Almost Periodic Solution for a Nabla BAM Neural Networks on Time Scales

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Abstract—**This paper is concerned with a BAM neural networks with nabla derivatives on time scales. Several sufficient conditions are obtained ensuring the existence and global exponential stability of almost periodic solution for the networks based on M-matrix theory and almost periodic functional hull theory. Some previous results are improved and extended in this paper and numerical examples are presented to illustrate the feasibility and effectiveness of the results.**

Index Terms—**BAM neural networks; exponential stability; almost periodic solution; hull theory; time scale.**

I. Introduction

In recent years, both continuous and discrete BAM neural networks with almost periodic coefficients have been extensively studied and applied in many different fields such as signal processing, pattern recognition, solving optimization problems, automatic control engineering and so on, one may see [1-6] and the references therein.

However, in applications, there are many neural networks whose development processes are more than just continuous or discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

A time scale is a nonempty arbitrary closed subset of reals. The theory of calculus on time scales was initiated by S. Hilger in his Ph.D. thesis in 1988 [7] in order to unify continuous and discrete analysis. The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors [8-10] powerfully.

Nowadays there have been some results on almost periodic solutions for neural networks on time scales, see, for example, [11-13]. But there are also many problems have not been solved well such as almost periodic solutions for some neural networks with nabla derivatives. Therefore, the study of existence and stability of almost periodic solution for neural networks on time scales need to be explored further.

Motivated by the statements above, in this paper, we shall study the following BAM neural networks on time scales:

$$\begin{cases} x_i^{\nabla}(t)=-a_i(t)x_i(t)+\sum\limits_{j=1}^{m}p_{ji}(t)f_j(y_j(t-\tau_{ji}(t))) \\ \qquad\qquad +I_i(t),\ t\in\mathbb{T},\ i=1,2,\cdots,n, \\ y_j^{\nabla}(t)=-b_j(t)y_j(t)+\sum\limits_{i=1}^{n}q_{ij}(t)g_i(x_i(t-\vartheta_{ij}(t))) \\ \qquad\qquad +L_j(t),\ t\in\mathbb{T},\ j=1,2,\cdots,m, \end{cases} \tag{1}$$

where $\mathbb{T}$ is a time scale, $x_i(t)$ and $y_j(t)$ are the activations of the ith neuron and the jth neuron, respectively. p_{ji}, q_{ij} are the connection weights at time t, $I_i(t)$ and $L_j(t)$ denote the external inputs at time t. g_i, f_j are the input-output functions (the activation functions). Time delays $\tau_{ji}(t), \vartheta_{ij}(t)$ correspond to finite speed of axonal transmission, $a_i(t), b_j(t)$ represent the rate with which the ith neuron and jth neuron will reset their potential to the resting state in isolation when they are disconnected from the network and the external inputs at time t. m, n correspond to the number of neurons in layers.

The initial conditions of system (1) are of the form

$$\begin{cases} x_i(s)=\phi_i(s),\ s\in[-\vartheta,0]\cap\mathbb{T}, \\ \vartheta=\max\limits_{1\le i\le n,1\le j\le m}\sup\limits_{t\in\mathbb{T}}\{\vartheta_{ij}(t)\},\ i=1,2,\cdots,n, \\ y_j(s)=\varphi_j(s),\ s\in[-\hat{\tau},0]\cap\mathbb{T}, \\ \hat{\tau}=\max\limits_{1\le i\le n,1\le j\le m}\sup\limits_{t\in\mathbb{T}}\{\tau_{ji}(t)\},\ j=1,2,\cdots,m, \end{cases}$$

where $\phi_i(\cdot)$ and $\varphi_i(\cdot)$ denote real-valued continuous functions defined on $[-\hat{\tau},0]\cap\mathbb{T}$ and $[-\vartheta,0]\cap\mathbb{T}$.

Throughout this paper, we make the following assumptions:

(H_1) Each $a_i(t)(i=1,2,\cdots,n,\ t\in\mathbb{T})$ and $b_j(t)(j=1,2,\cdots,m,\ t\in\mathbb{T})$ is positive, continuous and bounded function, and $-a_i\in\mathcal{R}^+, -b_j\in\mathcal{R}^+$.

(H_2) $p_{ji}(t), q_{ij}(t), I_i(t), L_j(t), 0<\vartheta_{ij}(t)<\vartheta, 0<\tau_{ji}(t)<\hat{\tau}$, are all nonnegative continuous bounded almost periodic functions on $\mathbb{T}$, $i=1,2,\cdots,n, j=1,2,\cdots,m$.

(H_3) $f_j, g_i\in C(\mathbb{R},\mathbb{R}), f_j, g_i\ge 0 (i=1,2,\cdots,n, j=1,2,\cdots,m)$ are Lipschitzian with Lipschitz constants $\eta_j, \lambda_i>0$,

$$|f_j(x)-f_j(y)|\le\eta_j|x-y|,\ |g_i(x)-g_i(y)|\le\lambda_i|x-y|.$$

For convenience, we denote $\bar{f}=\sup\limits_{t\in\mathbb{T}}|f(t)|$, $\underline{f}=\inf\limits_{t\in\mathbb{T}}|f(t)|$.

The main purpose of this paper is, by using some dynamic inequalities on time scales, to discuss the permanence of system (1), then by using the almost periodic functional hull theory on time scales, to establish criteria for the existence, uniqueness and global exponential stability of almost periodic solutions of system (1).

The organization of this paper is as follows: In Section 2, we introduce some notations and definitions and prove some preliminary results needed in the later sections. In Section 3, based on M-matrix theory, some sufficient conditions are obtained to ensure that the solution of (1) is globally exponentially stable. In Section 4, by using almost periodic functional hull theory, we show that the almost periodic system (1) has a unique globally exponentially stable strictly positive almost periodic solution. In Section 5, two examples

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are given to illustrate that our results are feasible and more general.

II. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. The backwards graininess function $\nu : \mathbb{T}_k \to [0, +\infty)$ is defined by $\nu(t) = t - \rho(t)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and its right-side limits exist at right-dense points in $\mathbb{T}$.

The function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if $1-\nu(t)p(t) \neq 0$ for all $t \in \mathbb{T}_k$. The set of all ν-regressive and ld-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}_\nu^+ = \{p \in \mathcal{R}_\nu : 1 - \nu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If p is a ν-regressive function, then the nabla exponential function $\hat{e}_r$ is defined by

$$\hat{e}_p(t, s) = \exp\left\{\int_s^t \hat{\xi}_{\nu(\tau)}(p(\tau))\nabla\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\hat{\xi}_h(z) = \begin{cases} -\frac{\mathrm{Log}(1-hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Lemma 1. (see [14]) *If $p \in \mathcal{R}_\nu$, and $a, b, c \in \mathbb{T}$, then*
(i) $\hat{e}_0(t, s) \equiv 1$ and $\hat{e}_p(t, t) \equiv 1$;
(ii) $\hat{e}_p(\rho(t), s) = (1 - \nu(t)p(t))\hat{e}_p(t, s)$;
(iii) $\hat{e}_p(t, s)\hat{e}_p(s, r) = \hat{e}_p(t, r)$;
(iv) $(\hat{e}_p(t, s))^\nabla = p(t)\hat{e}_p(t, s)$;
(v) $\int_a^b p(t)\hat{e}_p(c, \rho(t))\nabla t = \hat{e}_p(c, a) - \hat{e}_p(c, b)$.

For more details about the calculus on time scales, see [14].

Definition 1. (see [15]) *A time scale T is called an almost periodic time scale if*

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2. (see [15]) *Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{E}^n)$ is called an almost periodic function if the ε-translation set of function f*

$$E\{\varepsilon, f\} = \{\tau \in \Pi : |f(t + \tau) - f(t)| < \varepsilon, \ for \ all \ t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ and

$$|f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

τ is called the ε-translation number of f.

Definition 3. (see [15]) *Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $\mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of function f*

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon, \ for \ all \ t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset S of D; that is for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t + \tau, x) - f(t, x)| < \varepsilon, \forall t \in \mathbb{T}, x \in S.$$

τ is called the ε-translation number of f.

Definition 4. (see [15]) *Let $f(t) \in C(\mathbb{T} \times D, \mathbb{E}^n)$, $H(f) = \{g : \mathbb{T} \times D \to \mathbb{E}^n | there \ exist \ \alpha \in \Pi such \ that \ T_\alpha f(t, x) = g(t, x) \ exists \ uniformly \ on \ \mathbb{T} \times \mathbb{T}\}$ is called the hull of f.*

Lemma 2. (see [14]) *Let $y, f \in \mathcal{R}_\nu(\mathbb{T})$ and $p \in \mathcal{R}_\nu^+$. Then*

$$y^\nabla(t) \leq (\geq) p(t)y(t) + f(t), \ \forall t \in \mathbb{T}$$

implies

$$y(t) \leq (\geq) y(t_0)\hat{e}_p(t, t_0) + \int_{t_0}^t \hat{e}_p(t, \sigma(\tau))f(\tau)\Delta\tau, \forall t \in \mathbb{T}.$$

Lemma 3. *Assume that the assumptions $(H_1) - (H_3)$ are satisfied. Any solution $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ of system (1) is uniformly bounded on $[0, +\infty)_\mathbb{T}$.*

Proof: From system (1), for any $t \in [0, +\infty)_\mathbb{T}$, we have

$$\begin{aligned} x_i^\nabla(t) &\leq -\underline{a}_i x_i(t) + \sum_{j=1}^m \bar{p}_{ji}\bar{f}_j + \bar{I}_i, \\ y_j^\nabla(t) &\leq -\underline{b}_j y_j(t) + \sum_{i=1}^n \bar{q}_{ij}\bar{g}_i + \bar{L}_j, \end{aligned} \tag{2}$$

and

$$\begin{aligned} x_i^\nabla(t) &\geq -\bar{a}_i x_i(t) + \sum_{j=1}^m \underline{p}_{ji}\underline{f}_j + \underline{I}_i, \\ y_j^\nabla(t) &\geq -\bar{b}_j y_j(t) + \sum_{i=1}^n \underline{q}_{ij}\underline{g}_i + \underline{L}_j. \end{aligned} \tag{3}$$

Then, from (4), by Lemma 3 and Lemma 1(v), we have

$$\begin{aligned} & x_i(t) \\ \leq \ & x_i(t_0)\hat{e}_{-\underline{a}_i}(t, t_0) \\ & + \int_{t_0}^t \hat{e}_{-\underline{a}_i}(t, \sigma(s))\left[\sum_{j=1}^m \bar{p}_{ji}\bar{f}_j + \bar{I}_i\right]\Delta s \\ \leq \ & x_i(t_0)\hat{e}_{-\underline{a}_i}(t, t_0) \\ & + \left[-\frac{\sum_{j=1}^m \bar{p}_{ji}\bar{f}_j + \bar{I}_i}{\underline{a}_i}\right](\hat{e}_{-\underline{a}_i}(t, t_0) - 1) \\ = \ & \hat{e}_{-\underline{a}_i}(t, t_0)\left[x_i(t_0) - \frac{\sum_{j=1}^m \bar{p}_{ji}\bar{f}_j + \bar{I}_i}{\underline{a}_i}\right] + \frac{\sum_{j=1}^m \bar{p}_{ji}\bar{f}_j + \bar{I}_i}{\underline{a}_i} \\ \leq \ & \frac{\sum_{j=1}^m \bar{p}_{ji}\bar{f}_j + \bar{I}_i}{\underline{a}_i}, \ i = 1, 2, \cdots, n. \end{aligned}$$

Similarly, we can get

$$y_j(t) \le \frac{\sum_{i=1}^{n} \bar{q}_{ij}\bar{g}_i + \bar{L}_j}{\underline{b}_j}, \ j = 1, 2, \cdots, m.$$

On another side, from (5), by Lemma 3 and Lemma 1(v), then

$$\begin{aligned} & x_i(t) \\ \ge \ & x_i(t_0)\hat{e}_{-\bar{a}_i}(t, t_0) \\ & + \int_{t_0}^{t} \hat{e}_{-\bar{a}_i}(t, \sigma(s))\Big[\sum_{j=1}^{m} \underline{p}_{ji}\underline{f}_j + \underline{I}_i\Big]\Delta s \\ \ge \ & x_i(t_0)\hat{e}_{-\bar{a}_i}(t, t_0) \\ & + \Big[-\frac{\sum_{j=1}^{m} \underline{p}_{ji}\underline{f}_j + \underline{I}_i}{\bar{a}_i}\Big](\hat{e}_{-\bar{a}_i}(t, t_0) - 1) \\ = \ & \hat{e}_{-\bar{a}_i}(t, t_0)\Big[x_i(t_0) - \frac{\sum_{j=1}^{m} \underline{p}_{ji}\underline{f}_j + \underline{I}_i}{\bar{a}_i}\Big] + \frac{\sum_{j=1}^{m} \underline{p}_{ji}\underline{f}_j + \underline{I}_i}{\bar{a}_i} \\ \ge \ & \frac{\sum_{j=1}^{m} \underline{p}_{ji}\underline{f}_j + \underline{I}_i}{\bar{a}_i}, \ i = 1, 2, \cdots, n. \end{aligned}$$

Similarly, we can get

$$y_j(t) \ge \frac{\sum_{i=1}^{n} \underline{q}_{ij}\underline{g}_i + \underline{L}_j}{\bar{b}_j}, \ j = 1, 2, \cdots, m.$$

So, any solution $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ of system (1) is uniformly bounded on $[0, +\infty)_{\mathbb{T}}$. The proof is completed.

Lemma 4. *If the following conditions satisfy:*

(i) $D^+x_i^{\nabla}(t) \le \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}\bar{x}_j(t)$, $t \in [t_0, +\infty)_{\mathbb{T}}$, $i, j = 1, 2, \cdots, n$, *where* $a_{ij} \ge 0 (i \ne j)$, $b_{ij} \ge 0$, $\sum_{i=1}^{n} \bar{x}_i(t_0) > 0$, $\bar{x}_i(t) = \sup_{s \in [t-\tau_0, t]_{\mathbb{T}}} x_i(s)$, *and* $\tau_0 > 0$ *is a constant;*

(ii) $\widetilde{M} := -(a_{ij} + b_{ij})_{n\times n}$ *is an* M*-matrix;*

then there exists a constant $\gamma_i > 0$, $a > 0$, *such that the solutions of inequality (i) satisfies*

$$x_i(t) \le \gamma_i\Big(\sum_{j=1}^{n} \bar{x}_j(t_0)\Big)\hat{e}_{\ominus a}(t, t_0), \ \forall\ t \in (t_0, +\infty)_{\mathbb{T}}, \quad i = 1, 2, \cdots, n.$$

Proof: Assume that

$$\begin{aligned} G(t, x(t), \bar{x}(t)) \ = \ & (g_1(t, x(t), \bar{x}(t)), g_2(t, x(t), \bar{x}(t)), \\ & \cdots, g_n(t, x(t), \bar{x}(t))), \end{aligned}$$

where

$$g_i(t, x(t), \bar{x}(t)) = \Big(\sum_{j=1}^{n} a_{ij}x_i(t) + \sum_{j=1}^{n} b_{ij}\bar{x}_i(t)\Big), \quad i = 1, 2, \cdots, n.$$

By condition (i), then

$$D^+x_i^{\nabla}(t) \le g_i(t, x(t), \bar{x}(t)), \forall t \in [t_0, +\infty)_{\mathbb{T}}, \quad i = 1, 2, \cdots, n. \tag{4}$$

By condition (ii), there exists constants $\xi > 0$ and $d_i > 0$ $(i = 1, 2, \cdots, n)$ such that

$$\sum_{j=1}^{n}(a_{ij} + b_{ij})d_i < -\xi, \ i = 1, 2, \cdots, n.$$

Choose $0 < a \ll 1$, such that

$$ad_i + \sum_{j=1}^{n}(a_{ij}d_i + b_{ij}d_i\hat{e}_a(t, t - \tau_0)) < 0, \quad \forall t \in [t_0, +\infty)_{\mathbb{T}}, i = 1, 2, \cdots, n. \tag{5}$$

If $t \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, choose $F \gg 1$, such that

$$Fd_i\hat{e}_{\ominus a}(t, t_0) > 1, \ i = 1, 2, \cdots, n. \tag{6}$$

For any $\varepsilon > 0$, let

$$q_i(t) = Fd_i\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t, t_0).$$

From (5), for any $t \in [t_0, +\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} & D^+q_i^{\nabla}(t) \\ = \ & (\ominus a)Fd_i\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t, t_0) \\ \ge \ & -aFd_i\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t, t_0) \\ > \ & \sum_{j=1}^{n}\big(a_{ij}d_i + b_{ij}d_i\hat{e}_a(t, t - \tau_0)\big)F\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big) \\ & \times\hat{e}_{\ominus a}(t, t_0) \\ = \ & \sum_{j=1}^{n} a_{ij}d_iF\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t, t_0) \\ & + \sum_{j=1}^{n} b_{ij}d_iF\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t - \tau_0, t_0) \\ \ge \ & \sum_{j=1}^{n} a_{ij}q_i(t) + \sum_{j=1}^{n} b_{ij}\bar{q}_i(t) \\ = \ & g_i(t, q(t), \bar{q}(t)), \ i = 1, 2, \cdots, n, \end{aligned} \tag{7}$$

that is

$$D^+q_i^{\nabla}(t) > g_i(t, q(t), \bar{q}(t)), \ \forall t \in [t_0, +\infty)_{\mathbb{T}}, \quad i = 1, 2, \cdots, n. \tag{8}$$

For $t \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, by (6), we can get

$$\begin{aligned} q_i(t) &= Fd_i\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t, t_0) \\ &> \sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon, i = 1, 2, \cdots, n. \end{aligned}$$

Let $x_i(t) \le \sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon$, $t \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, then

$$q_i(t_0) > x_i(t_0), \ i = 1, 2, \cdots, n. \tag{9}$$

Together with (4), (8) and (9), by Lemma 1, we can get

$$x_i(t) < q_i(t) = Fd_i\Big(\sum_{j=1}^{n} \bar{x}_j(t_0) + \varepsilon\Big)\hat{e}_{\ominus a}(t, t_0), \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}, i = 1, 2, \cdots, n.$$

Let $\varepsilon \to 0^+$, $Fd_i = \gamma_i$, then

$$x_i(t) \le \gamma_i\Bigg(\sum_{j=1}^{n} \bar{x}_j(t_0)\Bigg)\hat{e}_{\ominus a}(t, t_0),\ \forall t \in (t_0, +\infty)_{\mathbb{T}},\quad i = 1, 2, \cdots, n.$$

The proof is completed.

Definition 5. *The almost periodic solution $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T$ of equation (1) is said to be exponentially stable, if there exist a positive α such that for any $\delta \in (-\infty, 0]_{\mathbb{T}}$, there exists $N = N(\delta) \ge 1$ such that the solution $z = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)^T$ satisfying*

$$\|z(t) - z^*(t)\| \le N\|\varphi(\delta) - z^*(\delta)\|\hat{e}_{\ominus\alpha}(t, \delta),\ t \in \mathbb{T}^+,$$

where $\delta \in [-\max\{\hat{\tau}, \vartheta\}, 0]$.

III. Global exponential stability

Suppose that $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T = (z_1^*, z_2^*, \cdots, z_{n+m}^*)^T$ is a solution of system (1). In this section, we will construct some suitable differential inequality to study the global exponential stability of this solution and we will use the following norm:

$$\begin{aligned}\|z\| &= \max_{1\le l\le n+m} \sup_{t\in\mathbb{T}} |z_l(t)| \\ &= \max\Bigg\{\max_{1\le i\le n} \sup_{t\in\mathbb{T}} |x_i(t)|, \max_{1\le j\le m} \sup_{t\in\mathbb{T}} |y_j(t)|\Bigg\}.\end{aligned}$$

Theorem 1. *Assume that $(H_1) - (H_3)$ hold and if*

$$\Upsilon := \begin{bmatrix} A & -PL \\ -Q\Lambda & B \end{bmatrix}_{(n+m)\times(n+m)}$$

is an M-matrix, where $A = \mathrm{diag}(\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n)_{n\times n}$, $B = \mathrm{diag}(\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_m)_{m\times m}$, $P = (\bar{p}_{ji})_{m\times n}, Q = (\bar{q}_{ij})_{n\times m}$, $L = \mathrm{diag}(\eta_1, \eta_2, \cdots, \eta_m), \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, then the solution of system (1) is globally exponentially stable.

Proof: Suppose that $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T$ is a solution of system (1), and $z = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)^T$ is another arbitrary solution. Then, system (1) can be written as

$$\begin{cases} \big(x_i(t) - x_i^*(t)\big)^\nabla = -a_i(t)x_i(t) + a_i(t)x_i^*(t) \\ + \sum\limits_{j=1}^{m} p_{ji}(t)(f_j(y_j(t - \tau_{ji}(t))) - f_j(y_j^*(t - \tau_{ji}(t)))), \\ \big(y_j(t) - y_j^*(t)\big)^\nabla = -b_j(t)y_j(t) + b_j(t)y_j^*(t) \\ + \sum\limits_{i=1}^{n} q_{ij}(t)(g_i(x_i(t - \vartheta_{ij}(t))) - g_i(x_i^*(t - \vartheta_{ij}(t)))). \end{cases} \tag{10}$$

The initial condition of (10) is $\psi(s) = (\phi_1(s), \cdots, \phi_n(s), \varphi_1(s), \cdots, \varphi_m(s))^T$.

Let $V(t) = |z(t) - z^*(t)|$, the upper right derivative $D^+V^\nabla(t)$ along the solutions of system (10) is as follows:

$$\begin{aligned} D^+V^\nabla(t) &= \mathrm{sign}(z(t) - z^*(t))(z(t) - z^*(t))^\nabla \\ &\le \begin{bmatrix} -A & 0 \\ 0 & -B \end{bmatrix} V(t) + \begin{bmatrix} 0 & PL \\ Q\Lambda & 0 \end{bmatrix} \overline{V}(t). \end{aligned}$$

According to Lemma 4, then there must exist constants $\alpha > 0,\ \gamma > 0$, for any $\delta \in [-\max\{\hat{\tau}, \vartheta\}, 0]$, and $l = 1, 2, \cdots, n+m$ such that

$$\begin{aligned}|z_l(t) - z_l^*(t)| \le \gamma \max\Bigg\{ &\max_{1\le i\le n} \sup_{-\vartheta\le\delta\le 0} |\phi_i(\delta) - x_i^*(\delta)|, \\ &\max_{1\le j\le m} \sup_{-\hat{\tau}\le\delta\le 0} |\varphi_j(\delta) - y_j^*(\delta)|\Bigg\}\hat{e}_{\ominus\alpha}(t, t_0),\end{aligned}$$

then

$$\begin{aligned} &\|z(t) - z^*(t)\| \\ \le\ & \frac{\gamma}{\hat{e}_{\ominus\alpha}(t_0, \delta)} \max\Bigg\{ \max_{1\le i\le n} \sup_{-\vartheta\le\delta\le 0} |\phi_i(\delta) - x_i^*(\delta)|, \\ & \max_{1\le j\le m} \sup_{-\hat{\tau}\le\delta\le 0} |\varphi_j(\delta) - y_j^*(\delta)|\Bigg\}\hat{e}_{\ominus\alpha}(t, \delta) \\ =\ & \frac{\gamma}{\hat{e}_{\ominus\alpha}(t_0, \delta)} \|\psi - z^*\| \hat{e}_{\ominus\alpha}(t, \delta). \end{aligned}$$

Let $N = N(\delta) = \frac{\gamma}{\hat{e}_{\ominus\alpha}(t_0,\delta)}$, then

$$\|z - z^*\| \le N\|\psi - z^*\|\hat{e}_{\ominus\alpha}(t, \delta).$$

From Definition 5, the solution $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T$ is globally exponentially stable. The proof is completed.

IV. Almost periodic solution

Suppose that $h(t)$ is an almost periodic function defined on $\mathbb{T}$. Let $H(h(t))$ denote the hull of $h(t)$.

Suppose that

$$\begin{aligned} &a_i^*(t) \in H(a_i(t)),\ b_j^*(t) \in H(b_j(t)), \\ &p_{ji}^*(t) \in H(p_{ji}(t)),\ q_{ij}^*(t) \in H(q_{ij}(t)), \\ &\tau_{ji}^*(t) \in H(\tau_{ji}(t)),\ \vartheta_{ij}^*(t) \in H(\vartheta_{ij}(t)), \\ &I_i^*(t) \in H(I_i(t)),\ L_j^*(t) \in H(L_j(t)) \end{aligned}$$

are selected such that there is a time sequence $\{t_n\}$:

$$\begin{aligned} &a_i(t + t_n) \to a_i^*(t),\ b_j(t + t_n) \to b_j^*(t), \\ &p_{ji}(t + t_n) \to p_{ji}^*(t),\ q_{ij}(t + t_n) \to q_{ij}^*(t), \\ &\tau_{ji}(t + t_n) \to \tau_{ji}^*(t),\ \vartheta_{ij}(t + t_n) \to \vartheta_{ij}^*(t), \\ &I_i(t + t_n) \to L_j^*(t),\ a_i(t + t_n) \to L_j^*(t) \end{aligned}$$

as $n \to +\infty$ and $t_n \to +\infty$ for all t on $\mathbb{T}$. Then we get a hull equation of system (1) as follows:

$$\begin{cases} x_i^\nabla(t) = -a_i^*(t)x_i(t) + \sum\limits_{j=1}^{m} p_{ji}^*(t)f_j(y_j(t - \tau_{ji}^*(t))) \\ \qquad +I_i^*(t),\ t \in \mathbb{T},\ i = 1, 2, \cdots, n, \\ y_j^\nabla(t) = -b_j^*(t)y_j(t) + \sum\limits_{i=1}^{n} q_{ij}^*(t)g_i(x_i(t - \vartheta_{ij}^*(t))) \\ \qquad +L_j^*(t),\ t \in \mathbb{T},\ j = 1, 2, \cdots, m, \end{cases} \tag{11}$$

According to the almost periodic theory, we can conclude that if system (1) satisfies $(H_1)-(H_3)$, then the hull equation (11) also satisfies $(H_1) - (H_3)$.

For convenience, we write functional differential equation (1) as the following almost periodic functional differential equation

$$z^\nabla(t) = F(t, z_t), \tag{12}$$

where $z = (x_1, \cdots, x_n, y_1, \cdots, y_m)$, $F(t, z_t) \in C(\mathbb{T} \times \Omega, \widetilde{\mathcal{S}})$ is an almost periodic function, and Ω is compact subset of $\mathbb{R}^{n+m}$.

Lemma 5. *If each of hull equation of system (12) has a unique strictly positive solution, then almost periodic system (1) has a unique strictly positive almost periodic solution.*

Proof: Suppose $\varphi(t)$ is a strictly positive solution of system (12) for t on $\mathbb{T}$. There exist sequences of real values $\hat{\alpha}$ and $\hat{\beta}$ which have common subsequence $\alpha \subset \hat{\alpha}$ and $\beta \subset \hat{\beta}$

such that $T_{\alpha+\beta} = T_\alpha T_\beta F(t, z_t)$ for t on $\mathbb{T}$ and $z \in \mathbb{R}^{n+m}$, $T_{\alpha+\beta}\varphi(t)$ and $T_\alpha T_\beta \varphi(t)$ exist uniformly on compact set of $\mathbb{T}$. Then $T_{\alpha+\beta}\varphi(t)$ and $T_\alpha T_\beta\varphi(t)$ are solutions of the following common hull equation of system (12)

$$z^{\nabla}(t) = T_{\alpha+\beta}F(t, z_t).$$

Therefore, we have $T_{\alpha+\beta}\varphi(t) = T_\alpha T_\beta\varphi(t)$ then $\varphi(t)$ is an almost periodic solution of system (12), that is $\varphi(t)$ is an almost periodic solution of system (1). Since $\alpha \subset \hat{\alpha} = \{\hat{\alpha_n}\}$ and $\hat{\alpha_n} \to +\infty$ as $n \to +\infty$, $T_\alpha F(t, z_t) = F(t, z_t)$ is uniformly tenable with respect to t on $\mathbb{T}$ and $z \in \mathbb{R}^{n+m}$. For the sequences $\hat{\alpha}$ and $\alpha \subset \hat{\alpha}$, we conclude that $T_\alpha\varphi(t) = \psi(t)$ is uniformly tenable with respect to t on $\mathbb{T}$ and $\psi(t) \in \mathbb{R}^{n+m}$. According to the uniqueness of solution and $T_\alpha\psi(t) = \psi(t)$ one obtains that $\varphi(t) = \psi(t)$. The proof is completed.

Lemma 6. *Suppose that conditions $(H_1)-(H_3)$ are satisfied, then there exists a bounded solution $z^*(t),\ t \in \mathbb{T}$ of system (1).*

Proof: Since $a_i(t),\ b_j(t),\ p_{ji}(t),\ q_{ij}(t),\ \tau_{ji}(t),\ \vartheta_{ij}(t),\ I_i(t),\ L_j(t)$ are nonnegative almost periodic functions, and with same sequence $\{t_n\}$, as $n \to +\infty$ and $t_n \to +\infty$ for all t on $\mathbb{T}$, and

$$\begin{aligned} &a_i(t+t_n) \to a_i(t),\ b_j(t+t_n) \to b_j(t),\\ &p_{ji}(t+t_n) \to p_{ji}(t),\ q_{ij}(t+t_n) \to q_{ij}(t),\\ &\tau_{ji}(t+t_n) \to \tau_{ji}(t),\ \vartheta_{ij}(t+t_n) \to \vartheta_{ij}(t),\\ &I_i(t+t_n) \to L_j(t),\ a_i(t+t_n) \to L_j(t) \end{aligned}$$

If $z(t)$ is a bounded solution of system (1) for $t \geq 0$ corresponding to the initial condition $\psi(t)$, then $z_n(t) = z(t+t_n)$ for $t \geq t_n$ satisfies

$$\begin{cases} x_{ni}^{\nabla}(t) = -a_i(t+t_n)x_{ni}(t) + \sum\limits_{j=1}^{m} p_{ji}(t+t_n) \\ \qquad\qquad \times f_j(y_{nj}(t-\tau_{ji}(t+t_n))) + I_i(t+t_n), \\ y_{nj}^{\nabla}(t) = -b_j(t+t_n)y_{nj}(t) + \sum\limits_{i=1}^{n} q_{ij}(t+t_n) \\ \qquad\qquad \times g_i(x_{ni}(t-\vartheta_{ij}(t+t_n))) + L_j(t+t_n). \end{cases}$$

Since $z_n(t)$ is bounded uniformly on $[t_n, \infty)_{\mathbb{T}}$, $n = 1, 2, \cdots$, which implies that $z(t+t_n)$ is also bounded uniformly on $[t_n, \infty)_{\mathbb{T}}$, $n = 1, 2, \cdots$. Hence $z_n(t)$ is bounded uniformly and equicontinuous. So, there exists a subsequence $\{t_n^1\}$ of $\{t_n\}$ with $t_n^1 > t_2$ such that $z(t+t_n^1) \to z^1(t)(n \to \infty)$ and $z^1(t)(t \in [-t_1, \infty))$ satisfies system (1). Similarly, proceeding by induction we have subsequence $\{t_n^n\}$ of $\{t_n^{n-1}\}$ such that $z(t+t_n^n) \to z^n(t)(n \to \infty)$ and $z^n(t)(t \in [-t_n, \infty))$ satisfies system (1). According to the diagonal procedure we have $z(t+t_n^n) \to z^*(t)(n \to \infty)$ and $z^n(t)(t \in [-t_n, \infty))$ converges uniformly on any compact set of $\mathbb{R}$, and z^* satisfies system (1).

Theorem 2. *If almost periodic system (1) satisfies $(H_1)-(H_3)$, then almost periodic system (1) has a unique strictly positive almost periodic solution which is globally exponentially stable.*

Proof: By Lemma 5, we only need to prove that each of hull equation of almost periodic system (1) has a unique strictly positive solution, hence we need firstly prove that each of hull equation of almost periodic system (1) has at least a strictly positive solution (the existence), then we further prove that each of hull equation of system (1) has a unique strictly positive solution (the uniqueness).

Now we prove the existence of strictly positive solution of any hull equation (11). According to the almost periodic hull basic theory, there exists a time sequence $\{t_n\}$:

$$\begin{aligned} &a_i(t+t_n) \to a_i^*(t),\ b_j(t+t_n) \to b_j^*(t),\\ &p_{ji}(t+t_n) \to p_{ji}^*(t),\ q_{ij}(t+t_n) \to q_{ij}^*(t),\\ &\tau_{ji}(t+t_n) \to \tau_{ji}^*(t),\ \vartheta_{ij}(t+t_n) \to \vartheta_{ij}^*(t),\\ &I_i(t+t_n) \to L_j^*(t),\ a_i(t+t_n) \to L_j^*(t) \end{aligned}$$

as $n \to +\infty$ and $t_n \to +\infty$ for all t on $\mathbb{T}$. Suppose $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ is any positive solution of hull equation (11). By the proof of Lemma 3, we have

$$0 < \inf_{t\in[0,+\infty)_{\mathbb{T}}} x_i(t) \leq \sup_{t\in[0,+\infty)_{\mathbb{T}}} x_i(t) < +\infty, \quad (13)$$

$$0 < \inf_{t\in[0,+\infty)_{\mathbb{T}}} y_j(t) \leq \sup_{t\in[0,+\infty)_{\mathbb{T}}} y_j(t) < +\infty. \quad (14)$$

Let $z_n(t) = z(t+t_n)$ for all $t \geq -t_n,\ n = 1, 2, \ldots$ such that

$$\begin{cases} x_{ni}^{\nabla}(t) = -a_i(t+t_n)x_{ni}(t) + \sum\limits_{j=1}^{m} p_{ji}(t+t_n) \\ \qquad\qquad \times f_j(y_{nj}(t-\tau_{ji}(t+t_n))) + I_i(t+t_n), \\ y_{nj}^{\nabla}(t) = -b_j(t+t_n)y_{nj}(t) + \sum\limits_{i=1}^{n} q_{ij}(t+t_n) \\ \qquad\qquad \times g_i(x_{ni}(t-\vartheta_{ij}(t+t_n))) + L_j(t+t_n). \end{cases} \quad (15)$$

From inequality (13), (14) and assumptions $(H_1)-(H_3)$, there exists a positive constant vector K which is independent of n such that $z_n^{\nabla}(t) \leq K$ for all $t \geq -t_n,\ i = 1, 2, \ldots$. Therefore, for any positive integer r, sequence $\{z_n(t) : n \geq r\}$ is uniformly bounded and equicontinuous on $[-t_n, +\infty)_{\mathbb{T}}$. According to Ascoli-Arzela Theorem, one concludes that there exists a time subsequence $\{t_k\}$ of $\{t_n\}$ such that sequence $\{z_k(t)\}$ not only converges on t on $\mathbb{T}$, but also converges uniformly on any compact set of $\mathbb{T}$ as $k \to +\infty$. Suppose $\lim\limits_{k\to+\infty} z_k(t) = z^*(t) = (x_1^*(t), \cdots, x_n^*(t), y_1^*(t), \cdots, y_m^*(t))$, then $z^*(t)$ is continuous on $\mathbb{T}$, and we have

$$0 < \inf_{t\in(-\infty,+\infty)_{\mathbb{T}}} x_i^*(t) \leq \sup_{t\in(-\infty,+\infty)_{\mathbb{T}}} x_i^*(t) < +\infty,$$

$$0 < \inf_{t\in(-\infty,+\infty)_{\mathbb{T}}} y_j^*(t) \leq \sup_{t\in(-\infty,+\infty)_{\mathbb{T}}} y_j^*(t) < +\infty.$$

From differential equation (15) and assumptions $(H_1)-(H_3)$, we can easily see that $z^*(t)$ is a solution of hull equation (11), hence each of hull equation of almost periodic system (1) has at least a strictly positive solution.

In the following section, we will prove the uniqueness of strictly positive solution for any hull equation (11). Suppose that the hull equation (11) has two arbitrary strictly positive solutions $z_1^*(t) = (x_1^*(t), \cdots, x_n^*(t), y_1^*(t), \cdots, y_m^*(t))$ and $z_2^*(t) = (\hat{x}_1^*(t), \cdots, \hat{x}_n^*(t), \hat{y}_1^*(t), \cdots,\ \hat{y}_m^*(t))$. Now we define the same Lyapunov functional in section 3, then we can get

$$0 \leq \|z_1^* - z_2^*\| \leq N\|\psi - z_2^*\|\hat{e}_{\ominus\alpha}(t, \delta) \to 0, \text{ as } t \to +\infty.$$

so, it is proved that any hull equation of system (1) has a unique strictly positive solution.

Summarizing the inference above, we know that any hull equation of system (1) has a unique strictly positive solution.

By Lemma 5 and Theorem 1, almost periodic system (1) has a unique strictly positive almost periodic solution which is globally exponentially stable. The proof is completed.

V. Numerical examples

Consider the following BAM neural networks

$$\begin{cases} x_i^{\nabla}(t) = -a_i(t)x_i(t) + \sum_{j=1}^{2} p_{ji}(t)f_j(y_j(t-\tau_{ji}(t))) \\ \qquad +I_i(t),\ t\in\mathbb{T},\ i=1,2, \\ y_j^{\nabla}(t) = -b_j(t)y_j(t) + \sum_{i=1}^{2} q_{ij}(t)g_i(x_i(t-\vartheta_{ij}(t))) \\ \qquad +L_j(t),\ t\in\mathbb{T},\ j=1,2, \end{cases} \tag{16}$$

where $I_i(t) = \sin t + 1$, $L_j(t) = \cos t + 1$, $g_i(x_i(t-\vartheta_{ij}(t))) = \frac{1}{2}\sin(x_i - \tau(t)) + 1$, $f_j(y_j(t-\tau_{ji}(t))) = \cos(y_j - \tau(t)) + 1$, $t \in \mathbb{T}$, $\lambda_i = \frac{1}{2}$, $\eta_j = 1$, $i = j = 1, 2$.

Example 1: $\mathbb{T} = \mathbb{R}$, $\tau(t) = 3|\cos t| + 1$, take

$$a_1(t) = b_2(t) = 3 - \sin t,\ a_2(t) = b_1(t) = 3 - \cos t,$$

then

$$\underline{a}_1 = \underline{a}_2 = \underline{b}_1 = \underline{b}_2 = 2, \bar{a}_1 = \bar{a}_2 = \bar{b}_1 = \bar{b}_2 = 4.$$

Let

$$\begin{aligned} p_{11}(t) &= 0.05\sin t + 1,\ p_{12}(t) = 0.1\cos t + 1, \\ p_{21}(t) &= 0.15\cos t + 1,\ p_{22}(t) = 0.05\sin t + 1, \\ q_{11}(t) &= 0.25\sin t + 1,\ q_{12}(t) = 0.05\cos t + 1, \\ q_{21}(t) &= 0.05\cos t + 1,\ q_{22}(t) = 0.5\sin t + 1. \end{aligned}$$

Then

$$\Upsilon = \begin{bmatrix} 2 & 0 & -1.05 & -1.1 \\ 0 & 2 & -1.15 & -1.05 \\ -0.625 & -0.525 & 2 & 0 \\ -0.525 & -0.75 & 0 & 2 \end{bmatrix}.$$

It is easy to see that the conditions $(H_1)-(H_3)$ hold and Υ is an M-matrix. According to Theorems 1 and 2, system (16) has one unique almost periodic solution, which is globally exponentially stable.

Example 2: $\mathbb{T} = \mathbb{Z}$, $\tau(t) = |\sin t| + 0.5$, take

$$\begin{aligned} a_1(t) &= b_2(t) = 0.5 + 0.01\sin t, \\ a_2(t) &= b_1(t) = 0.5 + 0.01\cos t, \end{aligned}$$

then

$$\underline{a}_1 = \underline{a}_2 = \underline{b}_1 = \underline{b}_2 = 0.4,\ \bar{a}_1 = \bar{a}_2 = \bar{b}_1 = \bar{b}_2 = 0.6.$$

Let

$$\begin{aligned} p_{11}(t) &= 0.05\sin t + 0.2,\ p_{12}(t) = 0.1\cos t + 0.2, \\ p_{21}(t) &= 0.15\cos t + 0.2,\ p_{22}(t) = 0.05\sin t + 0.2 \\ q_{11}(t) &= 0.15\sin t + 0.2,\ q_{12}(t) = 0.05\cos t + 0.2, \\ q_{21}(t) &= 0.05\cos t + 0.2,\ q_{22}(t) = 0.15\sin t + 0.2. \end{aligned}$$

Then

$$\Upsilon = \begin{bmatrix} 0.49 & 0 & -0.25 & -0.3 \\ 0 & 0.49 & -0.35 & -0.25 \\ -0.225 & -0.125 & 0.49 & 0 \\ -0.125 & -0.25 & 0 & 0.49 \end{bmatrix}.$$

It is easy to see that the conditions $(H_1)-(H_3)$ hold and Υ is an M-matrix. According to Theorems 1 and 2, system (16) has one unique almost periodic solution, which is globally exponentially stable.

VI. Conclusion

Two problems for a BAM neural networks with nabla derivatives on time scales have been studied, namely, existence and exponential stability of positive almost periodic solution on time scales. It is important to notice that the methods used in this paper can be extended to other types of neural networks [16-18]. Future work will include neural networks for dynamic system modeling and analysis on time scales.

References

[1] R. Samidurai, R. Sakthivel and S. Anthoni, "Global asymptotic stability of BAM neural networks with mixed delays and impulses," *Appl. Math. Comput.*, vol. 212, pp. 113-119, 2009.

[2] Y. Li, "Global exponential stability of BAM neural networks with delays and impulses," *Chaos, Solitons and Fractals*, vol. 24, pp. 279-285, 2005.

[3] Y. Liu and W. Tang, "Existence and exponential stability of periodic solution for BAM neural networks with periodic coefficients and delays," *Neurocomputing*, vol. 69, 2152-2160, 2006.

[4] M. Gao and B. Cui, "Global robust exponential stability of discrete-time interval BAM neural networks with time-varying delays," *Appl. Math. Model.*, vol. 33, pp. 1270-1284, 2009.

[5] X. Liu, M. Tang, R. Martin and X. Liu, "Discrete-time BAM neural networks with variable delays," *Phys. Lett. A*, vol. 367, pp. 322-330, 2007.

[6] J. Liang, J. Cao and D. Ho, "Discrete-time bidirectional associative memory neural networks with variable delays," *Phys. Lett. A*, vol. 335, pp. 226-234, 2005.

[7] S. Hilger, *Ein Masskettenkalkl mit Anwendug auf Zentrumsmanning-faltigkeiten [Ph.D. thesis]*, Universität Würzburg, 1988.

[8] M. Hu, L. Wang and Z. Wang, "Positive periodic solutions in shifts $\delta_{\pm}$ for a class of higher-dimensional functional dynamic equations with impulses on time scales," *Abstr. Appl. Anal.*, vol. 2014, Article ID 509052.

[9] Y. Li and C. Wang, "Almost periodic functions on time scales and applications," *Discrete Dyn. Nat. Soc.*, vol. 2011, Article ID 727068.

[10] M. Fazly and M. Hesaaraki, "Periodic solutions for predator-prey systems with Beddington-DeAngelis functional response on time scales," *Nonlinear Anal. RWA*, vol. 9, No. 3, pp. 1224-1235, 2008.

[11] M. Hu and L. Wang, "Almost periodic solution of neutral-type neural networks with time delay in the leakage term on time scales," *WSEAS Trans. Syst.*, vol. 13, pp. 231-241, 2014.

[12] B. Du, Y. Liu, H. Batarfi and F. Alsaadi, "Almost periodic solution for a neutral-type neural networks with distributed leakage delays on time scales," *Neurocomputing*, vol. 173, pp. 921-929, 2016.

[13] H. Zhou, Z. Zhou and W. Jiang, "Almost periodic solutions for neutral type BAM neural networks with distributed leakage delays on time scales," *Neurocomputing*, vol. 157, pp. 223-230, 2015.

[14] M. Bohner and A. Peterson, *Dynamic equations on time scales: An Introduction with Applications*. Birkhauser, Boston, 2001.

[15] Y. Li and C. Wang, "Uniformly Almost Periodic Functions and Almost Periodic Solutions to Dynamic Equations on Time Scales," *Abstr. Appl. Anal.*, vol. 2011, Article ID 341520.

[16] J. Liu, Q. Zhang and Z. Luo, "Dynamical analysis of fuzzy cellular neural networks with periodic coefficients and time-varying delays," IAENG International Journal of Applied Mathematics, vol. 46, no. 2, pp. 298-304, 2016.

[17] C. Xu, and P. Li, "Matrix measure strategies for stability of cellular neural networks with proportional delays," IAENG International Journal of Applied Mathematics, vol. 46, no. 3, pp. 305-310, 2016.

[18] L. Yang, Y. Yang, Y. Li and T. Zhang, "Almost periodic solution for a Lotka-Volterra recurrent neural networks with harvesting terms on time scales," Engineering Letters, vol. 24, no. 4, pp. 455-460, 2016.