Some Characterizations of Fuzzy Bi-ideals and Fuzzy Quasi-ideals of Semigroups

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Abstract—The aims of this paper are to characterize fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups. We define certain subsets of semigroups $S$, $[0, 1]$ and $S \times [0, 1]$. The relationships between sets of fuzzy points and the certain subsets of the semigroup $S \times [0, 1]$ are discussed. In the main results, characterizations of fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups are investigated by using the certain subsets of semigroups $S$, $[0, 1]$ and $S \times [0, 1]$.

Index Terms—fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy quasi-ideals, semigroups.

I. INTRODUCTION

The fundamental concepts of fuzzy sets have been proposed by Zadeh [24] since 1965. These concepts were applied in many areas such as: medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory. In 1971, the concepts of fuzzy sets were transferred to fuzzy groups by Rosenfeld [19]. The study of fuzzy subsemigroups of semigroups was introduced by Kuroki (see [10]-[15]). He investigated some properties of fuzzy subsemigroups, fuzzy ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals, fuzzy quasi-ideals of semigroups. After that many types of fuzzy algebraic structures have been introduced and investigated.

The fuzzy ideals and bi-ideals of semigroups have been applied in characterizing the duo semigroup, the simple semigroup and semilattices of subsemigroups [11]. He also investigated characterizations of regular semigroups and both intra-regular and left quasi-regular semigroups in terms of fuzzy generalized bi-ideals [13]. Completely regular semigroups and a semilattice of groups are characterized by using fuzzy semiprime quasi-ideals [15]. In [1], Ahsan et al. studied some properties of fuzzy quasi-ideals of semigroups and used their properties to characterize regular and intra-regular semigroups. Shabir et al. [22] introduced certain types of fuzzy bi-ideals, called prime, strongly prime, and semiprime fuzzy bi-ideals, and characterized semigroups in terms of their semiprime and strongly prime fuzzy bi-ideals. Concepts of fuzzy (generalized) bi-ideals and fuzzy quasi-ideals of semigroups play an important role in the study of types of semigroups. Some authors studied similar types of fuzzy subsets of other algebraic structures seen in [3]-[8], [17], [18], [20], [21] and [23].

Our propose of this work is to promote and develop fuzzy semigroup theory and related structures by using fuzzy subsemigroups, fuzzy (generalized) bi-ideals and fuzzy quasi-ideals. We define the certain subsets of $S$, $[0, 1]$ and $S \times [0, 1]$ and investigate their properties. In particular, we define a certain subset $U(R : a)$ of $S$ where $R$ is a subset of $S \times [0, 1]$ and this set is a general concept of the upper level set of a fuzzy set. We also describe relationships between sets of fuzzy points and the certain subsets of $S \times [0, 1]$. In the main results of this paper, we characterize fuzzy subsemigroups, fuzzy generalized bi-ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups by using the certain subsets of $S$, $[0, 1]$ and $S \times [0, 1]$. Moreover, we show that any fuzzy subset of $S$ is a fuzzy bi-ideal (resp., a fuzzy generalized bi-ideal, a fuzzy quasi-ideal) if and only if there exists the unique chain of bi-ideals (resp., generalized bi-ideals, quasi-ideals) of $S$ together with two special conditions.

II. PRELIMINARIES

In this section, we give some definitions, notations and results of semigroups and fuzzy semigroups.

Throughout this paper, $S$ stand for a semigroup unless otherwise specified. A nonempty subset $A$ of $S$ is called a subsemigroup of $S$ if $AA \subseteq A$. A nonempty subset $G$ of $S$ is called a generalized bi-ideal of $S$ if $GSG \subseteq G$. A subsemigroup $B$ of $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$. Then a nonempty subset $B$ of $S$ is a bi-ideal of $S$ if and only if $B$ is both a subsemigroup and generalized bi-ideal of $S$. A nonempty subset $Q$ of $S$ is called a quasi-ideal of $S$ if $QS \cap SQ \subseteq Q$.

A fuzzy subset $f$ [24] of $S$ is described as a function $f : S \rightarrow [0, 1]$ and its image is denoted by $\text{Im} f = \{ f(x) \mid x \in S \}$. The set of all fuzzy subsets of $S$ is denoted by $(S)$ and let $f, g \in (S)$. We define the order relation $\leq$ on $(S)$ as follows $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$. For $x \in S$, define

$$F_x = \{ (y, z) \in S \times S \mid x = yz \}.$$

The fuzzy subsets $f \land g$, $f \lor g$ and $f \circ g$ of $S$ are defined as follows:

$$(f \land g)(x) = \min \{ f(x), g(x) \}$$

and

$$(f \lor g)(x) = \sup \{ \min \{ f(y), g(z) \} \mid (y, z) \in F_x \},$$

if $F_x \neq \emptyset$;

0, otherwise

for all $x \in S$. Then $(S, \circ)$ is a semigroup [16]. For any $\alpha \in (0, 1)$ and $x \in S$, a fuzzy subset $x_\alpha$ of $S$ is called a fuzzy point [9] in $S$ if for all $y \in S$

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

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Then the set of all fuzzy points $FP(S)$ of $S$ is a subsemigroup of $F(S)$ [9].

The following definitions are important types of fuzzy subsystems of semigroups which will be used in this paper.

**Definition II.1.** [10] A fuzzy subset $f$ of $S$ is called a **fuzzy subsemigroup of** $S$ if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

**Definition II.2.** [12] A fuzzy subset $f$ of $S$ is called a **fuzzy generalized bi-ideal of** $S$ if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$.

**Definition II.3.** [10] A fuzzy subsemigroup $f$ of $S$ is called a **fuzzy bi-ideal of** $S$ if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$.

Then a fuzzy subset $f$ of $S$ is a fuzzy bi-ideal of $S$ if and only if $f$ is both a fuzzy generalized bi-ideal and a fuzzy subsemigroup of $S$.

**Definition II.4.** [11] A fuzzy subset $f$ of $S$ is called a **fuzzy quasi-ideal of** $S$ if $(f \circ \chi_S)(\chi_S \circ f) \leq f \leq f \circ \chi_S$ while $\chi_S(x) = 1$ for all $x \in S$.

Define a binary operation “$*$” on $S \times [0, 1]$ by for all $(x, \alpha), (y, \beta) \in S \times [0, 1]$

$$(x, \alpha) * (y, \beta) = (xy, \min\{\alpha, \beta\}).$$

Then $(S \times [0, 1], *)$ is a semigroup [2].

**Remark II.5.** For every subsemigroup $A$ of $S$ and every nonempty subset $\Delta$ of $[0, 1]$, we have $(A \times \Delta, *)$ is a subsemigroup of $(S \times \Delta, *)$. In what follows, let $S \times \Delta$ denote the semigroup $(S \times \Delta, *)$ throughout this paper.

Let $f$ be a fuzzy subset of $S$, $A \subseteq S$, $\alpha \in [0, 1]$, $\Delta \subseteq [0, 1]$ and $R \subseteq S \times [0, 1]$. We define the the certain subsets of $S \times [0, 1]$, $S$ and $[0, 1]$, respectively as follows:

$$(A \times \Delta)^f = \{(x, \alpha) \in A \times \Delta \mid f(x) \geq \alpha\},$$

$$U(R : \alpha) = \{x \in S \mid (x, \beta) \in R \text{ and } \alpha \leq \beta \}$$

for some $\beta \in [0, 1]$, and

$$(Imf)_\alpha = \{\beta \in Imf \mid \beta \geq \alpha\}.$$  

In particular, if $R$ is a fuzzy subset of $S$, then $U(R : \alpha) = \{x \in S \mid R(x) \geq \alpha\}$. Thus $U(R : \alpha)$ is a general concept of the upper level set of a fuzzy set. For $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$, we have $U(R : \alpha) \subseteq U(R : \beta)$ for $R \subseteq R(x)$.

The following propositions are easy to prove.

**Proposition II.6.** Let $f$ be a fuzzy subset of a semigroup $S$. Then

(i) $(Imf)_\alpha \subseteq Imf$ for all $\alpha \in [0, 1]$.

(ii) $U(f : \alpha) = \bigcup_{\gamma \in (Imf)_\alpha} f^{-1}(\gamma)$ for all $\alpha \in [0, 1]$.

(iii) $[S \times \Delta]^f = \bigcup_{\gamma \in \Delta} (U(f : \gamma) \times \{\gamma\})$ for every subset $\Delta \subseteq [0, 1]$.

(iv) If $\Delta \subseteq [0, 1]$ and $R := [S \times \Delta]^f$, then $U(R : \alpha) = U(f : \alpha)$ for all $\alpha \in \Delta$.

**Proposition II.7.** Let $S$ be a semigroup and let $\Delta$ be a nonempty subset of $[0, 1]$. If $R$ is a subsemigroup of $S \times \Delta$ and $\alpha \in \Delta$, then $U(R : \alpha)(\alpha(\neq 0))$ is a subsemigroup of $S$.

**Proposition II.8.** Let $S$ be a semigroup and let $\Delta$ be a nonempty subset of $[0, 1]$. If $R$ is a generalized bi-ideal of $S \times \Delta$ and $\alpha \in \Delta$, then $U(R : \alpha)(\alpha(\neq 0))$ is a generalized bi-ideal of $S$.

**Proposition II.9.** Let $S$ be a semigroup and let $\Delta$ be a nonempty subset of $[0, 1]$. If $R$ is a bi-ideal of $S \times \Delta$ and $\alpha \in \Delta$, then $U(R : \alpha)(\alpha(\neq 0))$ is a bi-ideal of $S$.

**Proposition II.10.** Let $S$ be a semigroup and let $\Delta$ be a nonempty subset of $[0, 1]$. If $R$ is a quasi-ideal of $S \times \Delta$ and $\alpha \in \Delta$, then $U(R : \alpha)(\alpha(\neq 0))$ is a quasi-ideal of $S$.

**III. FUZZY SUBSEMGROUPS**

In this section, we characterize fuzzy subsemigroups of a semigroup $S$ by using the certain subsets of $S$, $[0, 1]$, $FP(S)$ and $S \times [0, 1]$.

For the following theorem, we investigate characterizations of fuzzy subsemigroups of $S$ via the certain subsets of $[0, 1]$ and $S \times [0, 1]$.

**Theorem III.1.** Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are equivalent.

(i) $f$ is a fuzzy subsemigroup of $S$.

(ii) For every subsemigroup $A$ of $S$ and $\Delta \subseteq [0, 1]$, $[A \times \Delta]^f(\Delta(\neq 0))$ is a subsemigroup of $S \times \Delta$.

(iii) $[S \times \Delta]^f$ is a subsemigroup of $S \times \Delta$ where $Imf \subseteq \Delta \subseteq [0, 1]$.

(iv) $(Imf)(a, b) \subseteq (Imf)(a, b) \cup (Imf)(b, b)$ for all $a, b \in S$.

**Proof:** ((i) $\Rightarrow$ (ii)) Let $A$ be a subsemigroup of $S$, $\Delta \subseteq [0, 1]$ and $(a, \alpha), (b, \beta) \in [A \times \Delta]^f$. Then $f(a) \geq \alpha, f(b) \geq \beta$ and $\min\{\alpha, \beta\} \in \Delta$. Since $f$ is a fuzzy subsemigroup of $S$, we have $ab \in A$ and

$$f(ab) \geq \min\{f(a), f(b)\} \geq \min\{\alpha, \beta\}.$$  

Thus $(a, \alpha) * (b, \beta) \in [A \times \Delta]^f$. Hence $[A \times \Delta]^f$ is a subsemigroup of $S \times \Delta$.

((ii) $\Rightarrow$ (iii)) It is clear.

((iii) $\Rightarrow$ (i)) Let $Imf \subseteq \Delta \subseteq [0, 1]$ and $a, b \in S$. Then $(a, f(a)), (b, f(b)) \in [S \times Imf]^f \subseteq [S \times \Delta]^f$. By assumption (iii), we have $[S \times \Delta]^f$ is a subsemigroup of $S \times \Delta$. Thus $(a, f(a)) * (b, f(b)) \in [S \times \Delta]^f$. Hence $f(ab) \geq \min\{f(a), f(b)\}$.

((iv) $\Rightarrow$ (i)) Let $a, b \in S$ and $\alpha \in (Imf)(a, b)$, then $\alpha \geq f(ab)$. By assumption (iv), we have $\alpha \geq f(ab) \geq \min\{f(a), f(b)\}$. Thus $\alpha \geq f(a)$ or $\alpha \geq f(b)$. Hence $\alpha \in (Imf)(a, b)$. Therefore

$$(Imf)(a, b) \subseteq (Imf)(a, b) \cup (Imf)(b, b).$$

((iv) $\Rightarrow$ (i)) It is straightforward.

By applying Theorem III.1, we have Corollary III.2.

**Corollary III.2.** Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are equivalent.

(i) $f$ is a fuzzy subsemigroup of $S$.

(ii) $[S \times (0, 1)]^f(\neq 0)$ is a subsemigroup of $S \times (0, 1)$.

(iii) $[S \times Imf]^f$ is a subsemigroup of $S \times Imf$.

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(iv) \( [S \times [0, 1]]^f \) is a subsemigroup of \( S \times [0, 1] \).

**Example III.3.** Let \( S = \{a, b, c, d\} \) and define a binary operation \(*_S^f\) on \( S \) as follows:

\[
\begin{array}{c|cccc}
  & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & a \\
c & a & a & b & a \\
d & a & a & b & b \\
\end{array}
\]

Then \((S, \cdot)\) is a semigroup (see [16]). Let \( f \) be a fuzzy subset of \( S \) such that

\[
f(a) = f(b) = 0.8, \quad f(c) = 0.4, \quad f(d) = 0.3.
\]

Thus, by routine calculations, we can check that \([S \times Imf]^f = \{(a, 0.8), (b, 0.8), (a, 0.4), (b, 0.4), (c, 0.4), (a, 0.3), (b, 0.3), (c, 0.3), (d, 0.3)\}\) is a subsemigroup of \( S \times Imf \).

By Corollary III.2, we have that \( f \) is a fuzzy subsemigroup of \( S \).

Next, we show a relation between the sets \([S \times (0, 1)]^f\) and \( f := \{x_\alpha \in FP(S) \mid f(x) \geq \alpha\} \) in Proposition III.4 whose proof is straightforward and omitted.

**Proposition III.4.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \([S \times (0, 1)]^f\) is a subsemigroup of \([S \times (0, 1)] f\) if and only if \( f \) is a subsemigroup of \( FP(S) \).

By Corollary III.2 and Proposition III.4, we immediately get Corollary III.5.

**Corollary III.5.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \( f \) is a fuzzy subsemigroup of \( S \) if and only if \( f(\emptyset) \neq \emptyset \) is a subsemigroup of \( FP(S) \).

In the following result, an equivalent condition for any fuzzy subsemigroup of a semigroup \( S \) is discussed via the chain of subsemigroups of \( S \).

**Theorem III.6.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \( f \) is a fuzzy subsemigroup of \( S \) if and only if there exists the unique chain \( \{A_\alpha \mid \alpha \in Imf\} \) of subsemigroups of \( S \) such that

(i) \( f^{-1}(\alpha) \subseteq A_\alpha \) for all \( \alpha \in Imf \) and

(ii) for all \( \alpha, \beta \in Imf \), if \( \alpha < \beta \) then \( A_\beta \subseteq A_\alpha \) and \( A_\beta \cap f^{-1}(\alpha) = \emptyset \).

**Proof:** (\( \Rightarrow \)) For each \( \alpha \in Imf \), we choose \( A_\alpha = U(f : \alpha) \). By Proposition II.6 (ii), Proposition II.7 and Theorem III.1, we obtain that \( \{A_\alpha \mid \alpha \in Imf\} \) is a chain of subsemigroups of \( S \). By Proposition II.6 (ii), we have the conditions \( i \) and \( ii \). Suppose that \( \{B_\alpha \mid \alpha \in Imf\} \) is a chain of subsemigroups of \( S \) with the conditions \( i \) and \( ii \), and \( \alpha \in Imf \).

Let \( a \in B_\alpha \). If \( f(a) < \alpha \) then by the condition \( ii \), we have \( B_\alpha \cap f^{-1}(f(a)) = \emptyset \). Since \( a \in f^{-1}(f(a)) \), we get \( B_\alpha \cap f^{-1}(f(a)) = \emptyset \). This is a contradiction. Thus \( f(a) \geq \alpha \) and so \( a \in U(f : \alpha) = A_\alpha \). Hence \( B_\alpha \subseteq A_\alpha \).

Let \( a \in A_\alpha \). Then since \( A_\alpha = U(f : \alpha) \), we get \( f(a) \geq \alpha \). By the conditions \( i \) and \( ii \), we have

\[
a \in f^{-1}(f(a)) \subseteq B_{f(a)} \subseteq B_\alpha.
\]

Hence \( A_\alpha \subseteq B_\alpha \).

Therefore \( A_\alpha = B_\alpha \).

(\( \Leftarrow \)) Let \( (a, \alpha), (b, \beta) \in [S \times Imf]^f \). Then \( f(a) \geq \alpha, f(b) \geq \beta \) and \( \min\{\alpha, \beta\} \in Imf \). By the conditions \( i \) and \( ii \), we have

\[
a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\min\{\alpha, \beta\}}
\]

and similarly \( b \in A_{\min\{\alpha, \beta\}} \). Since \( \{A_\alpha \mid \alpha \in Imf\} \) is a chain of subsemigroups of \( S \), we get \( ab \in A_{\min\{\alpha, \beta\}} \). If \( f(ab) < \min\{\alpha, \beta\} \), then by the condition \( ii \), we have \( A_{\min\{\alpha, \beta\}} \cap f^{-1}(f(ab)) = \emptyset \) which contradicts with \( ab \in A_{\min\{\alpha, \beta\}} \). Thus \( f(ab) \geq \min\{\alpha, \beta\} \).

Hence \((a, \alpha) \ast (b, \beta) \in [S \times Imf]^f \). Therefore \([S \times Imf]^f \) is a subsemigroup of \( S \times Imf \). By Corollary III.2, we have \( f \) is a fuzzy subsemigroup of \( S \).

**Remark III.7.** In the proof of Theorem III.6, the unique chain of subsemigroups of \( S \) satisfying the conditions \( i \) and \( ii \) is \( \{U(f : \alpha) \mid \alpha \in Imf\} \).

We use the consequence of Theorem III.6 and Remark III.7 to get a form of a fuzzy subsemigroup of a semigroup which its image is finite.

**Corollary III.8.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( Imf = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) such that \( \alpha_1 > \alpha_2 > \ldots > \alpha_n \). Then \( f \) is a fuzzy subsemigroup of \( S \) if and only if \( \{U(f : \alpha) \mid \alpha \in \{1, 2, \ldots, n\}\} \) is the chain of subsemigroups of \( S \) such that

\[
f(x) = \begin{cases} 
\alpha_n & \text{if } x \in U(f : \alpha_n) \setminus U(f : \alpha_{n-1}) \\
\alpha_{n-1} & \text{if } x \in U(f : \alpha_{n-1}) \setminus U(f : \alpha_{n-2}) \\
\vdots \\
\alpha_2 & \text{if } x \in U(f : \alpha_2) \setminus U(f : \alpha_1) \\
\alpha_1 & \text{if } x \in U(f : \alpha_1)
\end{cases}
\]

for all \( x \in S \).

**Corollary III.9.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( Imf \subseteq \Delta \subseteq [0, 1] \). The following statements are equivalent.

(i) \( f \) is a fuzzy subsemigroup of \( S \).

(ii) There exists a subsemigroup \( R \) of \( S \times \Delta \) such that \( U(R : \alpha) = U(f : \alpha) \) for all \( \alpha \in \Delta \).

(iii) \( U(f : \alpha)(\not= \emptyset) \) is a subsemigroup of \( S \) for all \( \alpha \in \Delta \).

**Proof:** (\( \Rightarrow \)) Choose \( R = [S \times \Delta]^f \) and use Theorem III.1 and Proposition II.6 (iv).

(\( \Rightarrow \)) (\( \Rightarrow \)) It follows from Proposition II.7.

(\( \Rightarrow \)) (\( \Rightarrow \)) Apply Theorem III.6.

**IV. FUZZY (GENERALIZED) BI-IDEALS**

In this section, we characterize fuzzy generalized bi-ideals and fuzzy bi-ideals of a semigroup \( S \) by using the certain subsets of \( S, [0, 1], FP(S) \) and \( S \times [0, 1] \).

For the following theorem, we investigate characterizations of fuzzy generalized bi-ideals of \( S \) via the certain subsets of \( [0, 1] \) and \( S \times [0, 1] \).

**Theorem IV.1.** Let \( f \) be a fuzzy subset of a semigroup \( S \). The following statements are equivalent.

(i) \( f \) is a fuzzy generalized bi-ideal of \( S \).

(ii) For every generalized bi-ideal \( A \) of \( S \) and \( \Delta \subseteq [0, 1], [A \times \Delta]^f(\not= \emptyset) \) is a generalized bi-ideal of \( S \times \Delta \).

(iii) \([S \times \Delta]^f \) is a generalized bi-ideal of \( S \times \Delta \) where \( Imf \subseteq \Delta \subseteq [0, 1] \).

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(iv) \((Imf)_{f(axb)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}\) for all \(a, b, x \in S\).

Proof: ((i) \Rightarrow (ii)) Let \(A\) be a generalized bi-ideal of \(S, \Delta \subseteq [0, 1], (x, (x, y)) \in S \times \Delta\) and \((a, \alpha), (b, \beta) \in [A \times \Delta]^f\). Then \(\min\{\alpha, \beta, \gamma\} \in \Delta\) and

\[
\min\{f(a), f(b)\} \geq \min\{\alpha, \beta\} \geq \min\{\alpha, \beta, \gamma\}.
\]

Since \(f\) is a fuzzy generalized bi-ideal of \(S\) and \(A\) is a generalized bi-ideal of \(S\), we get \(axb \in A\) and

\[
f(axb) \geq \min\{f(a), f(b)\} \geq \min\{\alpha, \beta, \gamma\}.
\]

Thus \((a, \alpha) \ast (x, \gamma) \ast (b, \beta) \in [A \times \Delta]^f\). Hence \([A \times \Delta]^f\) is a generalized bi-ideal of \(S \times \Delta\).

((iii) \Rightarrow (ii)) It is clear.

((iv) \Rightarrow (i)) Let \(Imf \subseteq \Delta \subseteq [0, 1]\) and \(a, b, x \in S\). Then \((x, f(a)) \in S \times \Delta\) and \((a, f(a)), (b, f(b)) \in [S \times \Delta]^f\). By assumption (iii), we have \((a, f(a)) \ast (x, f(a)) \ast (b, f(b)) \in [S \times \Delta]^f\). Thus \(f(axb) \geq \min\{f(a), f(b)\}\).

((i) \Rightarrow (iv)) Let \(a, b, x \in S\) and \(\alpha \in (Imf)_{f(axb)}\). Then \(\alpha \geq f(axb)\). By assumption (i), we have \(\alpha \geq \min\{f(a), f(b)\}\). Thus \(\alpha \in (Imf)_{f(a)} \cup (Imf)_{f(b)}\) Hence

\[
(Imf)_{f(axb)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}.
\]

((iv) \Rightarrow (i)) It is straightforward.

By Theorem III.1 and Theorem IV.1, we immediately have the following theorem.

**Theorem IV.2.** Let \(f\) be a fuzzy subset of a semigroup \(S\). Then the following statements are equivalent.

(i) \(f\) is a fuzzy bi-ideal of \(S\).

(ii) For every bi-ideal \(A\) of \(S\) and \(\Delta \subseteq [0, 1]\), \([A \times \Delta]^f\) is a bi-ideal of \(S \times \Delta\).

(iii) \([S \times \Delta]^f\) is a bi-ideal of \(S \times \Delta\) where \(Imf \subseteq \Delta \subseteq [0, 1]\).

(iv) \((Imf)_{f(axb)} \subseteq (Imf)_{f(a)} \cup (Imf)_{f(b)}\) for all \(a, b, x \in S\).

By using and applying Theorem IV.1 and Theorem IV.2, we have Corollary IV.3.

**Corollary IV.3.** Let \(f\) be a fuzzy subset of a semigroup \(S\). Then the following statements are equivalent.

(i) \(f\) is a fuzzy (generalized) bi-ideal of \(S\).

(ii) \([S \times (0, 1)]^f\) is a (generalized) bi-ideal of \(S \times (0, 1]\).

(iii) \([S \times Imf]^f\) is a (generalized) bi-ideal of \(S \times Imf\).

(iv) \([S \times 0, 1]^f\) is a (generalized) bi-ideal of \(S \times 0, 1]\).

**Example IV.4.** Let \(S = \{a, b, c, d\}\) be a semigroup under the same binary operation in Example III.3.

(i) Let \(f\) be a fuzzy subset of \(S\) such that \(f(a) = 0.7, f(b) = 0.5, f(c) = 0.6, f(d) = 0.4\).

By routine calculations, we can check that

\[
[S \times Imf]^f = \{(a, 0.7), (a, 0.6), (a, 0.5), (a, 0.4), (b, 0.5), (b, 0.4), (c, 0.6), (c, 0.5), (c, 0.4), (d, 0.4)\}
\]

is a generalized bi-ideal of \(S \times Imf\) but it is not a bi-ideal of \(S \times Imf\) because

\((c, 0.6) \ast (c, 0.6) = (b, 0.6) \notin [S \times Imf]^f\),

\[f^{-1}(a) \subseteq A_{\alpha}, \text{for all } a \in Imf\]

and

\[\text{for all } \alpha, \beta \in Imf, \text{ if } \alpha < \beta \text{ then } A_{\alpha} \subseteq A_{\beta} \text{ and } A_{\beta} \cap f^{-1}(\alpha) = \emptyset.\]

Proof: \((\Rightarrow)\) Choose \(A_{\alpha} = U(f : \alpha)\) for all \(\alpha \in Imf\).

By Proposition II.6 (iv), Proposition II.8 and Theorem IV.1, we get that \(\{A_{\alpha} : \alpha \in Imf\}\) is a chain of generalized bi-ideals of \(S\) satisfying the conditions i) and ii). For the proof of uniqueness, it is similar to the proof of uniqueness of Theorem III.6.

\((\Leftarrow)\) Let \((a, \alpha), (b, \beta) \in [S \times Imf]^f\) and \((x, \gamma) \in S \times Imf\).

Then \(\min\{\alpha, \beta, \gamma\} \in Imf\) and

\[
\min\{f(a), f(b)\} \geq \min\{\alpha, \beta\} \geq \min\{\alpha, \beta, \gamma\}.
\]

By the conditions i) and ii), we have

\[a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\min\{\alpha, \beta, \gamma\}}.\]

Similarly, we have \(b \in A_{\min\{\alpha, \beta, \gamma\}}\). Since \(A_{\min\{\alpha, \beta, \gamma\}}\) is a generalized bi-ideal of \(S\), we have \(a \in A_{\min\{\alpha, \beta, \gamma\}}\).

If \(f(axb) < \min\{\alpha, \beta, \gamma\}\), then by the condition ii), we have \(A_{\min\{\alpha, \beta, \gamma\}} \cap f^{-1}(f(axb)) = \emptyset\) which contradicts
with \( axb \in A_{\min(\alpha, \beta, \gamma)} \cap f^{-1}(f(axb)) \). Thus \( f(axb) \geq \min(\alpha, \beta, \gamma) \). Hence \( (a, \alpha) * (x, \gamma) * (b, \beta) \in [S \times 1mf] \).

Therefore \([S \times 1mf] f\) is a generalized bi-ideal of \( S \times 1mf \).

By Corollary IV.3, we get \( f \) is a fuzzy generalized bi-ideal of \( S \).

In the following corollary, we show a form of a fuzzy generalized bi-ideal \( f \) of a semigroup where \( 1mf \) is finite.

**Corollary IV.9.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( 1mf = \{a_1, a_2, ..., a_n\} \) such that \( a_1 > a_2 > ... > a_n \). Then \( f \) is a fuzzy generalized bi-ideal of \( S \) if and only if

\[
\left\{ \begin{array}{ll}
\alpha_n & \text{if } x \in U(f : a_n) \setminus U(f : a_{n-1}) \\
\alpha_{n-1} & \text{if } x \in U(f : a_{n-2}) \setminus U(f : a_n-1) \\
& \vdots \\
\alpha_2 & \text{if } x \in U(f : a_2) \setminus U(f : a_1) \\
\alpha_1 & \text{if } x \in U(f : a_1)
\end{array} \right.
\]

for all \( x \in S \).

**Corollary IV.10.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( 1mf \subseteq \Delta \subseteq [0,1] \). The following statements are equivalent.

(i) \( f \) is a fuzzy generalized bi-ideal of \( S \).

(ii) There exists a generalized bi-ideal \( \mathcal{R} \) of \( S \times \Delta \) such that 

\[
U(\mathcal{R} : \alpha) = U(f : \alpha) \text{ for all } \alpha \in \Delta.
\]

(iii) \( U(f : \alpha)(\not= \emptyset) \) is a generalized bi-ideal of \( S \) for all \( \alpha \in \Delta \).

**Proof:** (i) \( \Rightarrow \) (ii) Choose \( \mathcal{R} = [S \times \Delta] f \) and use Theorem IV.1 and Proposition II.6 (iv).

(ii) \( \Rightarrow \) (iii) It follows from Proposition II.8.

(iii) \( \Rightarrow \) (i) Apply Theorem IV.8.

In the following three results, we characterize fuzzy bi-ideals of semigroups.

**Theorem IV.11.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \( f \) is a fuzzy bi-ideal of \( S \) if and only if there exists the unique chain \( \{A_\alpha \mid \alpha \in 1mf\} \) of bi-ideals of \( S \) such that

(i) \( f^{-1}(\alpha) \subseteq A_\alpha \) for all \( \alpha \in 1mf \) and

(ii) for all \( \alpha, \beta \in 1mf \), if \( \alpha < \beta \) then \( A_\beta \subset A_\alpha \) and \( A_\beta \cap f^{-1}(\alpha) = \emptyset \).

**Proof:** It follows from Theorem III.6 and Theorem IV.8.

**Corollary IV.12.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( 1mf = \{a_1, a_2, ..., a_n\} \) such that \( a_1 > a_2 > ... > a_n \). Then \( f \) is a fuzzy bi-ideal of \( S \) if and only if \( U(f : a_i) \mid i \in \{1, 2, ..., n\} \) is the chain of bi-ideals of \( S \) such that

\[
\left\{ \begin{array}{ll}
\alpha_n & \text{if } x \in U(f : a_n) \setminus U(f : a_{n-1}) \\
\alpha_{n-1} & \text{if } x \in U(f : a_{n-2}) \setminus U(f : a_n-1) \\
& \vdots \\
\alpha_2 & \text{if } x \in U(f : a_2) \setminus U(f : a_1) \\
\alpha_1 & \text{if } x \in U(f : a_1)
\end{array} \right.
\]

for all \( x \in S \).

**Proof:** It follows from Corollary III.8 and Corollary IV.9.

**Corollary IV.13.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( 1mf \subseteq \Delta \subseteq [0,1] \). The following statements are equivalent.

(i) \( f \) is a fuzzy bi-ideal of \( S \).

(ii) There exists a bi-ideal \( \mathcal{R} \) of \( S \times \Delta \) such that 

\[
U(\mathcal{R} : \alpha) = U(f : \alpha) \text{ for all } \alpha \in \Delta,
\]

(iii) \( U(f : \alpha)(\not= \emptyset) \) is a bi-ideal of \( S \) for all \( \alpha \in \Delta \).

**Proof:** It follows from Corollary III.9 and Corollary IV.10.

V. FUZZY QUASI-IDEALS OF SEMIGROUPS

In this section, characterizations of fuzzy quasi-ideals of a semigroup \( S \) are studied by using the certain subsets of \( S \), \([0,1], FP(S) \) and \( S \times [0,1] \).

In Theorem V.1, we characterize fuzzy quasi-ideals of \( S \) via the certain subsets of \([0,1] \) and \( S \times [0,1] \).

**Theorem V.1.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then the following statements are equivalent.

(i) \( f \) is a fuzzy quasi-ideal of \( S \).

(ii) For every quasi-ideal \( A \) of \( S \) and \( \Delta \subseteq [0,1] \), \([A \times \Delta]^f(\not= \emptyset) \) is a quasi-ideal of \( S \times \Delta \).

(iii) \([S \times \Delta]^f(\not= \emptyset) \) is a quasi-ideal of \( S \times \Delta \) where \( 1mf \subseteq \Delta \subseteq [0,1] \).

(iv) For all \( a \in S \) such that \( F_a \not= \emptyset \), we have

\[
(1mf)f(a) \subseteq \bigcap_{(x,y) \in F_a}(Imf)_{f(x,y)}.
\]

**Proof:** (i) \( \Rightarrow \) (ii) Let \( A \) be a quasi-ideal of \( S \), \( \Delta \subseteq [0,1] \) and \( (a, \alpha) \in (S \times \Delta) \cap ([A \times \Delta]^f \cap ([A \times \Delta]^f \times S \times \Delta)) \).

Then there exist \((x_1, \beta_1), (x_2, \beta_2) \in S \times \Delta \) and \((a_1, \gamma_1), (a_2, \gamma_2) \in [A \times \Delta]^f \) such that

\[
(a, \alpha) = (x_1, \beta_1) * (a_1, \gamma_1) = (a_2, \beta_2) * (x_2, \gamma_2).
\]

Thus \( f(a_1) \geq \gamma_1 \geq \min(\gamma_1, \beta_1) = \alpha \) and \( f(a_2) \geq \gamma_2 \geq \min(\gamma_2, \beta_2) = \alpha \). Since \( A \) is a quasi-ideal of \( S \), we have \( a \in A \). Consider

\[
(x_S \circ f)(a) = \sup \{\min(x_S(x), f(y)) \mid (x, y) \in F_a\} \geq \min(x_S(x), f(a)) = f(a) \geq \alpha.
\]

Similarly, \((f \circ x_S)(a) \not= \min(f(a_2), x_S(x_2)) \geq \alpha \).

By assumption (i), we get

\[
f(a) \geq ((x_S \circ f) \wedge (f \circ x_S))(a) \geq \alpha.
\]

Hence \((a, \alpha) \in [A \times \Delta]^f \). Therefore \([A \times \Delta]^f(\not= \emptyset) \) is a quasi-ideal of \( S \times \Delta \).

((iii) \( \Rightarrow \) (ii)) It is clear.

((ii) \( \Rightarrow \) (i)) Let \( 1mf \subseteq \Delta \subseteq [0,1] \). Suppose that \( (\chi_S \circ f) \wedge (f \circ \chi_S))(a) \geq f(a) \) for some \( a \in S \). Thus

\[
\sup_{(x,y) \in F_a}(f(x) \mid (x, y) \in F_a) > f(a),
\]

\[
\sup_{(x,y) \in F_a}(f(y) \mid (x, y) \in F_a) > f(a).
\]
Hence \( f(x_1) > f(a) \) and \( f(y_2) > f(a) \) for some \((x_1, y_1), (x_2, y_2) \in F_a \). Clearly, \((x_1, f(x_1)), (y_2, f(y_2)) \in [S \times \Delta]^f \) and \((y_1, f(y_2)), (x_2, f(x_1)) \in S \times \Delta \). Then, 
\[
\begin{align*}
(a, \min\{f(x_1), f(y_2)\}) &= (x_1, f(x_1)) \ast (y_1, f(y_2)) \\
&= (x_2, f(x_1)) \ast (y_2, f(y_2)).
\end{align*}
\]
By assumption (iii), we have 
\[
(a, \min\{f(x_1), f(y_2)\}) \in [S \times \Delta]^f.
\]
Therefore \( f(a) \geq f(x_1) \) or \( f(a) \geq f(y_2) \). It is a contradiction. Hence \((\chi_S \circ f) \land (f \circ \chi_S)(a) \leq f(a) \) for all \( a \in S \), that is \( f \) is a fuzzy quasi-ideal of \( S \).

(ii) \( \Rightarrow \) (iv) Let \( a \in S \) and \( F_a \neq \emptyset \). Then for all \((x, y) \in F_a \),
\[
(f \circ \chi_S)(a) = \sup\{\min\{f(x), \chi_S(y)\} \mid (x, y) \in F_a\} \geq f(x).
\]
Similarly, we have that \((\chi_S \circ f)(a) \geq f(y)\) for all \((x, y) \in F_a \). By assumption (ii), we have 
\[
f(a) \geq \min\{(f \circ \chi_S)(a), (\chi_S \circ f)(a)\}.
\]
Consider the following cases:

Case 1: \( f(a) \geq (f \circ \chi_S)(a) \). Then \( f(a) \geq f(x) \) for all \((x, y) \in F_a \). Thus \((Imf) f(a) \subseteq (Imf) f(x) \) for all \((x, y) \in F_a \). Hence \((Imf) f(a) \subseteq \bigcap (Imf) f(x) \).

Case 2: \( f(a) \geq (\chi_S \circ f)(a) \). Its proof is similar to the proof of Case 1.

Therefore, \((Imf) f(a) \subseteq \bigcap (Imf) f(x) \).

(iii) \( \Rightarrow \) (i) It is straightforward.

By Theorem V.1, we get Corollary V.2.

**Corollary V.2.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then the following statements are equivalent.

(i) \( f \) is a fuzzy quasi-ideal of \( S \).

(ii) \[ S \times (0, 1] f(\neq 0) \] is a quasi-ideal of \( S \times (0, 1] \).

(iii) \[ S \times Imf \] is a quasi-ideal of \( S \times Imf \).

(iv) \[ S \times [0, 1] f \] is a quasi-ideal of \( S \times [0, 1] \).

**Example V.3.** Let \( S = \{0, a, b, c\} \) be a semigroup with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \( f \) be a fuzzy subset of \( S \) such that 
\[
f(0) = f(a) = 0.8, \quad f(b) = f(c) = 0.3.
\]
Thus \( S \times Imf \) is a quasi-ideal of \( S \times (0, 1] \) by Corollary V.2, we get \( f \) is a fuzzy quasi-ideal of \( S \).

**Proposition V.4.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \( S \times [0, 1] f \) is a quasi-ideal of \( S \times [0, 1] \) if and only if \( f^{-1}(\neq 0) \) is a quasi-ideal of \( FP(S) \).

**Proof:** It is straightforward.

**Corollary V.5.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \( f \) is a fuzzy quasi-ideal of \( S \) if and only if \( f(\neq 0) \) is a quasi-ideal of \( FP(S) \).

**Proof:** It follows from Corollary V.2 and Proposition V.4.

In Theorem V.6, we give a characterization of a fuzzy quasi-ideal of a semigroup \( S \) by the chain of quasi-ideals of \( S \).

**Theorem V.6.** Let \( f \) be a fuzzy subset of a semigroup \( S \). Then \( f \) is a fuzzy quasi-ideal of \( S \) if and only if there exists the unique chain \( \{A_\alpha \mid \alpha \in Imf\} \) of quasi-ideals of \( S \) such that

(i) \( f(\neq 0) \subseteq A_\alpha \) for all \( \alpha \in Imf \)

(ii) for all \( \alpha, \beta \in Imf \), if \( \alpha < \beta \) then \( A_\beta \subseteq A_\alpha \) and \( A_\beta \cap f^{-1}(\neq 0) = 0 \).

**Proof:** (\( \Rightarrow \)) Choose \( A_\alpha = U(f : \alpha) \) for all \( \alpha \in Imf \). By Proposition II.10, Proposition II.6 (iv) and Theorem V.1, we get \( \{A_\alpha \mid \alpha \in Imf\} \) is a chain of quasi-ideals of \( S \) satisfying the conditions i) and ii). For the proof of uniqueness, it is similar to the proof of uniqueness of Theorem III.6.

(\( \Leftarrow \)) Let \( (a, \alpha) \in (S \times Imf \times [S \times Imf])^f \) \( \cap ([S \times Imf]^f \times S \times Imf) \). Then

\[
(a, \alpha) = (x_1, (\beta_1 \ast (a, \gamma_1)) = (a_2, \beta_2) \ast (x_2, \gamma_2)
\]
for some \((a_1, \gamma_1), (a_2, \gamma_2) \in [S \times Imf]^f \) and \((x_1, \beta_1), (x_2, \beta_2) \in S \times Imf \). Thus 
\[
f(x_1) \geq \gamma_1 \geq \alpha \quad \text{and} \quad f(x_2) \geq \gamma_2 \geq \alpha.
\]
By the conditions i) and ii), we see that 
\[
x_1 \in f^{-1}(f(x_1)) \subseteq A_{f(x_1)} \subseteq A_\alpha.
\]
and similarly \( x_2 \in A_\alpha \). Since \( A_\alpha \) is a quasi-ideal of \( S \), we have \( A_\alpha \subseteq A_\alpha \). Thus \( f(\neq 0) \geq \alpha \). Indeed, if \( f(\neq 0) \alpha \) then by the condition ii), we get \( A_\alpha \cap f^{-1}(f(\neq 0)) = 0 \) which is a contradiction with \( a \in A_\alpha \cap f^{-1}(f(\neq 0)) \). Therefore \( (a, \alpha) \in [S \times Imf]^f \). Consequently, \([S \times Imf]^f \) is a quasi-ideal of \( S \times Imf \). By Corollary V.2, we have \( f \) is a fuzzy quasi-ideal of \( S \).

In the proof of Theorem V.6, the unique chain of quasi-ideals of \( S \) satisfying the conditions i) and ii) is \( \{U(f : \alpha) \mid \alpha \in Imf\} \). Then we get a form of a fuzzy quasi-ideal of \( S \) which its image is finite.

**Corollary V.7.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( Imf = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) such that \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \). Then \( f \) is a fuzzy quasi-ideal of \( S \) if and only if \( \{U(f : \alpha_i) \mid i \in \{1, 2, \ldots, n\}\} \) is the chain of quasi-ideals of \( S \) such that

\[
\begin{align*}
\alpha_n & \quad \text{if} \ x \in U(f : \alpha_n) \setminus U(f : \alpha_{n-1}) \\
\alpha_{n-1} & \quad \text{if} \ x \in U(f : \alpha_{n-1}) \setminus U(f : \alpha_{n-2}) \\
o & \quad \text{if} \ x \in U(f : \alpha_2) \setminus U(f : \alpha_1) \\
\alpha_1 & \quad \text{if} \ x \in U(f : \alpha_1)
\end{align*}
\]
for all \( x \in S \).

**Corollary V.8.** Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( Imf \subseteq \Delta \subseteq [0, 1] \). The following statements are equivalent.

(i) \( f \) is a fuzzy quasi-ideal of \( S \).

(ii) There exists a quasi-ideal \( R \) of \( S \times \Delta \) such that \( U(R : \alpha) = U(f : \alpha) \) for all \( \alpha \in \Delta \).

(iii) \( U(f : \alpha)(\neq 0) \) is a quasi-ideal of \( S \) for all \( \alpha \in \Delta \).

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Proof: (i) \( \Rightarrow \) (ii) Choose \( \mathcal{R} = [S \times \Delta]^f \) and use Theorem V.1 and Proposition II.6 (iv).

(iii) \( \Rightarrow \) (iii) It follows from Proposition II.10.

(iv) \( \Rightarrow \) (i) Apply Theorem V.6.

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