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A New Branch and Reduce Approach for Solving Generalized Linear Fractional Programming

Yong-Hong Zhang, and Chun-Feng Wang

Abstract—**In this paper, for solving generalized linear fractional programming (GLFP), a new branch-and-reduce approach is presented. Firstly, an equivalent problem (EP) of GLFP is given; then, a new linear relaxation technique is proposed; finally, the problem EP is reduced to a sequence of linear programming problems by using the new linear relaxation technique. Meanwhile, to improve the convergence speed of our algorithm, two reducing techniques are presented. The proposed algorithm is proved to be convergent, and some experiments are provided to show its feasibility and efficiency.**

Index Terms—**Linear relaxation; Global optimization; Generalized linear fractional programming; Reducing technique; Branch and bound.**

I. Introduction

THIS paper considers the following generalized linear fractional programming (GLFP) problem:

$$\text{GLFP}\left\{\begin{array}{ll} \min & \sum\limits_{i=1}^{p}\frac{\sum\limits_{j=1}^{n}c_{ij}x_j+d_i}{\sum\limits_{j=1}^{n}e_{ij}x_j+f_i} \\ \text{s.t.} & x\in D=\{x\in R^n \mid Ax\le b\}, \end{array}\right.$$

where $p\ge 2,\ A\in R^{m\times n},\ b\in R^m,\ c_{ij},\ d_i,\ e_{ij},\ f_i$ are arbitrary real numbers, D is bounded with $intD\neq\emptyset$, and for $\forall x\in D,\ \sum\limits_{j=1}^{n}e_{ij}x_j+f_i\neq 0,\ i=1,\cdots,p,\ j=1,\cdots,n.$

Among fractional programming, the problem GLFP is a special class of optimization problems, which has attracted the interest of practitioners for many years. The main reason is that many applications in various fields can be put into the form GLFP, including transportation scheme, manufacturing, economic benefits, and multi-objective bond portfolio [1-5], etc.

In addition, since the problem GLFP may be not (quasi)convex, it may has multiple local optimal solutions, and many of which fail to be globally optimal, that is, the problem GLFP posses significant theoretical and computational difficulties. So, it also has attracted the interest of many researchers.

During the past years, for $x\in D$, with the assumption that $\sum\limits_{j=1}^{n}c_{ij}x_j+d_i\ge 0,\ \sum\limits_{j=1}^{n}e_{ij}x_j+f_i>0$, many algorithms have been presented for solving special cases of problem GLFP. For example, when $p=1$, by using variable transformation, Charnes and Cooper proposed a method [6]. When $p=2$, Konno et al. proposed an effective parametric algorithm [7], which can be used to solve large scale problem. When $p=3$, Konno and Abe developed a heuristic algorithm[8]. When $p\ge 3$, through utilizing an updated objective function method, Bitran and Novaes presented an approach, which solves a sequence of linear programming problems depending on updating the local gradient of the fractional objective function at successive points [9]. By using an equivalent transformation and a linearization technique, Shen and Wang proposed a branch and bound algorithm for solving a sum of linear ratios problem with coefficients [10]. Dur et al. introduced the algorithm (DMIHR) and applied it to the class of fractional optimization problem [11]. By adopting monotonic transformation technique, Shen et al. proposed a method to solve quadratic ratios fractional program [12]. By using the characteristics of exponential and logarithmic functions, Wang et al. presented a twice linearization technique [13]. Through using suitable transformation, Benson proposed a method, which has a potential to solve GLFP by some well known techniques [14]. By solving an equivalent concave minimum problem of the original problem, Benson put forward a new branch and bound algorithm [15]. Under the assumption that $\sum\limits_{j=1}^{n}c_{ij}x_j+d_i\ge 0,\ \sum\limits_{j=1}^{n}e_{ij}x_j+f_i\neq 0$, a branch and bound algorithm was developed [16]. Recently, two global optimization algorithms were proposed for solving GLFP with the only assumption that $\sum\limits_{j=1}^{n}e_{ij}x_j+f_i\neq 0$ [17,18].

The goal of this research has two-fold. First, we present a new linear relaxation method. Second, in order to improve the convergence speed of our algorithm, two reducing techniques are proposed, which can be used to eliminate the region in which the global minimum solution of GLFP does not exist.

The main features of this algorithm are (1) the model considered by this paper has a more general form, which only request $\sum\limits_{j=1}^{n}e_{ij}x_j+f_i\neq 0$; (2) a new linear relaxation method is proposed in order to obtain a lower bound for the global optimal value of problem EP over partitioned subsets, which is more convenient in the computation than the methods [10,14,18]; (2) two reducing techniques are presented, which can be used to improved the convergence speed of the proposed algorithm; (3) numerical experiments are given to show the feasibility and efficiency of our algorithm.

This paper is structured as follows. In Section 2, an equivalent problem EP of problem GLFP is derived. In Section 3, a new linear relaxation technique is presented for

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generating the linear relaxation programming problem LRP of problem EP, which can provide a lower bound for the optimal value of problem EP. In order to improve the convergence speed of our algorithm, two reducing techniques are presented in Section 4. In Section 5, the global optimization algorithm is described, and the convergence of this algorithm is established. Numerical results are reported to show the feasibility and efficiency of our algorithm in Section 6.

II. Equivalent problem (EP) of GLFP

In problem GLFP, for $\forall x \in D$, since $\sum_{j=1}^{n} e_{ij}x_j + f_i \neq 0$, by the intermediate value theorem, we have $\sum_{j=1}^{n} e_{ij}x_j + f_i > 0$ or $\sum_{j=1}^{n} e_{ij}x_j + f_i < 0$. In addition, through using the techniques of [15,16], $\sum_{j=1}^{n} c_{ij}x_j + d_i \geq 0$ always can be satisfied. Therefore, without loss of generality, we assume that $\sum_{j=1}^{n} c_{ij}x_j + d_i \geq 0$ always holds.

For $\forall x \in D$, let

$$I_+ = \{i \mid \sum_{j=1}^{n} e_{ij}x_j + f_i > 0,\ i = 1, \cdots, p\},$$
$$I_- = \{i \mid \sum_{j=1}^{n} e_{ij}x_j + f_i < 0,\ i = 1, \cdots, p\}.$$

Compute $l_j^0 = \min_{x\in D} x_j$, $u_j^0 = \max_{x\in D} x_j (j = 1, \cdots, n)$, and define rectangle $H^0 = [l^0, u^0]$. Furthermore, we compute $\underline{y}_i^0 = \frac{1}{\sum_{j=1}^{n} \max\{e_{ij}l_j^0, e_{ij}u_j^0\}+f_i}$, $\overline{y}_i^0 = \frac{1}{\sum_{j=1}^{n} \min\{e_{ij}l_j^0, e_{ij}u_j^0\}+f_i}$, $i = 1, \cdots, p$, and construct rectangle $Y^0 = [\underline{y}^0, \overline{y}^0]$. Through introducing new variables $y_i (i = 1, \cdots, p)$, the problem GLFP can be converted into an equivalent problem EP as follows:

$$\text{EP}(H^0)\begin{cases} v(H^0) = \min \varphi_0(x, y) = \sum_{i=1}^{p} y_i(\sum_{j=1}^{n} c_{ij}x_j + d_i) \\ \text{s.t.}\ \ \varphi_i(x, y) = y_i(\sum_{j=1}^{n} e_{ij}x_j + f_i) \geq 1,\ i \in I_+, \\ \varphi_i(x, y) = y_i(\sum_{j=1}^{n} e_{ij}x_j + f_i) \leq 1,\ i \in I_-, \\ x \in D \bigcap H^0,\ y \in Y^0. \end{cases}$$

The key equivalence result for problem GLFP and EP(H^0) is given by the following theorem.

Theorem 1 If (x^*, y^*) is a global optimal solution of problem EP(H^0), then x^* is a global optimal solution of problem GLFP. Conversely, if x^* is a global optimal solution of problem GLFP, then (x^*, y^*) is a global optimal solution of problem EP(H^0), where $y_i^* = \frac{1}{\sum_{j=1}^{n} e_{ij}x_j^*+f_i}$, $i = 1, \cdots, p$.

Proof. From the definitions of problems GLFP and EP(H^0), the conclusion can be obtained easily, so it is omitted.

By Theorem 1, in order to globally solve problem GLFP, we may solve problem EP(H^0) instead. Therefore, in the following we only consider how to solve the problem EP(H^0).

III. Linear relaxation programming (LRP) problem

Let $H = \{x \mid l \leq x \leq u\}$ denote the initial box H^0 or modified box as defined for some partitioned subproblem in a branch and bound scheme. This section will show how to construct the problem LRP for problem EP over H.

For convenience in expression, let

$$\begin{aligned} T_i^{c+} &= \{j \mid c_{i,j} > 0,\ j = 1, \cdots, n\}, \\ T_i^{c-} &= \{j \mid c_{i,j} < 0,\ j = 1, \cdots, n\}, \\ T_i^{e+} &= \{j \mid e_{i,j} > 0,\ j = 1, \cdots, n\}, \\ T_i^{e-} &= \{j \mid e_{i,j} < 0,\ j = 1, \cdots, n\}. \end{aligned}$$

To derive the problem LRP of problem EP, for $i = 1, \cdots, p$, we first compute $\underline{y}_i = \frac{1}{\sum_{j=1}^{n} \max\{e_{ij}l_j, e_{ij}u_j\}+f_i}$, $\overline{y}_i = \frac{1}{\sum_{j=1}^{n} \min\{e_{ij}l_j, e_{ij}u_j\}+f_i}$, and consider the term x_jy_i in the interval $[l_j, u_j]$ and $[\underline{y}_i, \overline{y}_i]$. Since $x_j - l_j \geq 0$, $y_i - \underline{y}_i \geq 0$, we have

$$(x_j - l_j)(y_i - \underline{y}_i) \geq 0,$$

that is

$$x_jy_i - x_j\underline{y}_i - l_jy_i + l_j\underline{y}_i \geq 0.$$

Furthermore, we have

$$x_jy_i \geq x_j\underline{y}_i + l_jy_i - l_j\underline{y}_i. \tag{1}$$

Meanwhile, since $x_j - l_j \geq 0$, $y_i - \overline{y}_i \leq 0$, we have

$$(x_j - l_j)(y_i - \overline{y}_i) \leq 0,$$

Furthermore, we can obtain

$$x_jy_i \leq x_j\overline{y}_i + l_jy_i - l_j\overline{y}_i. \tag{2}$$

Based on (1) and (2), we can derive the problem LRP of problem EP. Towards this end, first, consider the objective function $\varphi_0(x, y)$, we have

$$\begin{aligned} \varphi_0(x, y) &= \sum_{i=1}^{p} y_i(\sum_{j=1}^{n} c_{ij}x_j + d_i) \\ &= \sum_{i=1}^{p}(\sum_{j\in T_i^{c+}} c_{ij}x_jy_i + \sum_{j\in T_i^{c-}} c_{ij}x_jy_i) + \sum_{i=1}^{p} d_iy_i \\ &\geq \sum_{i=1}^{p}(\sum_{j\in T_i^{c+}} c_{ij}(x_j\underline{y}_i + l_jy_i - l_j\underline{y}_i) \\ &+ \sum_{j\in T_i^{c-}} c_{ij}(x_j\overline{y}_i + l_jy_i - l_j\overline{y}_i)) + \sum_{i=1}^{p} d_iy_i \\ &= \varphi_0^l(x, y). \end{aligned}$$

$$\begin{aligned} \varphi_0(x, y) &= \sum_{i=1}^{p} y_i(\sum_{j=1}^{n} c_{ij}x_j + d_i) \\ &= \sum_{i=1}^{p}(\sum_{j\in T_i^{c+}} c_{ij}x_jy_i + \sum_{j\in T_i^{c-}} c_{ij}x_jy_i) + \sum_{i=1}^{p} d_iy_i \\ &\leq \sum_{i=1}^{p}(\sum_{j\in T_i^{c+}} c_{ij}(x_j\overline{y}_i + l_jy_i - l_j\overline{y}_i) \\ &+ \sum_{j\in T_i^{c-}} c_{ij}(x_j\underline{y}_i + l_jy_i - l_j\underline{y}_i)) + \sum_{i=1}^{p} d_iy_i \\ &= \varphi_0^u(x, y). \end{aligned}$$

Then, consider the constraint functions $\varphi_i(x,y)$, $i = 1,\cdots,p$. For $i \in I_i^+$, by using (1) and (2), we have

$$\begin{aligned}\varphi_i(x,y) &= \sum_{j=1}^{n} e_{ij}x_jy_i + f_iy_i \\ &\le \sum_{j\in T_i^{e+}} e_{ij}(x_j\overline{y}_i + l_jy_i - l_j\overline{y}_i) \\ &+ \sum_{j\in T_i^{e-}} e_{ij}(x_j\underline{y}_i + l_jy_i - l_j\underline{y}_i) + f_iy_i = \varphi_i^u(x,y).\end{aligned} \tag{3}$$

For $i \in I_i^-$, we have

$$\begin{aligned}\varphi_i(x,y) &= \sum_{j=1}^{n} e_{ij}x_jy_i + f_iy_i \\ &\ge \sum_{j\in T_i^{e+}} e_{ij}(x_j\underline{y}_i + l_jy_i - l_j\underline{y}_i) \\ &+ \sum_{j\in T_i^{e-}} e_{ij}(x_j\overline{y}_i + l_jy_i - l_j\overline{y}_i) + f_iy_i = \varphi_i^l(x,y).\end{aligned} \tag{4}$$

From the above discussion, the linear relaxation programming problem LRP can be established as follows, which provides a lower bound for the optimal value $v(H)$ of problem EP(H):

$$\text{LRP}(H)\left\{\begin{array}{lll} LB(H) = & \min & \varphi_0^l(x,y) \\ & \text{s.t.} & \varphi_i^u(x,y) \ge 1,\ i \in I_+, \\ & & \varphi_i^l(x,y) \le 1,\ i \in I_-, \\ & & x \in D\bigcap H,\ y \in Y = [\underline{y},\overline{y}]. \end{array}\right.$$

Obviously, if $\overline{H} \subseteq H \subseteq H^0$, then $LB(\overline{H}) \ge LB(H)$.

IV. Reducing technique

To improve the convergence speed of this algorithm, we present two reducing techniques, which can be used to eliminate the region in which the global optimal solution of problem EP(H^0) does not exist.

Assume that UB and LB are the current known upper bound and lower bound of the optimal value $v(H^0)$ of the problem EP(H^0). Let

$$\begin{aligned}
&\alpha_j = \sum_{i=1}^{p}\xi_i,\ \text{where}\ \xi_i = \left\{\begin{array}{ll} c_{ij}\underline{y}_i, & \text{if } c_{ij} > 0, \\ c_{ij}\overline{y}_i, & \text{if } c_{ij} < 0, \end{array}\right. \\
&\beta_i = \sum_{j=1}^{n} c_{ij}l_j + e_i, \\
&\Lambda_1 = -\sum_{i=1}^{p}\Big(\sum_{j\in T_i^{c+}} c_{ij}l_j\underline{y}_i + \sum_{j\in T_i^{c-}} c_{ij}l_j\overline{y}_i\Big), \\
&\Omega_1 = \sum_{i=1}^{p}\min\{\beta_i\underline{y}_i, \beta_i\overline{y}_i\}, \\
&\gamma_k = UB - \sum_{j=1,j\neq k}^{n}\min\{\alpha_jl_j, \alpha_ju_j\} - \Omega_1 - \Lambda_1, \\
&\theta_j = \sum_{i=1}^{p}\eta_i,\ \text{where}\ \eta_i = \left\{\begin{array}{ll} c_{ij}\overline{y}_i, & \text{if } c_{ij} > 0, \\ c_{ij}\underline{y}_i, & \text{if } c_{ij} < 0, \end{array}\right. \\
&\Lambda_2 = -\sum_{i=1}^{p}\Big(\sum_{j\in T_i^{c+}} c_{ij}l_j\overline{y}_i + \sum_{j\in T_i^{c-}} c_{ij}l_j\underline{y}_i\Big), \\
&\Omega_2 = \sum_{i=1}^{p}\max\{\beta_i\underline{y}_i, \beta_i\overline{y}_i\}, \\
&\tau_k = LB - \sum_{j=1,j\neq k}^{n}\max\{\theta_jl_j, \theta_ju_j\} - \Omega_2 - \Lambda_2,
\end{aligned}$$

The reducing techniques are derived as in the following theorems.

Theorem 2 For any subrectangle $H \subseteq H^0$ with $H_j = [l_j, u_j]$, if there exists some index $k \in \{1,2,\cdots,n\}$ such that $\alpha_k > 0$ and $\gamma_k < \alpha_ku_k$, then there is no globally optimal solution of problem EP(H^0) over H^1; if $\alpha_k < 0$ and $\gamma_k < \alpha_kl_k$, for some k, then there is no globally optimal solution of problem EP(H^0) over H^2, where

$$H^1 = (H_j^1)_{n\times 1} \subseteq H,\ \text{with}\ H_j^1 = \left\{\begin{array}{ll} H_j, & j \neq k, \\ (\frac{\gamma_k}{\alpha_k}, u_k]\bigcap H_k, & j = k, \end{array}\right.$$

$$H^2 = (H_j^2)_{n\times 1} \subseteq H,\ \text{with}\ H_j^2 = \left\{\begin{array}{ll} H_j, & j \neq k, \\ [l_k, \frac{\gamma_k}{\alpha_k})\bigcap H_k, & j = k. \end{array}\right.$$

proof First, we show that for all $x \in H^1$, $\varphi_0(x,y) > UB$. Consider the kth component x_k of x. Since $x_k \in (\frac{\gamma_k}{\alpha_k}, u_k]$, it follows that

$$\frac{\gamma_k}{\alpha_k} < x_k \le u_k.$$

From $\alpha_k > 0$, we have $\gamma_k < \alpha_kx_k$. For all $x \in H^1$, by the above inequality and the definition of γ_k, it implies that

$$UB - \sum_{j=1,j\neq k}^{n}\min\{\alpha_jl_j, \alpha_ju_j\} - \Omega_1 - \Lambda_1 < \alpha_kx_k,$$

that is

$$\begin{aligned}UB &< \sum_{j=1,j\neq k}^{n}\min\{\alpha_jl_j, \alpha_ju_j\} + \alpha_kx_k + \Omega_1 + \Lambda_1 \\ &\le \sum_{j=1}^{n}\alpha_jx_j + \sum_{i=1}^{p}\beta_iy_i + \Lambda_1 = \varphi_0^l(x,y).\end{aligned}$$

Thus, for all $x \in H^1$, we have $\varphi_0(x,y) \ge \varphi_0^l(x,y) > UB \ge v(H^0)$, i.e. for all $x \in H^1$, $\varphi_0(x,y)$ is always greater than the optimal value $v(H^0)$ of the problem EP(H^0). Therefore, there can not exist globally optimal solution of problem EP(H^0) over H^1.

For all $x \in H^2$, if there exists some k such that $\alpha_k < 0$ and $\gamma_k < \alpha_kl_k$, from arguments similar to the above, it can be derived that there is no globally optimal solution of problem EP(H^0) over H^2

Theorem 3 For any subrectangle $H \subseteq H^0$ with $H_j = [l_j, u_j]$, if there exists some index $k \in \{1,2,\cdots,n\}$ such that $\theta_k > 0$ and $\tau_k > \theta_kl_k$, then there is no globally optimal solution of problem EP(H^0) over H^3; if $\theta_k < 0$ and $\tau_k > \theta_ku_k$, for some k, then there is no globally optimal solution of problem EP(H^0) over H^4, where

$$H^3 = (H_j^3)_{n\times 1} \subseteq H,\ \text{with}\ H_j^3 = \left\{\begin{array}{ll} H_j, & j \neq k, \\ [l_k, \frac{\tau_k}{\theta_k})\bigcap H_k, & j = k, \end{array}\right.$$

$$H^4 = (H_j^4)_{n\times 1} \subseteq H,\ \text{with}\ H_j^4 = \left\{\begin{array}{ll} H_j, & j \neq k, \\ (\frac{\tau_k}{\theta_k}, u_k]\bigcap H_k, & j = k. \end{array}\right.$$

proof First, we show that for all $x \in H^3$, $\varphi_0(x,y) < LB$. Consider the kth component x_k of x. By the assumption and the definitions of θ_k and τ_k, we have

$$l_k \le x_k < \frac{\tau_k}{\theta_k}.$$

Note that $\theta_k > 0$, we have $\tau_k > \theta_kx_k$. For all $x \in H^3$, by the above inequality and the definition of τ_k, it implies that

$$\begin{aligned}LB &> \sum_{j=1,j\neq k}^{n}\max\{\theta_jl_j, \theta_ju_j\} + \theta_kx_k + \Omega_2 + \Lambda_2 \\ &\ge \sum_{j=1}^{n}\theta_jx_j + \sum_{i=1}^{p}\beta_iy_i + \Lambda_2 = \varphi_0^u(x,y) \ge \varphi_0(x,y).\end{aligned}$$

Thus, for all $x \in H^3$, we have $v(H^0) \geq LB > \varphi_0(x,y)$. Therefore, there can not exist globally optimal solution of problem EP(H^0) over H^3.

For all $x \in H^4$, if there exists some k such that $\theta_k < 0$ and $\tau_k < \theta_k u_k$, from arguments similar to the above, it can be derived that there is no globally optimal solution of problem EP(H^0) over H^4

V. Algorithm and its convergence

In this section, based on the former results, we present the branch and bound algorithm to solve problem EP(H^0). This method need to solve a sequence of linear relaxation programming problems over partitioned subsets of H^0 in order to find a global optimal solution.

A. Branching rule

During each iteration of the algorithm, the branching process creates a more refined partition that cannot yet be excluded from further consideration in searching for a global optimal solution for problem EP(H^0), which is a critical element in guaranteeing convergence. This paper chooses a simple and standard bisection rule, which is sufficient to ensure convergence since it drives the intervals shrinking to a singleton for all the variables along any infinite branch of the branch and bound tree.

Consider any node subproblem identified by rectangle $H = \{x \in R^n \mid l_j \leq x_j \leq u_j,\ j = 1, \cdots, n\} \subseteq H^0$. The branching rule is as follows:

(i) let $k = \text{argmax}\{l_j - u_j \mid j = 1, \cdots, n\}$;

(ii) let $\pi_k = (l_k + u_k)/2$;

(iii) let

$$H^1 = \{x \in R^n \mid l_j \leq x_j \leq u_j,\ j \neq k, l_k \leq x_k \leq \pi_k\},$$
$$H^2 = \{x \in R^n \mid l_j \leq x_j \leq u_j,\ j \neq k, \pi_k \leq x_k \leq u_k\}.$$

Through using this branching rule, the rectangle H is partitioned into two subrectangles H^1 and H^2.

B. Branch and bound algorithm

Based upon the results and operations given above, this subsection summarizes the basic steps of the proposed algorithm.

Let $LB(H^k)$ be the optimal function value of LRP over the subrectangle $H = H^k$, and (x^k, y^k) be an element of the corresponding argmin.

Algorithm statement

Step 1. Choose $\epsilon \geq 0$. Find an optimal solution x^0 and the optimal value $LB(H^0)$ for problem LRP(H) with $H = H^0$. Set $LB_0 = LB(H^0)$, $y_i^0 = \frac{1}{\sum_{j=1}^n e_{ij}x_j^0 + f_i},\ i = 1, \cdots, p,$ and $UB_0 = \varphi_0(x^0, y^0)$. If $UB_0 - LB_0 \leq \epsilon$, then stop: (x^0, y^0) and x^0 are ϵ-optimal solutions of problems EP(H^0) and GLFP, respectively. Otherwise, set $Q_0 = \{H^0\},\ F = \emptyset,\ k = 1$, and go to Step 2.

Step 2. Set $LB_k = LB_{k-1}$. Subdivide H^{k-1} into two subrectangles $H^{k,1},\ H^{k,2}$ via the branching rule. Set $F = F \bigcup \{H^{k-1}\}$.

Step 3. Set $t = t + 1$. If $t > 2$, go to Step 5. Otherwise, continue.

Step 4. If $LB(H^{k,t}) > UB_k$, set $F = F \bigcup \{H^{k,t}\}$, and go to Step 3. Otherwise, set

$$y_i^{k,t} = \frac{1}{\sum_{j=1}^n e_{ij}x_j^{k,t} + f_i},\ i = 1, \cdots, p.$$

Let $UB_k = \min\{UB_k, \varphi_0(x^{k,t}, y^{k,t})\}$. If $UB_k = \varphi_0(x^{k,t}, y^{k,t})$, set $x^k = x^{k,t},\ y^k = y^{k,t}$, go to Step 3.

Step 5. Set

$$F = F \bigcup \{H \in Q_{k-1} \mid UB_k \leq LB(H)\},$$
$$Q_k = \{H \mid H \in (Q_{k-1} \bigcup \{H^{k,1}, H^{k,2}\}), H \notin F\}.$$

Step 6. Set $LB_k = \min\{LB(H) \mid H \in Q_k\}$. Let H^k be the subrectangle which satisfies that $LB_k = LB(H^k)$. If $UB_k - LB_k \leq \epsilon$, stop, (x^k, y^k) and x^k are global ϵ-optimal solutions of problems EP(H^0) and GLFP, respectively. Otherwise, set $k = k + 1$, and go to Step 2.

C. Convergence analysis

In this subsection, we give the global convergence properties of the above algorithm.

Theorem 4 The above algorithm either terminates finitely with a globally ϵ-optimal solution, or generates an infinite sequence $\{(x^k, y^k)\}$ of iteration such that along any infinite branch of the branch and bound tree, which any accumulation point is a globally optimal solution of problem EP(H^0).

Proof When the algorithm is finite, by the algorithm, it terminates at some step $k \geq 0$. Upon termination, it follows that

$$UB_k - LB_k \leq \epsilon.$$

From Step 1 and Step 6 in the algorithm, a feasible solution (x^k, y^k) for the problem EP(H^0) can be found, and the following relation holds

$$\varphi_0(x^k, y^k) - LB_k \leq \epsilon.$$

By Section 3, we have

$$LB_k \leq v(H^0).$$

Since (x^k, y^k) is a feasible solution of problem EP(H^0), $\varphi_0(x^k, y^k) \geq v(H^0)$. Taken together above, it implies that

$$v(H^0) \leq \varphi_0(x^k, y^k) \leq LB_k + \epsilon \leq v(H^0) + \epsilon,$$

and so (x^k, y^k) is a global ϵ-optimal solution to the problem EP(H^0) in the sense that

$$v(H^0) \leq \varphi_0(x^k, y^k) \leq v(H^0) + \epsilon.$$

If the algorithm is infinite, then an infinite sequence $\{(x^k, y^k)\}$ will be generated. Since the feasible region of EP(H^0) is bounded, the sequence $\{(x^k, y^k)\}$ must be has a convergence subsequence. Without loss of generality, set $\lim_{k\to\infty}(x^k, y^k) = (x^*, y^*)$, then we have

$$\lim_{k\to\infty} y_i^k = y_i^* = \frac{1}{\sum_{j=1}^n e_{ij}x_j^* + f_i},$$
$$\lim_{k\to\infty} (\sum_{j=1}^n c_{ij}x_j^k + d_i) = \sum_{j=1}^n c_{ij}x_j^* + d_i.$$

Furthermore, we have

$$
\begin{aligned}
&\lim_{k\to\infty} UB_k = \lim_{k\to\infty} \varphi_0(x^k, y^k) = \varphi_0(x^*, y^*),\\
&\lim_{k\to\infty} LB_k = \lim_{k\to\infty} (\sum_{i=1}^{p}(\sum_{j\in T_i^{c+}} c_{ij}(x_j^k \underline{y}_i + l_j y_i^k - l_j \underline{y}_i)\\
&\qquad + \sum_{j\in T_i^{c-}} c_{ij}(x_j^k \overline{y}_i + l_j y_i^k - l_j \overline{y}_i) + \sum_{i=1}^{p} d_i y_i^k\\
&\qquad = \sum_{i=1}^{p} \frac{\sum_{j=1}^{n} c_{ij}x_j^* + d_i}{\sum_{j=1}^{n} e_{ij}x_j^* + f_i} = \varphi_0(x^*, y^*).
\end{aligned}
$$

Therefore, we have $\lim_{k\to\infty}(UB_k - LB_k) = 0$.

This implies that (x^*, y^*) is a global optimal solution of problem EP(H^0). By Theorem 1, x^* is a global optimal solution of problem GLFP.

VI. Numerical experiments

In this section, to verify the performance of the proposed algorithm, some numerical experiments are reported, and compared with several latest algorithms[10,13,17-20]. The algorithm is implemented by Matlab 7.1, and all test problems are carried out on a Pentium IV (3.06 GHZ) microcomputer. The simplex method is applied to solve the linear relaxation programming problems.

The results of problems 1-7 are summarized in Table I, where the following notations have been used in row headers: ϵ: convergence error; Iter: number of algorithm iterations.

Table II summarizes our computational results of Example 8. For this test problem, ϵ is set to $1e{-}3$. In Table II, $\mathrm{Ave.Iter}$ represents the average number of iterations; $\mathrm{Ave.Time}$ stands for the average CPU time of the algorithm in seconds, which are obtained by randomly running our algorithm for 10 test problems.

Example 1[18]

$$
\begin{array}{ll}
\max & 0.9 \times \dfrac{-x_1 + 2x_2 + 2}{3x_1 - 4x_2 + 5} + (-0.1) \times \dfrac{4x_1 - 3x_2 + 4}{-2x_1 + x_2 + 3}\\
\text{s.t.} & x_1 + x_2 \le 1.5,\\
& x_1 - x_2 \le 0,\\
& 0 \le x_1 \le 1,\ 0 \le x_2 \le 1.
\end{array}
$$

Example 2[13,18,19]

$$
\begin{array}{ll}
\max & \dfrac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50} + \dfrac{3x_1 + 4x_3 + 50}{4x_1 + 4x_2 + 5x_3 + 50}\\
& +\dfrac{x_1 + 2x_2 + 5x_3 + 50}{x_1 + 5x_2 + 5x_3 + 50} + \dfrac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50}\\
\text{s.t.} & 2x_1 + x_2 + 5x_3 \le 10,\\
& x_1 + 6x_2 + 3x_3 \le 10,\\
& 5x_1 + 9x_2 + 2x_3 \le 10,\\
& 9x_1 + 7x_2 + 3x_3 \le 10,\\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0.
\end{array}
$$

Example 3[20]

$$
\begin{array}{ll}
\min & \dfrac{-x_1 + 2x_2 + 2}{3x_1 - 4x_2 + 5} + \dfrac{4x_1 - 3x_2 + 4}{-2x_1 + x_2 + 3}\\
\text{s.t.} & x_1 + x_2 \le 1.5,\\
& x_1 - x_2 \le 0,\\
& 0 \le x_1 \le 1,\ 0 \le x_2 \le 1.
\end{array}
$$

Example 4[19,20]

$$
\begin{array}{ll}
\max & \dfrac{3x_1 + 5x_2 + 3x_3 + 50}{3x_1 + 4x_2 + 5x_3 + 50} + \dfrac{3x_1 + 4x_2 + 50}{4x_1 + 3x_2 + 2x_3 + 50}\\
& +\dfrac{4x_1 + 2x_2 + 4x_3 + 50}{5x_1 + 4x_2 + 3x_3 + 50}\\
\text{s.t.} & 6x_1 + 3x_2 + 3x_3 \le 10,\\
& 10x_1 + 3x_2 + 8x_3 \le 10,\\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.
\end{array}
$$

Example 5[10]

$$
\begin{array}{ll}
\max & \dfrac{63x_1 - 18x_2 + 39}{13x_1 + 26x_2 + 13} + \dfrac{13x_1 + 26x_2 + 13}{37x_1 + 73x_2 + 13}\\
& +\dfrac{37x_1 + 73x_2 + 13}{13x_1 + 13x_2 + 13} + \dfrac{13x_1 + 13x_2 + 13}{63x_1 - 18x_2 + 39}\\
\text{s.t.} & 5x_1 - 3x_2 = 3,\\
& 1.5 \le x_1 \le 3.
\end{array}
$$

Example 6[10]

$$
\begin{array}{ll}
\max & \dfrac{3x_1 + 4x_2 + 50}{3x_1 + 5x_2 + 4x_3 + 50} - \dfrac{3x_1 + 5x_2 + 3x_3 + 50}{5x_1 + 5x_2 + 4x_3 + 50}\\
& -\dfrac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50} - \dfrac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50}\\
\text{s.t.} & 6x_1 + 3x_2 + 3x_3 \le 10,\\
& 10x_1 + 3x_2 + 8x_3 \le 10,\\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.
\end{array}
$$

Example 7[17]

$$
\begin{array}{ll}
\max & \dfrac{37x_1 + 73x_2 + 13}{13x_1 + 13x_2 + 13} + \dfrac{63x_1 - 16x_2 + 39}{13x_1 + 26x_2 + 13}\\
\text{s.t.} & 5x_1 - 3x_2 = 3,\\
& 1.5 \le x_1 \le 3.
\end{array}
$$

Example 8

$$
\begin{array}{ll}
\min & \sum_{i=1}^{p} \dfrac{\sum_{j=1}^{n} c_{ij}x_j + d_i}{\sum_{j=1}^{n} e_{ij}x_j + f_i}\\
\text{s.t.} & x \in D = \{x \in R^n \mid Ax \le b\},
\end{array}
$$

where the elements of the matrix $A \in R^{m\times n},\ c_{ij},\ e_{i,j} \in R$ are randomly generated in the interval [0,1]. All constant terms of denominators and numerators are the same number, which randomly generated in [1,100]. The elements of $b \in R^m$ are equal to 1. This agrees with the way random numbers are generated in [13].

TABLE I: Computational results of Examples 1-7

Example	ϵ	Methods	Optimal solution	Optimal value	Iter
1	1e-9	[18]	(0.0, 1.0)	3.575	1
	1e-9	*ours*	(0.0, 1.0)	3.575	1
2	1e-6	[13]	(1.1111, 1.365e-5, 1.351e-5)	4.081481	39
	1e-5	[20]	(0.0013, 0.0, 0.0)	4.087412	1640
	1e-9	[18]	(1.1111, 0.0, 0.0)	4.0907	1289
	1e-9	*ours*	(1.1111, 0.0, 0.0)	4.0907	28
3	1e-8	[20]	(0.0, 0.283935547)	1.623183358	71
	1e-8	*ours*	(0.0, 0.283935547)	1.623183358	65
4	1e-5	[19]	(0.0, 1.6725, 0.0)	3.0009	1033
	1e-8	[20]	(0.0, 3.3333, 0.0)	3.00292	119
	1e-8	*ours*	(0.0, 3.3333, 0.0)	3.00292	77
5	1e-6	[10]	(3.0, 4.0)	3.2917	9
	1e-6	*ours*	(3.0, 4.0)	3.2917	8
6	1e-6	[10]	(-1.838e-16, 3.3333, 0.0)	1.9	8
	1e-6	*ours*	(0.0, 3.3333, 0.0)	1.9	8
7	1e-4	[17]	(3.0, 4.0)	5.0	32
	1e-4	*ours*	(3.0, 4.0)	5.0	20

From Table I, it can be seen that, for Examples 1-7, our algorithm can determine the global optimal solution effectively than that of the references [10,13,17-20].

TABLE II: Computational results of Example 8

(p, m, n)	Ave.Time	Ave.Iter
(5,30,30)	0.1534	2.6
(5,50,50)	0.30	2.8
(5,100,100)	2.7641	3.4
(10,30,30)	0.1890	2.9
(10,50,50)	0.4329	4.6
(10,100,100)	3.2490	6.2

From Table II, the computational results show that our algorithm performs well on the test problems, and can solve them in a reasonable amount of time. Meanwhile, we find that, with the size of the problem becoming large, the average number of iterations and the average CPU time do not increase quickly.

The results in Tables I and II show that our algorithm is both feasible and efficient.

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