Some Characterizations of Fuzzy Bi-ideals and Fuzzy Quasi-ideals of Semigroups

Pongpun Julath[1] and Manoj Siripitukdet[1,2]

Abstract—**The aims of this paper are to characterize fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups. We define certain subsets of semigroups S, $[0,1]$ and $S\times[0,1]$. The relationships between sets of fuzzy points and the certain subsets of the semigroup $S\times[0,1]$ are discussed. In the main results, characterizations of fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups are investigated by using the certain subsets of semigroups S, $[0,1]$ and $S\times[0,1]$.**

Index Terms—**fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy quasi-ideals, semigroups.**

I. Introduction

THE fundamental concepts of fuzzy sets have been proposed by Zadeh [24] since 1965. These concepts were applied in many areas such as: medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory. In 1971, the concepts of fuzzy sets were transferred to fuzzy groups by Rosenfeld [19]. The study of fuzzy subsystems of semigroups was introduced by Kuroki (see [10]-[15]). He investigated some properties of fuzzy subsemigroups, fuzzy ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals, fuzzy quasi-ideals of semigroups. After that many types of fuzzy algebraic structures have been introduced and investigated.

The fuzzy ideals and bi-ideals of semigroups have been applied in characterizing the duo semigroup, the simple semigroup and semilattices of subsemigroups [11]. He also investigated characterizations of regular semigroups and both intra-regular and left quasi-regular semigroups in terms of fuzzy generalized bi-ideals [13]. Completely regular semigroups and a semilattice of groups are characterized by using fuzzy semiprime quasi-ideals [15]. In [1], Ahsan et al. studied some properties of fuzzy quasi-ideals of semigroups and used their properties to characterize regular and intra-regular semigroups. Shabir et al. [22] introduced certain types of fuzzy bi-ideals, called prime, strongly prime, and semiprime fuzzy bi-ideals, and characterized semigroups in terms of their semiprime and strongly prime fuzzy bi-ideals. Concepts of fuzzy (generalized) bi-ideals and fuzzy quasi-ideals of semigroups play an important role in the study of types of semigroups. Some authors studied similar types of fuzzy subsets of other algebraic structures seen in [3]-[8], [17], [18], [20], [21] and [23].

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[1]Department of Mathematics, Faculty of Science Naresuan University, Phitsanulok 65000, Thailand. e-mail manojs@nu.ac.th, pongpunjulatha@hotmail.com

[2]Research Center for Academic Excellence in Mathematics, Naresuan University, Thailand.

Corresponding author email: manojs@nu.ac.th (M. Siripitukdet)

Our propose of this work is to promote and develop fuzzy semigroup theory and related structures by using fuzzy subsemigroups, fuzzy (generalized) bi-ideals and fuzzy quasi-ideals. We define the certain subsets of S, $[0,1]$ and $S\times[0,1]$ and investigate their properties. In particular, we define a certain subset $U(\mathcal{R}:\alpha)$ of S where $\mathcal{R}$ is a subset of $S\times[0,1]$ and this set is a general concept of the upper level set of a fuzzy set. We also describe relationships between sets of fuzzy points and the certain subsets of $S\times[0,1]$. In the main results of this paper, we characterize fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups by using the certain subsets of S, $[0,1]$ and $S\times[0,1]$. Moreover, we show that any fuzzy subset of S is a fuzzy bi-ideal (resp., a fuzzy generalized bi-ideal, a fuzzy quasi-ideal) if and only if there exists the unique chain of bi-ideals (resp., generalized bi-ideals, quasi-ideals) of S together with two special conditions.

II. Preliminaries

In this section, we give some definitions, notations and results of semigroups and fuzzy semigroups.

Throughout this paper, S stand for a semigroup unless otherwise specified. A nonempty subset A of S is called a **subsemigroup** of S if $AA\subseteq A$. A nonempty subset G of S is called a **generalized bi-ideal** of S if $GSG\subseteq G$. A subsemigroup B of S is called a **bi-ideal** of S if $BSB\subseteq B$. Then a nonempty subset B of S is a bi-ideal of S if and only if B is both a subsemigroup and generalized bi-ideal of S. A nonempty subset Q of S is called a **quasi-ideal** of S if $QS\cap SQ\subseteq Q$.

A **fuzzy subset** f [24] of S is described as a function $f:S\to[0,1]$ and its image is denoted by $Imf=\{f(x)\mid x\in S\}$. The set of all fuzzy subsets of S is denoted by $F(S)$ and let $f,g\in F(S)$. We define the order relation "$\leq$ " on $F(S)$ as follows $f\leq g$ if and only if $f(x)\leq g(x)$ for all $x\in S$. For $x\in S$, define

$$F_x=\{(y,z)\in S\times S\mid x=yz\}.$$

The fuzzy subsets $f\wedge g$ and $f\circ g$ of S are defined as follows:

$$(f\wedge g)(x) \quad = \quad \min\{f(x),g(x)\}$$

and

$$(f\circ g)(x)=\begin{cases}\sup\{\min\{f(y),g(z)\}\mid(y,z)\in F_x\},\\ \text{if } F_x\neq\emptyset;\\ 0,\text{ otherwise}\end{cases}$$

for all $x\in S$. Then $(F(S),\circ)$ is a semigroup [16]. For any $\alpha\in(0,1]$ and $x\in S$, a fuzzy subset x_α of S is called a **fuzzy point** [9] in S if for all $y\in S$

$$x_\alpha(y)=\begin{cases}\alpha, & \text{if } x=y;\\ 0, & \text{otherwise.}\end{cases}$$

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Then the set of all fuzzy points $FP(S)$ of S is a subsemigroup of $F(S)$ [9].

The following definitions are important types of fuzzy subsystems of semigroups which will be used in this paper.

Definition II.1. [10] A fuzzy subset f of S is called a **fuzzy subsemigroup** of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

Definition II.2. [12] A fuzzy subset f of S is called a **fuzzy generalized bi-ideal** of S if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$.

Definition II.3. [10] A fuzzy subsemigroup f of S is called a **fuzzy bi-ideal** of S if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$.

Then a fuzzy subset f of S is a fuzzy bi-ideal of S if and only if f is both a fuzzy generalized bi-ideal and a fuzzy subsemigroup of S.

Definition II.4. [11] A fuzzy subset f of S is called a **fuzzy quasi-ideal** of S if $(f \circ \chi_S) \wedge (\chi_S \circ f) \leq f$ where $\chi_S(x) = 1$ for all $x \in S$.

Define a binary operation "$*$" on $S \times [0,1]$ by for all $(x, \alpha), (y, \beta) \in S \times [0,1]$

$$(x, \alpha) * (y, \beta) = (xy, \min\{\alpha, \beta\}).$$

Then $(S \times [0,1], *)$ is a semigroup [2].

Remark II.5. For every subsemigroup A of S and every nonempty subset Δ of $[0,1]$, we have $(A \times \Delta, *)$ is a subsemigroup of $(S \times [0,1], *)$. In what follows, let $S \times \Delta$ denote the semigroup $(S \times \Delta, *)$ throughout this paper.

Let f be a fuzzy subset of S, $A \subseteq S$, $\alpha \in [0,1]$, $\Delta \subseteq [0,1]$ and $\mathcal{R} \subseteq S \times [0,1]$. We define the the certain subsets of $S \times [0,1]$, S and $[0,1]$, respectively as follows:

$$\begin{aligned} [A \times \Delta]^f &= \{(x, \alpha) \in A \times \Delta \mid f(x) \geq \alpha\}, \\ U(\mathcal{R} : \alpha) &= \{x \in S \mid (x, \beta) \in \mathcal{R} \text{ and } \alpha \leq \beta \\ &\quad\ \text{for some } \beta \in [0,1]\}, \\ (Imf)_\alpha &= \{\beta \in Imf \mid \beta \geq \alpha\}. \end{aligned}$$

In particular, if $\mathcal{R}$ is a fuzzy subset of S, then

$$U(\mathcal{R} : \alpha) = \{x \in S \mid \mathcal{R}(x) \geq \alpha\}.$$

Thus $U(\mathcal{R} : \alpha)$ is a general concept of the upper level set of a fuzzy set. For α, $\beta \in [0,1]$ with $\alpha \leq \beta$, we have $U(\mathcal{R} : \beta) \subseteq U(\mathcal{R} : \alpha)$. Hence $\{U(\mathcal{R} : \alpha) \mid \alpha \in [0,1]\}$ is a chain of subsets of S under the inclusion relation "$\subseteq$".

The following propositions are easy to prove.

Proposition II.6. *Let f be a fuzzy subset of a semigroup S. Then*

- (i) $(Imf)_\alpha \subseteq Imf$ *for all* $\alpha \in [0,1]$.
- (ii) $U(f : \alpha) = \bigcup_{\gamma \in (Imf)_\alpha} f^{-1}(\gamma) = f^{-1}((Imf)_\alpha)$ *for all* $\alpha \in [0,1]$.
- (iii) $[S \times \Delta]^f = \bigcup_{\gamma \in \Delta} (U(f : \gamma) \times \{\gamma\})$ *for every subset* Δ *of* $[0,1]$.
- (iv) *If* $\Delta \subseteq [0,1]$ *and* $\mathcal{R} := [S \times \Delta]^f$, *then* $U(\mathcal{R} : \alpha) = U(f : \alpha)$ *for all* $\alpha \in \Delta$.

Proposition II.7. *Let S be a semigroup and let Δ be a nonempty subset of $[0,1]$. If $\mathcal{R}$ is a subsemigroup of $S \times \Delta$ and $\alpha \in \Delta$, then $U(\mathcal{R} : \alpha)(\neq \emptyset)$ is a subsemigroup of S.*

Proposition II.8. *Let S be a semigroup and let Δ be a nonempty subset of $[0,1]$. If $\mathcal{R}$ is a generalized bi-ideal of $S \times \Delta$ and $\alpha \in \Delta$, then $U(\mathcal{R} : \alpha)(\neq \emptyset)$ is a generalized bi-ideal of S.*

Proposition II.9. *Let S be a semigroup and let Δ be a nonempty subset of $[0,1]$. If $\mathcal{R}$ is a bi-ideal of $S \times \Delta$ and $\alpha \in \Delta$, then $U(\mathcal{R} : \alpha)(\neq \emptyset)$ is a bi-ideal of S.*

Proposition II.10. *Let S be a semigroup and let Δ be a nonempty subset of $[0,1]$. If $\mathcal{R}$ is a quasi-ideal of $S \times \Delta$ and $\alpha \in \Delta$, then $U(\mathcal{R} : \alpha)(\neq \emptyset)$ is a quasi-ideal of S.*

III. Fuzzy Subsemigroups

In this section, we characterize fuzzy subsemigroups of a semigroup S by using the certain subsets of S, $[0,1]$, $FP(S)$ and $S \times [0,1]$.

For the following theorem, we investigate characterizations of fuzzy subsemigroups of S via the certain subsets of $[0,1]$ and $S \times [0,1]$.

Theorem III.1. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

- (i) *f is a fuzzy subsemigroup of S.*
- (ii) *For every subsemigroup A of S and $\Delta \subseteq [0,1]$, $[A \times \Delta]^f(\neq \emptyset)$ is a subsemigroup of $S \times \Delta$.*
- (iii) *$[S \times \Delta]^f$ is a subsemigroup of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0,1]$.*
- (iv) *$(Imf)_{f(ab)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}$ for all $a, b \in S$.*

Proof: $((i) \Rightarrow (ii))$ Let A be a subsemigroup of S, $\Delta \subseteq [0,1]$ and $(a, \alpha), (b, \beta) \in [A \times \Delta]^f$. Then $f(a) \geq \alpha$, $f(b) \geq \beta$ and $\min\{\alpha, \beta\} \in \Delta$. Since f is a fuzzy subsemigroup of S and A is a subsemigroup of S, we have $ab \in A$ and

$$f(ab) \geq \min\{f(a), f(b)\} \geq \min\{\alpha, \beta\}.$$

Thus $(a, \alpha) * (b, \beta) \in [A \times \Delta]^f$. Hence $[A \times \Delta]^f$ is a subsemigroup of $S \times \Delta$.

$((ii) \Rightarrow (iii))$ It is clear.

$((iii) \Rightarrow (i))$ Let $Imf \subseteq \Delta \subseteq [0,1]$ and $a, b \in S$. Then $(a, f(a)), (b, f(b)) \in [S \times Imf]^f \subseteq [S \times \Delta]^f$. By assumption (iii), we have $[S \times \Delta]^f$ is a subsemigroup of $S \times \Delta$. Thus $(a, f(a)) * (b, f(b)) \in [S \times \Delta]^f$. Hence $f(ab) \geq \min\{f(a), f(b)\}$.

$((i) \Rightarrow (iv))$ Let $a, b \in S$ and $\alpha \in (Imf)_{f(ab)}$. Then $\alpha \geq f(ab)$. By assumption (i), we have $\alpha \geq f(ab) \geq \min\{f(a), f(b)\}$. Thus $\alpha \geq f(a)$ or $\alpha \geq f(b)$. Hence $\alpha \in (Imf)_{f(a)}$ or $\alpha \in (Imf)_{f(b)}$. Therefore

$$(Imf)_{f(ab)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}.$$

$((iv) \Rightarrow (i))$ It is straightforward. ■

By applying Theorem III.1, we have Corollary III.2.

Corollary III.2. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

- (i) *f is a fuzzy subsemigroup of S.*
- (ii) *$[S \times (0,1]]^f(\neq \emptyset)$ is a subsemigroup of $S \times (0,1]$.*
- (iii) *$[S \times Imf]^f$ is a subsemigroup of $S \times Imf$.*

(iv) $[S \times [0,1]]^f$ *is a subsemigroup of* $S \times [0,1]$.

Example III.3. *Let* $S = \{a,b,c,d\}$ *and define a binary operation "$\cdot$" on* S *as follows :*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then $(S,\cdot)$ *is a semigroup (see [16]). Let* f *be a fuzzy subset of* S *such that*

$$f(a) = f(b) = 0.8, \quad f(c) = 0.4, \quad f(d) = 0.3.$$

Thus, by routine calculations, we can check that $[S \times Imf]^f = \{(a,0.8), (b,0.8), (a,0.4), (b,0.4), (c,0.4), (a,0.3), (b,0.3), (c,0.3), (d,0.3)\}$ *is a subsemigroup of* $S \times Imf$. *By Corollary III.2, we have* f *is a fuzzy subsemigroup of* S.

Next, we show a relation between the sets $[S \times (0,1]]^f$ and $\underline{f} := \{x_\alpha \in FP(S) \mid f(x) \geq \alpha\}$ in Proposition III.4 whose proof is straightforward and omitted.

Proposition III.4. *Let* f *be a fuzzy subset of a semigroup* S. *Then* $[S \times (0,1]]^f$ *is a subsemigroup of* $S \times (0,1]$ *if and only if* $\underline{f}$ *is a subsemigroup of* $FP(S)$.

By Corollary III.2 and Proposition III.4, we immediately get Corollary III.5.

Corollary III.5. *Let* f *be a fuzzy subset of a semigroup* S. *Then* f *is a fuzzy subsemigroup of* S *if and only if* $\underline{f}(\neq \emptyset)$ *is a subsemigroup of* $FP(S)$.

In the following result, an equivalent condition for any fuzzy subsemigroup of a semigroup S is discussed via the chain of subsemigroups of S.

Theorem III.6. *Let* f *be a fuzzy subset of a semigroup* S. *Then* f *is a fuzzy subsemigroup of* S *if and only if there exists the unique chain* $\{A_\alpha \mid \alpha \in Imf\}$ *of subsemigroups of* S *such that*

i) $f^{-1}(\alpha) \subseteq A_\alpha$ *for all* $\alpha \in Imf$ *and*
ii) *for all* $\alpha, \beta \in Imf$, *if* $\alpha < \beta$ *then* $A_\beta \subset A_\alpha$ *and* $A_\beta \cap f^{-1}(\alpha) = \emptyset$.

Proof: ($\Rightarrow$) For each $\alpha \in Imf$, we choose $A_\alpha = U(f : \alpha)$. By Proposition II.6 (iv), Proposition II.7 and Theorem III.1, we obtain that $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of subsemigroups of S. By Proposition II.6 (ii), we have the conditions $i)$ and $ii)$. Suppose that $\{B_\alpha \mid \alpha \in Imf\}$ is a chain of subsemigroups of S with the conditions $i)$ and $ii)$, and $\alpha \in Imf$.

Let $a \in B_\alpha$. If $f(a) < \alpha$ then by the condition $ii)$, we have $B_\alpha \cap f^{-1}(f(a)) = \emptyset$. Since $a \in f^{-1}(f(a))$, we get $B_\alpha \cap f^{-1}(f(a)) \neq \emptyset$. This is a contradiction. Thus $f(a) \geq \alpha$ and so $a \in U(f : \alpha) = A_\alpha$. Hence $B_\alpha \subseteq A_\alpha$.

Let $a \in A_\alpha$. Then since $A_\alpha = U(f : \alpha)$, we get $f(a) \geq \alpha$. By the conditions $i)$ and $ii)$, we have

$$a \in f^{-1}(f(a)) \subseteq B_{f(a)} \subseteq B_\alpha.$$

Hence $A_\alpha \subseteq B_\alpha$.
Therefore $A_\alpha = B_\alpha$.
($\Leftarrow$) Let $(a,\alpha), (b,\beta) \in [S \times Imf]^f$. Then $f(a) \geq \alpha$, $f(b) \geq \beta$ and $\min\{\alpha,\beta\} \in Imf$. By the conditions $i)$ and $ii)$, we have

$$a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\min\{\alpha,\beta\}}$$

and similarly $b \in A_{\min\{\alpha,\beta\}}$. Since $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of subsemigroups of S, we get $ab \in A_{\min\{\alpha,\beta\}}$. If $f(ab) < \min\{\alpha,\beta\}$, then by the condition $ii)$, we have $A_{\min\{\alpha,\beta\}} \cap f^{-1}(f(ab)) = \emptyset$ which contradicts with $ab \in A_{\min\{\alpha,\beta\}} \cap f^{-1}(f(ab))$. Thus $f(ab) \geq \min\{\alpha,\beta\}$. Hence $(a,\alpha) * (b,\beta) \in [S \times Imf]^f$. Therefore $[S \times Imf]^f$ is a subsemigroup of $S \times Imf$. By Corollary III.2, we have f is a fuzzy subsemigroup of S. ∎

Remark III.7. In the proof of Theorem III.6, the unique chain of subsemigroups of S satisfying the conditions $i)$ and $ii)$ is $\{U(f : \alpha) \mid \alpha \in Imf\}$.

We use the consequence of Theorem III.6 and Remark III.7 to get a form of a fuzzy subsemigroup of a semigroup which its image is finite.

Corollary III.8. *Let* f *be a fuzzy subset of a semigroup* S *and* $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ *such that* $\alpha_1 > \alpha_2 > ... > \alpha_n$. *Then* f *is a fuzzy subsemigroup of* S *if and only if* $\{U(f : \alpha_i) \mid i \in \{1,2,...,n\}\}$ *is the chain of subsemigroups of* S *such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in U(f:\alpha_n)\backslash U(f:\alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in U(f:\alpha_{n-1})\backslash U(f:\alpha_{n-2}) \\ \vdots & \\ \alpha_2 & \text{if } x \in U(f:\alpha_2)\backslash U(f:\alpha_1) \\ \alpha_1 & \text{if } x \in U(f:\alpha_1) \end{cases}$$

for all $x \in S$.

Corollary III.9. *Let* f *be a fuzzy subset of a semigroup* S *and* $Imf \subseteq \Delta \subseteq [0,1]$. *The following statements are equivalent.*

(i) f *is a fuzzy subsemigroup of* S.
(ii) *There exists a subsemigroup* $\mathcal{R}$ *of* $S \times \Delta$ *such that* $U(\mathcal{R} : \alpha) = U(f : \alpha)$ *for all* $\alpha \in \Delta$.
(iii) $U(f : \alpha)(\neq \emptyset)$ *is a subsemigroup of* S *for all* $\alpha \in \Delta$.

Proof: $((i) \Rightarrow (ii))$ Choose $\mathcal{R} = [S \times \Delta]^f$ and use Theorem III.1 and Proposition II.6 (iv).
$((ii) \Rightarrow (iii))$ It follows from Proposition II.7.
$((iii) \Rightarrow (i))$ Apply Theorem III.6. ∎

IV. Fuzzy (Generalized) Bi-Ideals

In this section, we characterize fuzzy generalized bi-ideals and fuzzy bi-ideals of a semigroup S by using the certain subsets of S, $[0,1]$, $FP(S)$ and $S \times [0,1]$.

For the following theorem, we investigate characterizations of fuzzy generalized bi-ideals of S via the certain subsets of $[0,1]$ and $S \times [0,1]$.

Theorem IV.1. *Let* f *be a fuzzy subset of a semigroup* S. *The following statements are equivalent.*

(i) f *is a fuzzy generalized bi-ideal of* S.
(ii) *For every generalized bi-ideal* A *of* S *and* $\Delta \subseteq [0,1]$, $[A \times \Delta]^f(\neq \emptyset)$ *is a generalized bi-ideal of* $S \times \Delta$.
(iii) $[S \times \Delta]^f$ *is a generalized bi-ideal of* $S \times \Delta$ *where* $Imf \subseteq \Delta \subseteq [0,1]$.

(*iv*) $(Imf)_{f(axb)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}$ *for all* $a, b, x \in S$.

Proof: ($(i) \Rightarrow (ii)$) Let A be a generalized bi-ideal of S, $\Delta \subseteq [0,1]$, $(x,\gamma) \in S \times \Delta$ and $(a,\alpha),(b,\beta) \in [A \times \Delta]^f$. Then $\min\{\alpha,\beta,\gamma\} \in \Delta$ and

$$\min\{f(a), f(b)\} \geq \min\{\alpha,\beta\} \geq \min\{\alpha,\beta,\gamma\}.$$

Since f is a fuzzy generalized bi-ideal of S and A is a generalized bi-ideal of S, we get $axb \in A$ and

$$f(axb) \geq \min\{f(a), f(b)\} \geq \min\{\alpha,\beta,\gamma\}.$$

Thus $(a,\alpha) * (x,\gamma) * (b,\beta) \in [A \times \Delta]^f$.
Hence $[A \times \Delta]^f$ is a generalized bi-ideal of $S \times \Delta$.
($(ii) \Rightarrow (iii)$) It is clear.
($(iii) \Rightarrow (i)$) Let $Imf \subseteq \Delta \subseteq [0,1]$ and $a, b, x \in S$. Then $(x, f(a)) \in S \times \Delta$ and $(a, f(a)), (b, f(b)) \in [S \times \Delta]^f$. By assumption (iii), we have $(a, f(a)) * (x, f(a)) * (b, f(b)) \in [S \times \Delta]^f$. Thus $f(axb) \geq \min\{f(a), f(b)\}$.
($(i) \Rightarrow (iv)$) Let $a, b, x \in S$ and $\alpha \in (Imf)_{f(axb)}$. Then $\alpha \geq f(axb)$. By assumption (i), we have $\alpha \geq f(axb) \geq \min\{f(a), f(b)\}$. Thus $\alpha \in (Imf)_{f(a)} \cup (Imf)_{f(b)}$. Hence

$$(Imf)_{f(axb)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}.$$

($(iv) \Rightarrow (i)$) It is straightforward. ■

By Theorem III.1 and Theorem IV.1, we immediately have the following theorem.

Theorem IV.2. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(*i*) *f is a fuzzy bi-ideal of S.*
(*ii*) *For every bi-ideal A of S and $\Delta \subseteq [0,1]$, $[A \times \Delta]^f (\neq \emptyset)$ is a bi-ideal of $S \times \Delta$.*
(*iii*) *$[S \times \Delta]^f$ is a bi-ideal of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0,1]$.*
(*iv*) *$(Imf)_{f(axb)} \cup (Imf)_{f(ab)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}$ for all $a, b, x \in S$.*

By using and applying Theorem IV.1 and Theorem IV.2, we have Corollary IV.3.

Corollary IV.3. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(*i*) *f is a fuzzy (generalized) bi-ideal of S.*
(*ii*) *$[S \times (0,1]]^f (\neq \emptyset)$ is a (generalized) bi-ideal of $S \times (0,1]$.*
(*iii*) *$[S \times Imf]^f$ is a (generalized) bi-ideal of $S \times Imf$.*
(*iv*) *$[S \times [0,1]]^f$ is a (generalized) bi-ideal of $S \times [0,1]$.*

Example IV.4. *Let $S = \{a, b, c, d\}$ be a semigroup under the same binary operation in Example III.3.*

(*i*) *Let f be a fuzzy subset of S such that $f(a) = 0.7, f(b) = 0.5, f(c) = 0.6, f(d) = 0.4$. By routine calculations, we can check that*

$$\begin{aligned}[S \times Imf]^f = \{&(a,0.7),(a,0.6),(a,0.5),\\ &(a,0.4),(b,0.5),(b,0.4),(c,0.6),\\ &(c,0.5),(c,0.4),(d,0.4)\}\end{aligned}$$

is a generalized bi-ideal of $S \times Imf$, but it is not a bi-ideal of $S \times Imf$ because

$$(c,0.6) * (c,0.6) = (b,0.6) \notin [S \times Imf]^f,$$

that is, $[S \times Imf]^f$ is not a subsemigroup of $S \times Imf$. By Corollary IV.3, we get f is a fuzzy generalized bi-ideal of S but not a fuzzy bi-ideal of S.

(*ii*) *Let g be a fuzzy subset of S such that $g(a) = 0.9, g(b) = g(c) = 0.3, g(d) = 0.1$. Thus $[S \times Img]^g = \{(a,0.9), (a,0.3), (a,0.1), (b,0.3), (b,0.1), (c,0.3), (c,0.1), (d,0.1)\}$ is a bi-ideal of $S \times Img$. By Corollary IV.3, we have g is a fuzzy bi-ideal of S.*

By Theorem III.1 and Theorem IV.1, the following corollary holds:

Corollary IV.5. *Let f be a fuzzy bi-ideal of a semigroup S, $A \subseteq S$ and $\Delta \subseteq [0,1]$ such that $[A \times \Delta]^f \neq \emptyset$. Then the following statements hold.*

(*i*) *If A is a subsemigroup of S, then $[A \times \Delta]^f$ is a subsemigroup of $S \times \Delta$.*
(*ii*) *If A is a generalized bi-ideal of S, then $[A \times \Delta]^f$ is a generalized bi-ideal of $S \times \Delta$.*

Next, we give a relation between a (generalized) bi-ideal $[S \times (0,1]]^f$ of $S \times (0,1]$ and a (generalized) bi-ideal $\underline{f}$ of $FP(S)$ in the following proposition. Its proof is straightforward and omitted.

Proposition IV.6. *Let f be a fuzzy subset of a semigroup S. Then $[S \times (0,1]]^f$ is a (generalized) bi-ideal of $S \times (0,1]$ if and only if $\underline{f}$ is a (generalized) bi-ideal of $FP(S)$.*

Corollary IV.7. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy (generalized) bi-ideal of S if and only if $\underline{f}(\neq \emptyset)$ is a (generalized) bi-ideal of $FP(S)$.*

In the following theorem, we characterize a fuzzy generalized bi-ideal of a semigroup S by the chain of generalized bi-ideals of S.

Theorem IV.8. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy generalized bi-ideal of S if and only if there exists the unique chain $\{A_\alpha \mid \alpha \in Imf\}$ of generalized bi-ideals of S such that*

i) *$f^{-1}(\alpha) \subseteq A_\alpha$ for all $\alpha \in Imf$ and*
ii) *for all $\alpha, \beta \in Imf$, if $\alpha < \beta$ then $A_\beta \subset A_\alpha$ and $A_\beta \cap f^{-1}(\alpha) = \emptyset$.*

Proof: ($\Rightarrow$) Choose $A_\alpha = U(f : \alpha)$ for all $\alpha \in Imf$. By Proposition II.6 (iv), Proposition II.8 and Theorem IV.1, we get that $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of generalized bi-ideals of S satisfying the conditions i) and ii). For the proof of uniqueness, it is similar to the proof of uniqueness of Theorem III.6.
($\Leftarrow$) Let $(a,\alpha),(b,\beta) \in [S \times Imf]^f$ and $(x,\gamma) \in S \times Imf$. Then $\min\{\alpha,\beta,\gamma\} \in Imf$ and

$$\min\{f(a), f(b)\} \geq \min\{\alpha,\beta\} \geq \min\{\alpha,\beta,\gamma\}.$$

By the conditions i) and ii), we have

$$a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\min\{\alpha,\beta,\gamma\}}.$$

Similarly, we have $b \in A_{\min\{\alpha,\beta,\gamma\}}$. Since $A_{\min\{\alpha,\beta,\gamma\}}$ is a generalized bi-ideal of S, we have $axb \in A_{\min\{\alpha,\beta,\gamma\}}$. If $f(axb) < \min\{\alpha,\beta,\gamma\}$, then by the condition ii), we have $A_{\min\{\alpha,\beta,\gamma\}} \cap f^{-1}(f(axb)) = \emptyset$ which contradicts with $axb \in A_{\min\{\alpha,\beta,\gamma\}} \cap f^{-1}(f(axb))$. Thus $f(axb) \geq \min\{\alpha, \beta, \gamma\}$. Hence $(a, \alpha) * (x, \gamma) * (b, \beta) \in [S \times Imf]^f$. Therefore $[S \times Imf]^f$ is a generalized bi-ideal of $S \times Imf$. By Corollary IV.3, we get f is a fuzzy generalized bi-ideal of S. ■

In the following corollary, we show a form of a fuzzy generalized bi-ideal f of a semigroup where Imf is finite.

Corollary IV.9. *Let f be a fuzzy subset of a semigroup S and $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 > \alpha_2 > ... > \alpha_n$. Then f is a fuzzy generalized bi-ideal of S if and only if $\{U(f : \alpha_i) \mid i \in \{1, 2, ..., n\}\}$ is the chain of generalized bi-ideals of S such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in U(f:\alpha_n)\backslash U(f:\alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in U(f:\alpha_{n-1})\backslash U(f:\alpha_{n-2}) \\ \vdots & \\ \alpha_2 & \text{if } x \in U(f:\alpha_2)\backslash U(f:\alpha_1) \\ \alpha_1 & \text{if } x \in U(f:\alpha_1) \end{cases}$$

for all $x \in S$.

Corollary IV.10. *Let f be a fuzzy subset of a semigroup S and $Imf \subseteq \Delta \subseteq [0, 1]$. The following statements are equivalent.*

- (i) *f is a fuzzy generalized bi-ideal of S.*
- (ii) *There exists a generalized bi-ideal $\mathcal{R}$ of $S \times \Delta$ such that $U(\mathcal{R} : \alpha) = U(f : \alpha)$ for all $\alpha \in \Delta$.*
- (iii) *$U(f : \alpha)(\neq \emptyset)$ is a generalized bi-ideal of S for all $\alpha \in \Delta$.*

Proof: $((i) \Rightarrow (ii))$ Choose $\mathcal{R} = [S \times \Delta]^f$ and use Theorem IV.1 and Proposition II.6 (iv).
$((ii) \Rightarrow (iii))$ It follows from Proposition II.8.
$((iii) \Rightarrow (i))$ Apply Theorem IV.8. ■

In the following three results, we characterize fuzzy bi-ideals of semigroups.

Theorem IV.11. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy bi-ideal of S if and only if there exists the unique chain $\{A_\alpha \mid \alpha \in Imf\}$ of bi-ideals of S such that*

- *i)* *$f^{-1}(\alpha) \subseteq A_\alpha$ for all $\alpha \in Imf$ and*
- *ii)* *for all $\alpha, \beta \in Imf$, if $\alpha < \beta$ then $A_\beta \subset A_\alpha$ and $A_\beta \cap f^{-1}(\alpha) = \emptyset$.*

Proof: It follows from Theorem III.6 and Theorem IV.8. ■

Corollary IV.12. *Let f be a fuzzy subset of a semigroup S and $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 > \alpha_2 > ... > \alpha_n$. Then f is a fuzzy bi-ideal of S if and only if $\{U(f : \alpha_i) \mid i \in \{1, 2, ..., n\}\}$ is the chain of bi-ideals of S such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in U(f:\alpha_n)\backslash U(f:\alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in U(f:\alpha_{n-1})\backslash U(f:\alpha_{n-2}) \\ \vdots & \\ \alpha_2 & \text{if } x \in U(f:\alpha_2)\backslash U(f:\alpha_1) \\ \alpha_1 & \text{if } x \in U(f:\alpha_1) \end{cases}$$

for all $x \in S$.

Proof: It follows from Corollary III.8 and Corollary IV.9. ■

Corollary IV.13. *Let f be a fuzzy subset of a semigroup S and $Imf \subseteq \Delta \subseteq [0, 1]$. The following statements are equivalent.*

- (i) *f is a fuzzy bi-ideal of S.*
- (ii) *There exists a bi-ideal $\mathcal{R}$ of $S \times \Delta$ such that $U(\mathcal{R} : \alpha) = U(f : \alpha)$ for all $\alpha \in \Delta$.*
- (iii) *$U(f : \alpha)(\neq \emptyset)$ is a bi-ideal of S for all $\alpha \in \Delta$.*

Proof: It follows from Corollary III.9 and Corollary IV.10. ■

V. Fuzzy Quasi-Ideals of Semigroups

In this section, characterizations of fuzzy quasi-ideals of a semigroup S are studied by using the certain subsets of S, $[0, 1]$, $FP(S)$ and $S \times [0, 1]$.

In Theorem V.1, we characterize fuzzy quasi-ideals of S via the certain subsets of $[0, 1]$ and $S \times [0, 1]$.

Theorem V.1. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

- (i) *f is a fuzzy quasi-ideal of S.*
- (ii) *For every quasi-ideal A of S and $\Delta \subseteq [0, 1]$, $[A \times \Delta]^f(\neq \emptyset)$ is a quasi-ideal of $S \times \Delta$.*
- (iii) *$[S \times \Delta]^f$ is a quasi-ideal of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0, 1]$.*
- (iv) *For all $a \in S$ such that $F_a \neq \emptyset$, we have*

$$(Imf)_{f(a)} \subseteq \bigcap_{(x,y)\in F_a} (Imf)_{f(x)}$$

or

$$(Imf)_{f(a)} \subseteq \bigcap_{(x,y)\in F_a} (Imf)_{f(y)}.$$

Proof: $((i) \Rightarrow (ii))$ Let A be a quasi-ideal of S, $\Delta \subseteq [0, 1]$ and

$$(a, \alpha) \in (S \times \Delta * [A \times \Delta]^f) \cap ([A \times \Delta]^f * S \times \Delta).$$

Then there exist $(x_1, \beta_1), (x_2, \beta_2) \in S \times \Delta$ and (a_1, γ_1), $(a_2, \gamma_2) \in [A \times \Delta]^f$ such that

$$(a, \alpha) = (x_1, \beta_1) * (a_1, \gamma_1) = (a_2, \beta_2) * (x_2, \gamma_2).$$

Thus $f(a_1) \geq \gamma_1 \geq \min\{\gamma_1, \beta_1\} = \alpha$ and $f(a_2) \geq \gamma_2 \geq \min\{\gamma_2, \beta_2\} = \alpha$. Since A is a quasi-ideal of S, we have $a \in A$. Consider

$$\begin{aligned} (\chi_S \circ f)(a) &= \sup\{\min\{\chi_S(x), f(y)\} \mid (x, y) \in F_a\} \\ &\geq \min\{\chi_S(x_1), f(a_1)\} \\ &= f(a_1) \\ &\geq \alpha. \end{aligned}$$

Similarly, $(f \circ \chi_S)(a) \geq \min\{f(a_2), \chi_S(x_2)\} \geq \alpha$.
By assumption (i), we get

$$f(a) \quad \geq \quad ((\chi_S \circ f) \wedge (f \circ \chi_S))(a) \geq \alpha.$$

Hence $(a, \alpha) \in [A \times \Delta]^f$. Therefore $[A \times \Delta]^f$ is a quasi-ideal of $S \times \Delta$.
$((ii) \Rightarrow (iii))$ It is clear.
$((iii) \Rightarrow (i))$ Let $Imf \subseteq \Delta \subseteq [0, 1]$. Suppose that $((\chi_S \circ f) \wedge (f \circ \chi_S))(a) > f(a)$ for some $a \in S$. Thus

$$\sup\{f(x) \mid (x, y) \in F_a\} > f(a),$$
$$\sup\{f(y) \mid (x, y) \in F_a\} > f(a).$$

Hence $f(x_1) > f(a)$ and $f(y_2) > f(a)$ for some (x_1, y_1), $(x_2, y_2) \in F_a$. Clearly, $(x_1, f(x_1)), (y_2, f(y_2)) \in [S \times \Delta]^f$ and $(y_1, f(y_2)), (x_2, f(x_1)) \in S \times \Delta$. Then,

$$\begin{aligned}(a, \min\{f(x_1), f(y_2)\}) &= (x_1, f(x_1)) * (y_1, f(y_2)) \\ &= (x_2, f(x_1)) * (y_2, f(y_2)).\end{aligned}$$

By assumption (iii), we have

$$(a, \min\{f(x_1), f(y_2)\}) \in [S \times \Delta]^f.$$

Therefore $f(a) \geq f(x_1)$ or $f(a) \geq f(y_2)$. It is a contradiction. Hence $((\chi_S \circ f) \wedge (f \circ \chi_S))(a) \leq f(a)$ for all $a \in S$, that is f is a fuzzy quasi-ideal of S.
$((i) \Rightarrow (iv))$ Let $a \in S$ and $F_a \neq \emptyset$. Then for all $(x, y) \in F_a$,

$$\begin{aligned}(f \circ \chi_S)(a) &= \sup\{\min\{f(x), \chi_S(y)\} \mid (x, y) \in F_a\} \\ &\geq f(x).\end{aligned}$$

Similarly, we have that $(\chi_S \circ f)(a) \geq f(y)$ for all $(x, y) \in F_a$. By assumption (i), we have

$$f(a) \geq \min\{(f \circ \chi_S)(a), (\chi_S \circ f)(a)\}.$$

Consider the following cases:

Case 1: $f(a) \geq (f \circ \chi_S)(a)$. Then $f(a) \geq f(x)$ for all $(x, y) \in F_a$. Thus $(Imf)_{f(a)} \subseteq (Imf)_{f(x)}$ for all $(x, y) \in F_a$. Hence $(Imf)_{f(a)} \subseteq \bigcap\limits_{(x,y) \in F_a} (Imf)_{f(x)}$.

Case 2: $f(a) \geq (\chi_S \circ f)(a)$. Its proof is similar to the proof of Case 1.
Therefore, $(Imf)_{f(a)} \subseteq \bigcap\limits_{(x,y) \in F_a} (Imf)_{f(y)}$.
$((iv) \Rightarrow (i))$ It is straightforward. ■

By Theorem V.1, we get Corollary V.2.

Corollary V.2. *Let f be a fuzzy subset of a semigroup S. Then the following statements are equivalent.*

(i) *f is a fuzzy quasi-ideal of S.*
(ii) *$[S \times (0, 1]]^f (\neq \emptyset)$ is a quasi-ideal of $S \times (0, 1]$.*
(iii) *$[S \times Imf]^f$ is a quasi-ideal of $S \times Imf$.*
(iv) *$[S \times [0, 1]]^f$ is a quasi-ideal of $S \times [0, 1]$.*

Example V.3. *Let $S = \{0, a, b, c\}$ be a semigroup with the following multiplication table:*

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

Let f be a fuzzy subset of S such that

$$f(0) = f(a) = 0.8, \quad f(b) = f(c) = 0.3.$$

Thus $[S \times Imf]^f = \{(0, 0.8), (a, 0.8), (0, 0.3), (a, 0.3), (b, 0.3), (c, 0.3)\}$ is a quasi-ideal of $S \times Imf$. By Corollary V.2, we get f is a fuzzy quasi-ideal of S.

Proposition V.4. *Let f be a fuzzy subset of a semigroup S. Then $[S \times (0, 1]]^f$ is a quasi-ideal of $S \times (0, 1]$ if and only if $\underline{f}$ is a quasi-ideal of $FP(S)$.*

Proof: It is straightforward. ■

Corollary V.5. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy quasi-ideal of S if and only if $\underline{f}(\neq \emptyset)$ is a quasi-ideal of $FP(S)$.*

Proof: It follows from Corollary V.2 and Proposition V.4. ■

In Theorem V.6, we give a characterization of a fuzzy quasi-ideal of a semigroup S by the chain of quasi-ideals of S.

Theorem V.6. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy quasi-ideal of S if and only if there exists the unique chain $\{A_\alpha \mid \alpha \in Imf\}$ of quasi-ideals of S such that*

i) *$f^{-1}(\alpha) \subseteq A_\alpha$ for all $\alpha \in Imf$ and*
ii) *for all $\alpha, \beta \in Imf$, if $\alpha < \beta$ then $A_\beta \subset A_\alpha$ and $A_\beta \cap f^{-1}(\alpha) = \emptyset$.*

Proof: $(\Rightarrow)$ Choose $A_\alpha = U(f : \alpha)$ for all $\alpha \in Imf$. By Proposition II.10, Proposition II.6 (iv) and Theorem V.1, we get $\{A_\alpha \mid \alpha \in Imf\}$ is a chain of quasi-ideals of S satisfying the conditions $i)$ and $ii)$. For the proof of uniqueness, it is similar to the proof of uniqueness of Theorem III.6.
$(\Leftarrow)$ Let $(a, \alpha) \in (S \times Imf * [S \times Imf]^f) \cap ([S \times Imf]^f * S \times Imf)$. Then

$$(a, \alpha) = (x_1, \beta_1) * (a_1, \gamma_1) = (a_2, \beta_2) * (x_2, \gamma_2)$$

for some $(a_1, \gamma_1), (a_2, \gamma_2) \in [S \times Imf]^f$ and (x_1, β_1), $(x_2, \beta_2) \in S \times Imf$. Thus

$$f(x_1) \geq \gamma_1 \geq \alpha \text{ and } f(x_2) \geq \gamma_2 \geq \alpha.$$

By the conditions $i)$ and $ii)$, we see that

$$x_1 \in f^{-1}(f(x_1)) \subseteq A_{f(x_1)} \subseteq A_\alpha$$

and similarly $x_2 \in A_\alpha$. Since A_α is a quasi-ideal of S, we have $a \in A_\alpha$. Thus $f(a) \geq \alpha$. Indeed, if $f(a) < \alpha$ then by the condition $ii)$, we get $A_\alpha \cap f^{-1}(f(a)) = \emptyset$ which is a contradiction with $a \in A_\alpha \cap f^{-1}(f(a))$. Therefore $(a, \alpha) \in [S \times Imf]^f$. Consequently, $[S \times Imf]^f$ is a quasi-ideal of $S \times Imf$. By Corollary V.2, we have f is a fuzzy quasi-ideal of S. ■

In the proof of Theorem V.6, the unique chain of quasi-ideals of S satisfying the conditions $i)$ and $ii)$ is $\{U(f : \alpha) \mid \alpha \in Imf\}$. Then we get a form of a fuzzy quasi-ideal of S which its image is finite.

Corollary V.7. *Let f be a fuzzy subset of a semigroup S and $Imf = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_1 > \alpha_2 > ... > \alpha_n$. Then f is a fuzzy quasi-ideal of S if and only if $\{U(f : \alpha_i) \mid i \in \{1, 2, ..., n\}\}$ is the chain of quasi-ideals of S such that*

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in U(f : \alpha_n) \backslash U(f : \alpha_{n-1}) \\ \alpha_{n-1} & \text{if } x \in U(f : \alpha_{n-1}) \backslash U(f : \alpha_{n-2}) \\ \vdots & \\ \alpha_2 & \text{if } x \in U(f : \alpha_2) \backslash U(f : \alpha_1) \\ \alpha_1 & \text{if } x \in U(f : \alpha_1) \end{cases}$$

for all $x \in S$.

Corollary V.8. *Let f be a fuzzy subset of a semigroup S and $Imf \subseteq \Delta \subseteq [0, 1]$. The following statements are equivalent.*

(i) *f is a fuzzy quasi-ideal of S.*
(ii) *There exists a quasi-ideal $\mathcal{R}$ of $S \times \Delta$ such that $U(\mathcal{R} : \alpha) = U(f : \alpha)$ for all $\alpha \in \Delta$.*
(iii) *$U(f : \alpha)(\neq \emptyset)$ is a quasi-ideal of S for all $\alpha \in \Delta$.*

Proof: $((i) \Rightarrow (ii))$ Choose $\mathcal{R} = [S \times \Delta]^f$ and use Theorem V.1 and Proposition II.6 (iv).
$((ii) \Rightarrow (iii))$ It follows from Proposition II.10.
$((iii) \Rightarrow (i))$ Apply Theorem V.6. ■

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