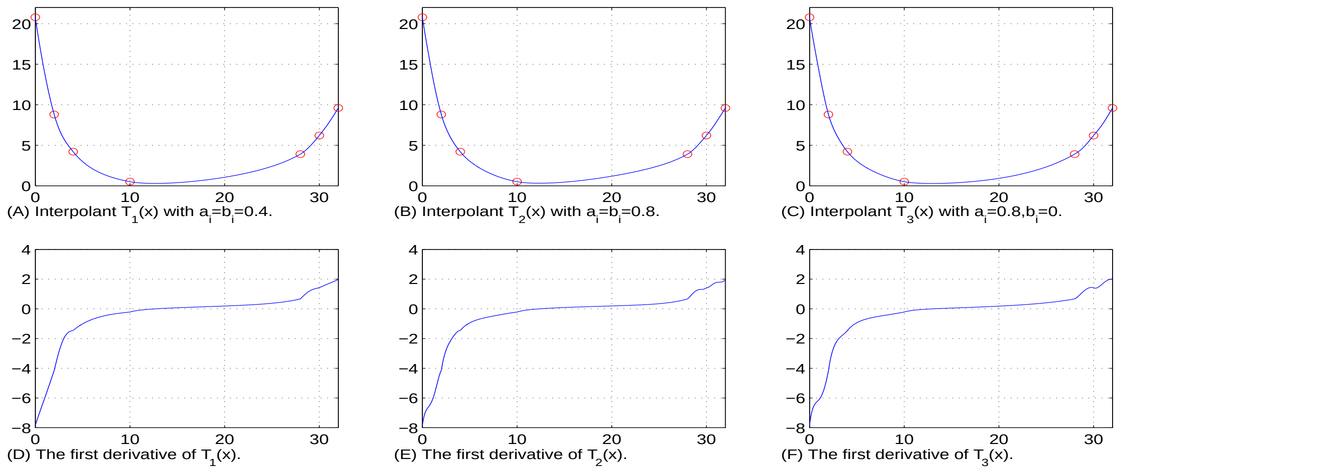

Fig. 1. Positivity-preserving interpolation curves generated by using the conditions (2) with different free parameters a_i and b_i for the positive data set given in Tab. I and the graphics of their first derivatives.

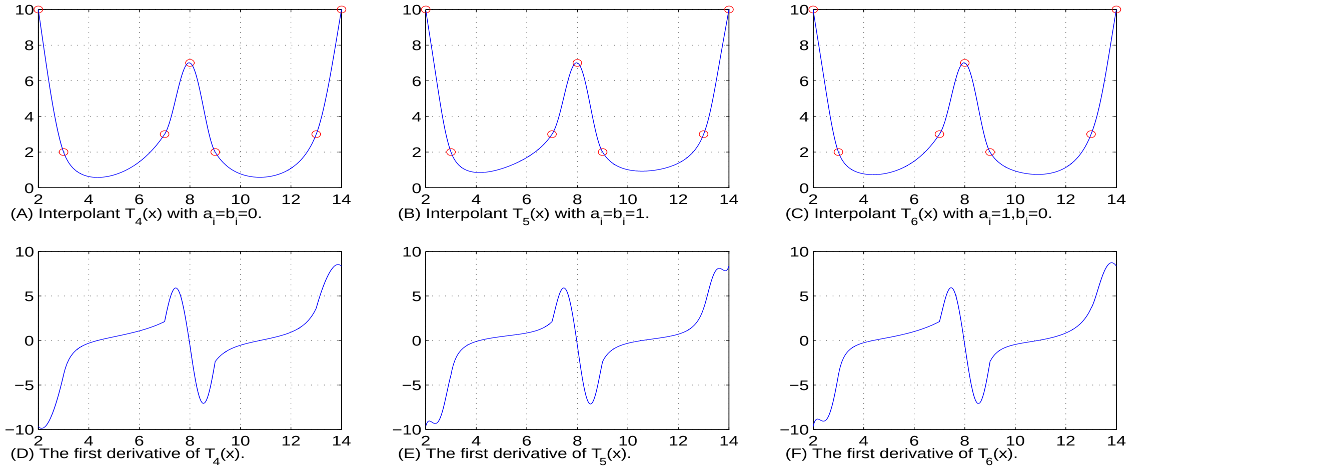

Fig. 2. Positivity-preserving interpolation curves generated by using the conditions (2) with different free parameters a_i and b_i for the positive data set given in Tab. II and the graphics of their first derivatives.

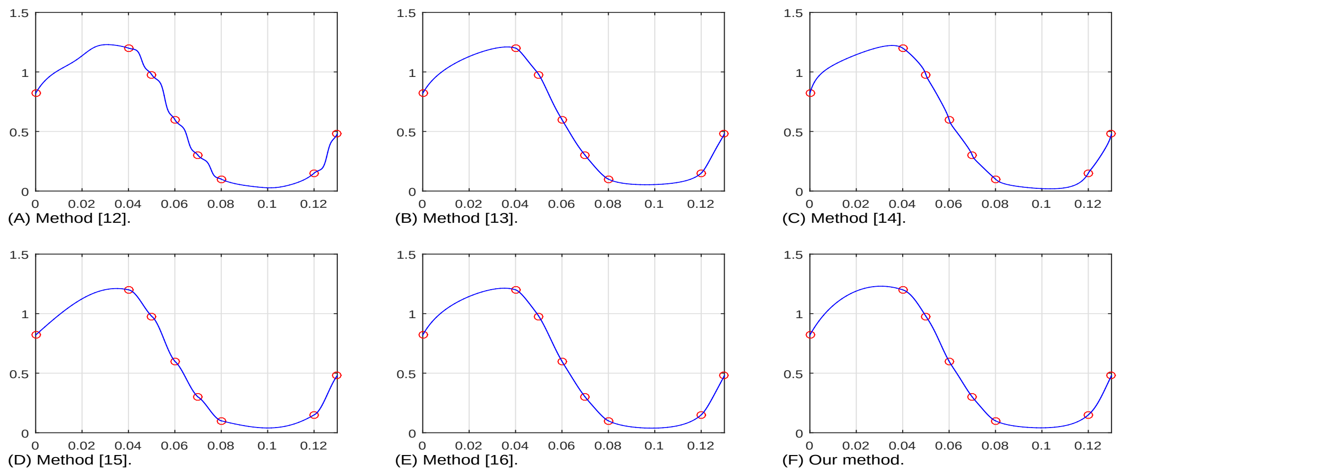

Fig. 3. C^1 positivity-preserving curves generated by different methods for the positive data set given in Tab. III. Their corresponding parameters for all segments are set as method [12]: $(w_i^0 = 0.2, w_i^1 = 0.6, w_i^2 = 0.3, w_i^3 = 0.1)$; method [13]: $(\mu_i = 4.1, \eta_i = 4.5)$; method [14]: $(w_i^0 = 0.2, w_i^1 = 0.8, w_i^2 = 0.8, w_i^3 = 0.3)$; method [15]: $(\alpha_i = 2, \beta_i = 1.5)$; method [16]: $(w_i^0 = 0.2, w_i^1 = 0.5, w_i^2 = 0.5, w_i^3 = 0.2)$; our method: $(\alpha_i = 1, \beta_i = 0)$.

TABLE IV
THE POSITIVE DATA SET GIVEN IN [14].

i	1	2	3	4	5	6	7	8	9	10
x_i	0	3.25	15	26.5	30	32	37	40	42.5	44
f_i	8.8	3	0.025	3.1	6.2	9.6	20	22.5	21.519	20

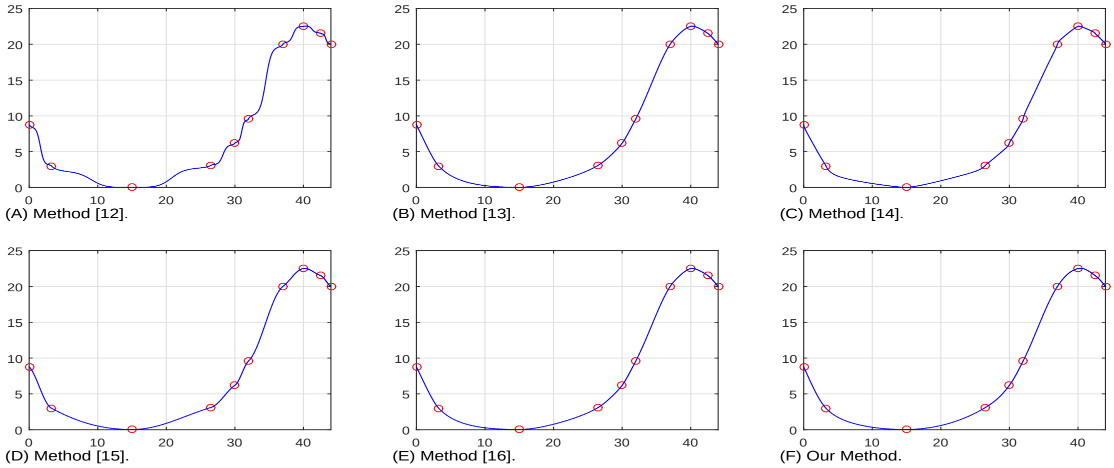

Fig. 4. C^1 positivity-preserving curves generated by different methods for the positive data set given in Tab. IV. Their corresponding parameters for all segments are set as method [12]: $(w_i^0 = 0.5, w_i^1 = 4, w_i^2 = 4, w_i^3 = 0.5)$; method [13]: $(\mu_i = 3, \eta_i = 3)$; method [14]: $(w_i^0 = 0.2, w_i^1 = 0.9, w_i^2 = 0.9, w_i^3 = 0.2)$; method [15]: $(\alpha_i = 2, \beta_i = 2)$; method [16]: $(w_i^0 = 2, w_i^1 = 4, w_i^2 = 4, w_i^3 = 2)$; our method: $(\alpha_i = 0.2, \beta_i = 0.1)$.

$F(x_{i+1}, y)$ are all C^1 continuous, we can easily conclude that the given blending rational cubic/linear trigonometric interpolation surface $F(x, y)$ is global C^1 continuous over the rectangular domain $[x_1, x_n] \times [y_1, y_m]$.

In most applications, the first partial derivatives $D_{i,j}^x$ and $D_{i,j}^y$ are not given and hence must be determined either from given data or by some other means. In this paper, we use the following arithmetic mean method to compute the first partial derivatives

$$
\begin{aligned}
&D_{1,j}^x = \Delta_{1,j}^x + \left(\Delta_{1,j}^x - \Delta_{2,j}^x\right) \tfrac{h_1^x}{h_1^x + h_2^x},\\
&D_{n,j}^x = \Delta_{n-1,j}^x + \left(\Delta_{n-1,j}^x - \Delta_{n-2,j}^x\right) \tfrac{h_{n-1}^x}{h_{n-2}^x + h_{n-1}^x},\\
&D_{i,j}^x = \tfrac{\Delta_{i-1,j}^x + \Delta_{i,j}^x}{2}, \quad i = 2, 3, \ldots, n-1; j = 1, 2, \ldots, m,\\
&D_{i,1}^y = \Delta_{i,1}^y + \left(\Delta_{i,1}^y - \Delta_{i,2}^y\right) \tfrac{h_1^y}{h_1^y + h_2^y},\\
&D_{i,m}^y = \Delta_{i,m-1}^y + \left(\Delta_{i,m-1}^y - \Delta_{i,m-2}^y\right) \tfrac{h_{m-1}^y}{h_{m-2}^y + h_{m-1}^y},\\
&D_{i,j}^y = \tfrac{\Delta_{i,j-1}^y + \Delta_{i,j}^y}{2}, \quad i = 1, 2, \ldots, n; j = 2, 3, \ldots, n-1,
\end{aligned}
$$

where $\Delta_{i,j}^x = (F_{i+1,j} - F_{i,j})/h_i^x$ and $\Delta_{i,j}^y = (F_{i,j+1} - F_{i,j})/h_j^y$. This arithmetic mean method is computationally economical and suitable for visualization of shaped data [21].

B. Positivity-preserving conditions

In this subsection, we want to develop simply schemes so that the C^1 interpolation surface $F(x, y)$ given in (3) can preserve the shape of 3D positive data on rectangular grides.

Let $\{(x_i, y_i, F_{i,j})\}$ be a positive data set defined over the rectangular grid $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 1, 2, \cdots, n-1$, $j = 1, 2, \cdots, m-1$ such that $F_{i,j} > 0, \forall i, j$. The interpolation surface $F(x, y)$ given in (3) preserves the shape of positive data if

$$F(x, y) > 0, \quad \forall (x, y) \in [x_1, x_n] \times [y_1, y_m].$$

For $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we want to rewrite the expression of the interpolation surface $F(x, y)$ given in 3 as the following form

$$
\begin{aligned}
&F(x, y) = b_0(s)F(x, y_j) + b_1(s)F(x, y_{j+1}) + b_0(t)F(x_i, y)\\
&+b_1(t)F(x_{i+1}, y) - b_0(t)b_0(s)F_{i,j} - b_0(t)b_1(s)F_{i,j+1}\\
&-b_1(t)b_0(s)F_{i+1,j} - b_1(t)b_1(s)F_{i+1,j+1}\\
&= b_0(s)\left[F(x, y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j}\right]\\
&+b_1(s)\left[F(x, y_{j+1}) - \tfrac{1}{2}b_0(t)F_{i,j+1} - \tfrac{1}{2}b_1(t)F_{i+1,j+1}\right]\\
&+b_0(t)\left[F(x_i, y) - \tfrac{1}{2}b_0(s)F_{i,j} - \tfrac{1}{2}b_1(s)F_{i,j+1}\right]\\
&+b_1(t)\left[F(x_{i+1}, y) - \tfrac{1}{2}b_0(s)F_{i+1,j} - \tfrac{1}{2}b_1(s)F_{i+1,j+1}\right].
\end{aligned} \tag{4}
$$

Without loss of generality, for any $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, since $b_0(z)$ and $b_1(z)$ are strict positive for any $z \in (0, 1)$, from (4), we can see that the interpolation surface $F(x, y)$ is positive everywhere in the domain $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ if the following constrains hold

$$
\begin{cases}
F(x, y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j} > 0,\\
F(x, y_{j+1}) - \tfrac{1}{2}b_0(t)F_{i,j+1} - \tfrac{1}{2}b_1(t)F_{i+1,j+1} > 0,\\
F(x_i, y) - \tfrac{1}{2}b_0(s)F_{i,j} - \tfrac{1}{2}b_1(s)F_{i,j+1} > 0,\\
F(x_{i+1}, y) - \tfrac{1}{2}b_0(s)F_{i+1,j} - \tfrac{1}{2}b_1(s)F_{i+1,j+1} > 0.
\end{cases} \tag{5}
$$

For $F(x, y_j) - \frac{1}{2}b_0(t)F_{i,j} - \frac{1}{2}b_1(t)F_{i+1,j}$, notice that $B_0(t; \alpha_{i,j}^x) + B_1(t; \alpha_{i,j}^x) = b_0(t)$ and $B_2(t; \beta_{i,j}^x) + B_3(t; \beta_{i,j}^x) = b_1(t)$, from (1), we have

$$
\begin{aligned}
&F(x, y_j) - \tfrac{1}{2}b_0(t)F_{i,j} - \tfrac{1}{2}b_1(t)F_{i+1,j}\\
&= \tfrac{1}{2}B_0(t; \alpha_{i,j}^x)F_{i,j} + B_1(t; \alpha_{i,j}^x)\left[\tfrac{F_{i,j}}{2} + \tfrac{2h_i^x}{\pi(1+\alpha_{i,j}^x)}D_{i,j}^x\right]\\
&+B_2(t; \beta_{i,j}^x)\left[\tfrac{F_{i+1,j}}{2} - \tfrac{2h_i^x}{\pi(1+\beta_{i,j}^x)}D_{i+1,j}^x\right]\\
&+\tfrac{1}{2}B_3(t; \beta_{i,j}^x)F_{i+1,j},
\end{aligned}
$$

Thus we can see that the following constrains are sufficient to ensure $F(x,y_j)-\frac{1}{2}b_0(t)F_{i,j}-\frac{1}{2}b_1(t)F_{i+1,j}>0$

$$\begin{cases} \alpha^x_{i,j}\geq 0, & \beta^x_{i,j}\geq 0,\\ \frac{F_{i,j}}{2}+\frac{2h^x_i}{\pi\left(1+\alpha^x_{i,j}\right)}D^x_{i,j}\geq 0, & \frac{F_{i+1,j}}{2}-\frac{2h^x_i}{\pi\left(1+\beta^x_{i,j}\right)}D^x_{i+1,j}\geq 0, \end{cases}$$

which bring forth the following sufficient conditions

$$\begin{cases} \alpha^x_{i,j}\geq \max\left\{-1-\frac{4h^x_iD^x_{i,j}}{\pi F_{i,j}},0\right\},\\ \beta^x_{i,j}\geq \max\left\{-1+\frac{4h^x_iD^x_{i+1,j}}{\pi F_{i+1,j}},0\right\}, \end{cases} \tag{6}$$

In the same manner, we can also derive similar sufficient conditions for $F(x,y_{j+1})-\frac{1}{2}b_0(t)F_{i,j+1}-\frac{1}{2}b_1(t)F_{i+1,j+1}>0$, $F(x_i,y)-\frac{1}{2}b_0(s)F_{i,j}-\frac{1}{2}b_1(s)F_{i,j+1}>0$ and $F(x_{i+1},y)-\frac{1}{2}b_0(s)F_{i+1,j}-\frac{1}{2}b_1(s)F_{i+1,j+1}>0$. In conclusion, for a positive data set, we can obtain the following sufficient conditions for $F(x,y)>0,\forall(x,y)\in[x_1,x_n]\times[y_1,y_n]$

$$\begin{cases} \alpha^x_{i,j}=\max\left\{-1-\frac{4h^x_iD^x_{i,j}}{\pi F_{i,j}},0\right\}+a^x_{i,j},\\ \beta^x_{i,j}=\max\left\{-1+\frac{4h^x_iD^x_{i+1,j}}{\pi F_{i+1,j}},0\right\}+b^x_{i,j},\\ \alpha^y_{i,j}=\max\left\{-1-\frac{4h^y_jD^y_{i,j}}{\pi F_{i,j}},0\right\}+a^y_{i,j},\\ \beta^y_{i,j}=\max\left\{-1+\frac{4h^y_jD^y_{i,j+1}}{\pi F_{i,j+1}},0\right\}+b^y_{i,j}, \end{cases} \tag{7}$$

where $i=1,2,\cdots,n-1$, $j=1,2,\cdots,m-1$ and $a^x_{i,j}$,$a^y_{i,j}$,$b^x_{i,j}$ and $b^y_{i,j}$ are arbitrary nonnegative real numbers and serve as free parameters.

C. Numerical examples and comparisons

We shall give several numerical examples to show that the proposed C^1 interpolation surface $F(x,y)$ given in (3) can be used to nicely visualize the shape of 3D positive data on rectangular grids. In the following figures, the given data points have been marked with solid black dots.

Fig. 5 shows the positivity-preserving interpolation surfaces $F_1(x,y)$ and $F_2(x,y)$ for the 3D positive data set given in Tab. V. From the results, we can see that both the two visually pleasing interpolation surfaces preserve the shape of the 3D positive data set given in Tab. V genially.

Fig. 6 shows the positivity-preserving interpolation surfaces $F_3(x,y)$ and $F_4(x,y)$ for the 3D positive data set given in Tab. VI. As can be seen from Fig. 6, both the two interpolation surfaces visualize the shape of the 3D positive data set given in Tab. IV well.

Fig. 7 shows the positivity-preserving interpolation surfaces $F_5(x,y)$ and $F_6(x,y)$ for the 3D positive data set given in Tab. VII. As can be seen from the Fig. 7, both the two interpolation surfaces visualize the positive shape of the data given in Tab. VII nicely and the shape of the interpolation surfaces can be adjusted conveniently by using the local free parameters.

Fig. 8 show the positivity-preserving interpolation surfaces generated by the method given in [12] and our method with a set of appropriate parameters for the positive data sets given in Tab. VIII. From the results, it can be seen that our blending cubci/linear trigonometric interpolation spline describes the positive data set more fairly than the method given in [12].

IV. Conclusion

As stated above, the constructed C^1 rational cubic/linear trigonometric interpolation spline is suitable for constructing positivity-preserving interpolation curves. To demonstrate our method, comparisons with the existing methods [12], [13], [14], [15], [16] are provided. The proposed blending rational cubic/linear trigonometric interpolation surfaces with four families of local free parameters can be C^1 continuous without making use of the first mixed partial derivatives at the data points. For 3D positive data on rectangular grids, by developing new constrain conditions on the boundary curves of each local interpolation surface patch, we theoretically deduce simple sufficient data dependent conditions on the local free parameters to generate positivity-preserving interpolation surfaces everywhere in the domains. The developed positivity-preserving schemes are not only local and computationally economical but also visually pleasant. The given method also allows extensions to generate C^1 positivity-preserving interpolation surfaces for 3D non-gridded data. These will be our future work.

TABLE VII
THE 3D POSITIVE DATA SET GIVEN IN [19].

y/x	-3	-2	-1	1	2	3
-3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124
-2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
-1	0.0404	0.1667	1.3333	1.3333	0.1667	0.0404
1	0.0404	0.1667	1.333	1.3333	0.1667	0.0404
2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124

TABLE VIII
THE 3D POSITIVE DATA SET GIVEN IN [12].

y/x	-3	-2	-1	0	1	2	3
-3	18	13	10	9	10	13	18
-2	13	8	5	4	5	8	13
-1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

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TABLE V
THE 3D POSITIVE DATA SET [20].

y/x	-3	-2	-1	0	1	2	3
-3	0.0401	0.0404	0.1755	1.0401	0.1755	0.0404	0.0401
-2	0.0583	0.0586	0.1936	1.0583	0.1936	0.0586	0.0583
-1	0.4078	0.4082	0.5432	1.4079	0.5432	0.4082	0.4078
0	1.04	1.0403	1.1753	2.04	1.1753	1.0403	1.04
1	0.4078	0.4082	0.5432	1.4079	0.5432	0.4082	0.4078
2	0.0583	0.0586	0.1936	1.0583	0.1936	0.0586	0.0583
3	0.0401	0.0404	0.1755	1.0401	0.1755	0.0404	0.0401

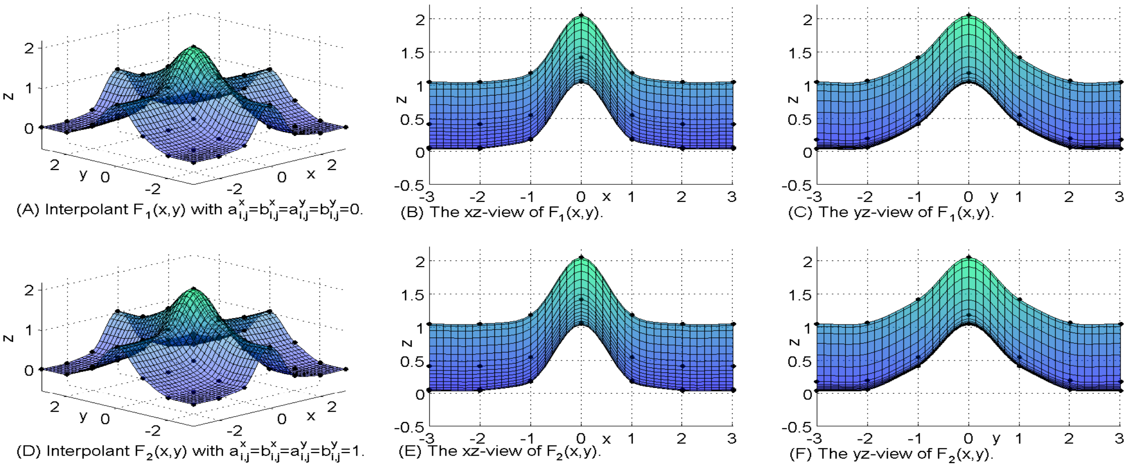

Fig. 5. C^1 positivity-preserving interpolation surfaces for the 3D positive data set given in Tab. V.

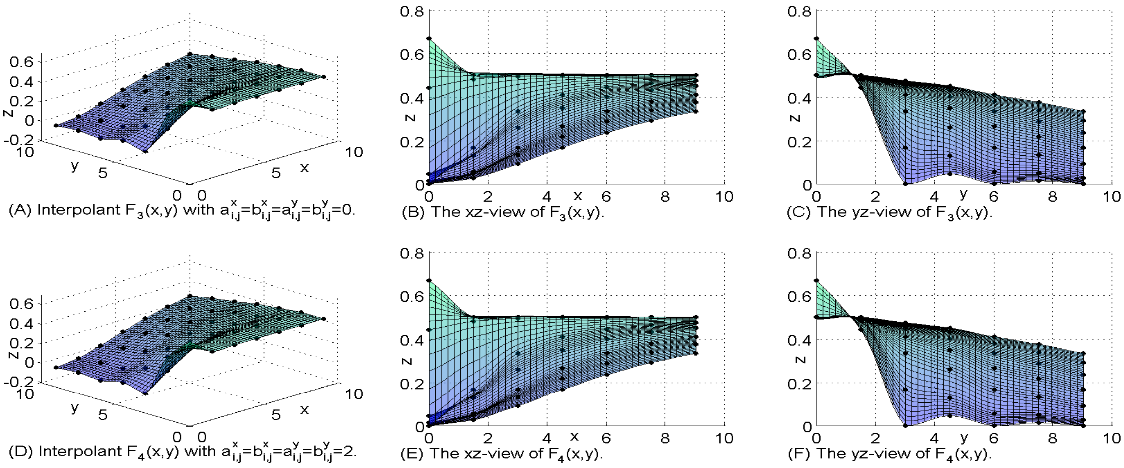

Fig. 6. C^1 positivity-preserving interpolation surfaces for the 3D positive data set given in Tab. VI.

TABLE VI
THE 3D POSITIVE DATA SET GIVEN IN [21].

y/x	0.0001	1.5	3	4.5	6	7.5	9
0.0001	0.6667	0.5	0.5	0.5	0.5	0.5	0.5
1.5	0.4422	0.4807	0.4936	0.4970	0.4982	0.4989	0.4992
3	0.0022	0.1681	0.3341	0.4095	0.4447	0.4631	0.4738
4.5	0.0472	0.1295	0.2603	0.3491	0.4006	0.4309	0.4497
6	0.0022	0.0575	0.1681	0.2657	0.3341	0.3793	0.4095
7.5	0.0156	0.0515	0.1331	0.2184	0.2876	0.3385	0.3752
9	0.0021	0.0283	0.0926	0.1681	0.2364	0.2916	0.3340

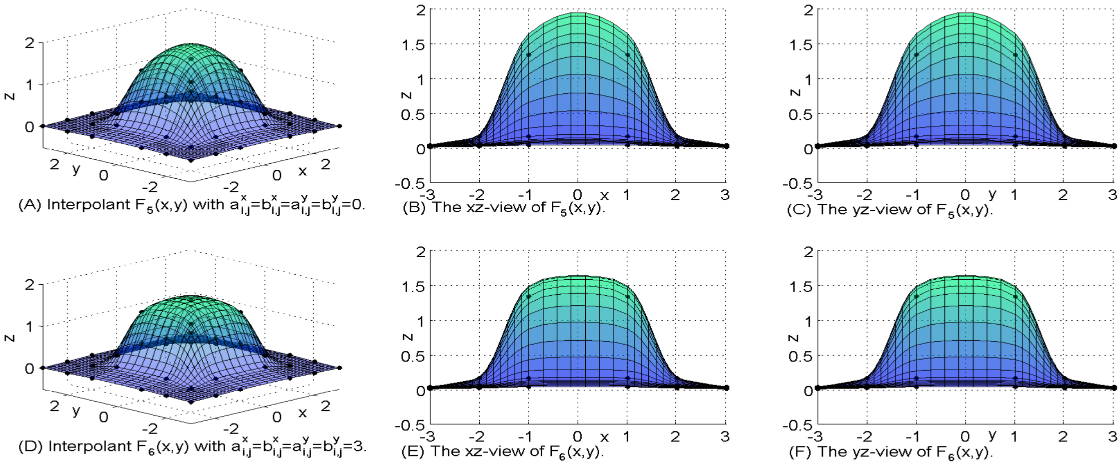

Fig. 7. C^1 positivity-preserving interpolation surfaces for the 3D positive data set given in Tab. VII.

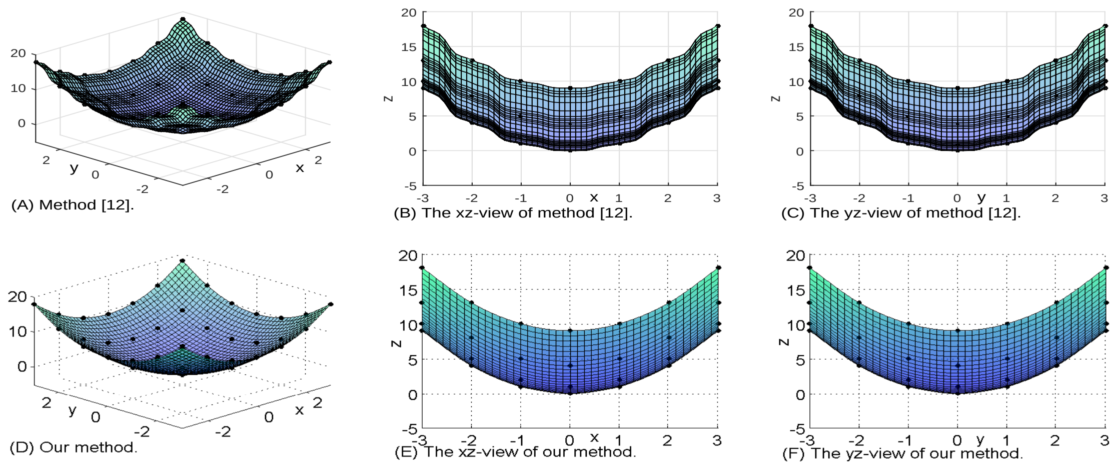

Fig. 8. C^1 positivity-preserving interpolation surfaces generated by different methods for the 3D positive data set given in Tab. VIII. Their corresponding parameters for all segments are set as method [12]: $(a_i = 1, \beta_i = 3, \gamma_i = 3, \delta_i = 1, \widehat{a}_j = 3, \widehat{\beta}_j = 3, \widehat{\gamma}_j = 3, \widehat{\delta}_j = 3)$; our method: $(\alpha^x_{i,j} = \beta^x_{i,j} = \alpha^y_{i,j} = \beta^y_{i,j} = 0)$.

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