Dynamics of Two-species Harvesting Model of Almost Periodic Facultative Mutualism with Discrete and Distributed Delays

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Abstract—**By using Mawhins continuous theorem of coincidence degree theory, some new sufficient conditions for the existence of at least four positive almost periodic solutions to a class of two-species harvesting model of facultative mutualism with both discrete and distributed delays are established. Further, by constructing a suitable Lyapunov functional, the local asymptotical stability for the model is also studied. An example is also given to illustrate the main result in this paper.**

Index Terms—**Multiplicity; Almost periodicity; Coincidence degree; Facultative mutualism; Local asymptotical stability.**

I. Introduction

Mutualism, an interaction of two-species of organisms that benefits both (see [1]), is found in many types of communities. Some notable examples can be found in papers [2-6]. Mutualism may be obligate or facultative. An obligate mutualist is a species which requires the presence of another species for its survival, e.g., some species of $Acacia$ require the ant $Pseudomyrmex$ in order to survive (see [7]). A facultative mutualist is one which benefits in some way from the association with another species but will survive in its absence, e.g., blue-green algae can grow and reproduce in the absence of zooplankton grazers, but growth and reproduction are enhanced by the presence of the zooplankton (see [8]). Despite the fact that mutualisms are common in nature, attempts to model such interactions mathematically are somewhat scant in the literature, in other words, in theoretical population biology mutualism has received very little attention compared to that given to predator-prey interactions or competition among species (see [9-13]). Inspired by a delayed single-species population growth model with so-called hereditary effect (see [14-16]) as follows:

$$y'(t) = y(t)\big[r(t) - a(t)y(t) - b(t)y(t-\mu(t))\big],$$

where the net birth rate $r(t)$, the self-inhibition rate $a(t)$, the reproduction rate $b(t)$, and the delay $\mu(t)$ are nonnegative continuous functions. Clearly, such a system involves a positive feedback term $b(t)y(t-\mu(t))$, which is due to gestation (see [17-18]).

In [10], Liu et al. proposed a delayed two-species system modelling "facultative mutualism" as follows:

$$\begin{cases} y_1'(t) = y_1(t)\big[r_1(t) - a_1(t)y_1(t) \\ \qquad\qquad -b_1(t)y_1(t-\mu_1(t)) + c_1(t)y_2(t-\nu_1(t))\big], \\ y_2'(t) = y_2(t)\big[r_2(t) - a_2(t)y_2(t) \\ \qquad\qquad -b_2(t)y_2(t-\mu_2(t)) + c_2(t)y_1(t-\nu_2(t))\big], \end{cases} \tag{1.1}$$

where r_i, a_i, b_i, c_i, μ_i and ν_i are continuous periodic functions, $i = 1, 2$. It is obvious that the type of ecological interaction corresponding to (1.1) is a two-species system of facultative mutualism, that is, each species can persist in the absence of the other, however, each species enhances the average growth rate of the other. From (1.1), we can see that the mutualism increases the average growth rates of two species with two different time delays $\nu_1(t)$ and $\nu_2(t)$. This assumption corresponds to the possible fact that the mutualism has not an effect when they species are infants, and hence they have to mature for some time before they are capable of increasing the average growth rate of two-species. In [10], some sufficient conditions are derived for the existence and globally asymptotic stability of positive periodic solutions of system (1.1), by using Mawhin's continuous theorem and Lyapunov functional.

In fact, time delays often occur and vary in an irregular fashion, and sometimes they may be not continuously differentiable. In addition, if at any time there are always individuals entering the mature population, the delay will be continuously distributed. Moreover, in many earlier studies, it has been shown that harvesting has a strong impact on dynamic evolution of a population, e.g., see [19-22]. So the study of the population dynamics with harvesting is becoming a very important subject in mathematical bio-economics. By considering the continuous distributed delay and harvesting term, the aim of this paper is to consider the following two-species harvesting model of facultative mutualism with both discrete and distributed delays:

$$\begin{cases} y_1'(t) = y_1(t)\big[r_1(t) - a_1(t)y_1(t) \\ \qquad\qquad -b_1(t)\int_{-\mu_1}^{0} k_1(s)y_1(t+s)\,\mathrm{d}s \\ \qquad\qquad +c_1(t)y_2(t-\nu_1(t))\big] - h_1(t), \\ y_2'(t) = y_2(t)\big[r_2(t) - a_2(t)y_2(t) \\ \qquad\qquad -b_2(t)\int_{-\mu_2}^{0} k_2(s)y_2(t+s)\,\mathrm{d}s \\ \qquad\qquad +c_2(t)y_1(t-\nu_2(t))\big] - h_2(t), \end{cases} \tag{1.2}$$

where $h_i\,(i = 1, 2)$ denote harvesting rate, r_i, a_i, b_i, c_i, ν_i and h_i are continuous nonnegative almost periodic functions.

For the last few years, by utilizing Mawhin's continuation theorem of coincidence degree theory, many scholars are concerning with the existence of multiple positive periodic solutions for some non-linear ecosystems with harvesting

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terms, e.g., see [23-27]. However, in real world phenomenon, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic [28-32] since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity in science and engineering [33-36]. Unlike the periodic oscillation, owing to the complexity of the almost periodic oscillation, it is hard to study the existence of positive almost periodic solutions of non-linear ecosystems by using Mawhin's continuation theorem. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning with the multiplicity of positive almost periodic solutions of system (1.2) by using Mawhin's continuation theorem. Stimulated by the above reason, the main purpose of this paper is to establish sufficient conditions for the existence of multiple positive almost periodic solutions to system (1.2) by applying Mawhin's continuation theorem of coincidence degree theory.

If $b_i \equiv 0$ and $\nu_i \equiv 0$, $i = 1, 2$, then system (1.2) reduces to the following form

$$\begin{cases} y_1'(t) &= y_1(t)\big[r_1(t) - a_1(t)y_1(t) \\ &\quad +c_1(t)y_2(t)\big] - h_1(t), \\ y_2'(t) &= y_2(t)\big[r_2(t) - a_2(t)y_2(t) \\ &\quad +c_2(t)y_1(t)\big] - h_2(t). \end{cases} \tag{1.3}$$

By using the continuation theorem of coincidence degree, Hu and Zhang [10] established the existence of four positive periodic solutions for system (1.4).

Related to a continuous function f, we use the following notations:

$$f^l = \inf_{s\in\mathbf{R}} f(s), \quad f^u = \sup_{s\in\mathbf{R}} f(s),$$

$$|f|_\infty = \sup_{s\in\mathbf{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, some sufficient conditions for the existence of at least four positive almost periodic solutions to system (1.2) are obtained by means of Mawhin's continuous theorem of coincidence degree theory. An example is also given to illustrate the main result of this paper.

II. Preliminaries

Mawhin's Continuous Theorem. ([40]) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold*:

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;

(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) $\deg\{JQN, \Omega\cap\mathrm{Ker}L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to \mathrm{Ker}L$ *is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Definition 1. ([39]) Let $x \in C(\mathbf{R}) = C(\mathbf{R}, \mathbf{R})$. x is said to be almost periodic on $\mathbf{R}$, if for $\forall\epsilon > 0$, the set

$$T(x, \epsilon) = \{\tau : |x(t + \tau) - x(t)| < \epsilon, \forall t \in \mathbf{R}\}$$

is relatively dense, i.e., for $\forall\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval length l, there exists a number $\tau = \tau(\epsilon) \in T(x, \epsilon)$ in this interval such that

$$|x(t + \tau) - x(t)| < \epsilon, \ \ \forall t \in \mathbf{R}.$$

τ is called to the ϵ-almost period of x, $T(x, \epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x, \epsilon)$.

Let $AP(\mathbf{R})$ denote the set of all real valued almost periodic functions on $\mathbf{R}$ and

$$AP(\mathbf{R}, \mathbf{R}^n) = \Big\{(x_1, x_2, \ldots, x_n)^T : \\ x_i \in AP(\mathbf{R}), i = 1, 2, \ldots, n, n \in \mathbf{N}^+\Big\}.$$

Lemma 1. ([32]) *Assume that $x \in AP(\mathbf{R}) \cap C^1(\mathbf{R})$ with $x' \in C(\mathbf{R})$, for $\forall\epsilon > 0$, there is a point $\xi_\epsilon \in [0, +\infty)$ such that*

$$x(\xi_\epsilon) \in [x^* - \epsilon, x^*] \quad \textit{and} \quad x'(\xi_\epsilon) = 0.$$

Lemma 2. ([32]) *Assume that $x \in AP(\mathbf{R}) \cap C^1(\mathbf{R})$ with $x' \in C(\mathbf{R})$, for $\forall\epsilon > 0$, there is a point $\eta_\epsilon \in [0, +\infty)$ such that*

$$x(\eta_\epsilon) \in [x_*, x_* + \epsilon] \quad \textit{and} \quad x'(\eta_\epsilon) = 0.$$

III. Multiplicity

Now we are in the position to present and prove our result on the existence of at least four positive almost periodic solutions for system (1.2).

Under the invariant transformation $(y_1, y_2)^T = (e^{x_1}, e^{x_2})^T$, system (1.2) reduces to

$$\begin{cases} x_1'(t) &= r_1(t) - a_1(t)e^{x_1(t)} \\ &\quad -b_1(t)e^{\int_{-\mu_1}^0 k_1(s)x_1(t+s)\,\mathrm{d}s} \\ &\quad +c_1(t)e^{x_2(t-\nu_1(t))} - \frac{h_1(t)}{e^{x_1(t)}}, \\ x_2'(t) &= r_2(t) - a_2(t)e^{x_2(t)} \\ &\quad -b_2(t)e^{\int_{-\mu_2}^0 k_2(s)x_2(t+s)\,\mathrm{d}s} \\ &\quad +c_2(t)e^{x_1(t-\nu_2(t))} - \frac{h_2(t)}{e^{x_2(t)}}. \end{cases} \tag{3.1}$$

For $x \in AP(\mathbf{R})$, we denote by

$$\bar{x} = m(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s,$$

$$a(x, \varpi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)e^{-\mathrm{i}\varpi s}\,\mathrm{d}s,$$

$$\Lambda(x) = \Big\{\varpi \in \mathbf{R} : \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\Big\}$$

the mean value, the Bohr transform and the set of Fourier exponents of x, respectively.

Set $\mathbb{X} = \mathbb{Y} = \mathbf{V}_1 \bigoplus \mathbf{V}_2$, where

$$\mathbf{V}_1 = \Big\{w = (x_1, x_2)^T \in AP(\mathbf{R}, \mathbf{R}^2) : \\ \forall\varpi \in \Lambda(x_1) \cup \Lambda(x_2), |\varpi| \geq \theta_0\Big\},$$

$$\mathbf{V}_2 = \big\{w = (x_1, x_2)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbf{R}\big\},$$

where θ_0 is a given positive constant. Define the norm

$$\|w\| = \max\left\{\sup_{s\in\mathbf{R}}|x_1(s)|, \sup_{s\in\mathbf{R}}|x_2(s)|\right\}, \ \ \forall w \in \mathbb{X} = \mathbb{Y}.$$

Lemma 3. ([32]) *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|$.*

Lemma 4. ([32]) *Let $L : \mathbb{X} \to \mathbb{Y}$, $Lw = L(x_1, x_2)^T = (x_1', x_2')^T$, then L is a Fredholm mapping of index zero.*

Lemma 5. ([32]) *Define $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ by*

$$\begin{aligned} Nw &= N\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} r_1(t) - a_1(t)e^{x_1(t)} - b_1(t)e^{\int_{-\mu_1}^0 k_1(s)x_1(t+s)\,\mathrm{d}s} \\ +c_1(t)e^{x_2(t-\nu_1(t))} - \frac{h_1(t)}{e^{x_1(t)}} \\ r_2(t) - a_2(t)e^{x_2(t)} - b_2(t)e^{\int_{-\mu_2}^0 k_2(s)x_2(t+s)\,\mathrm{d}s} \\ +c_2(t)e^{x_1(t-\nu_2(t))} - \frac{h_2(t)}{e^{x_2(t)}} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} Pw &= P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m(x_1) \\ m(x_2) \end{pmatrix} \\ &= \lim_{T\to\infty}\frac{1}{T}\begin{pmatrix} \int_0^T x_1(s)\,\mathrm{d}s \\ \int_0^T x_2(s)\,\mathrm{d}s \end{pmatrix} = Qw, \ \ \forall w \in \mathbb{X} = \mathbb{Y}. \end{aligned}$$

Then N is L-compact on $\bar{\Omega}$(Ω is an open and bounded subset of $\mathbb{X}$).

Theorem 1. *Assume that*

(H_0) $h_i^l > 0,\ i = 1, 2$
(H_1) $a_1^l a_2^l > c_1^u c_2^u$,
(H_2) $r_1^l - b_1^u e^{\rho_1} + c_1^l e^{\rho_4} > 2\sqrt{a_1^u h_1^u}$, $r_2^l - b_2^u e^{\rho_2} + c_2^l e^{\rho_3} > 2\sqrt{a_2^u h_2^u}$, *where*

$$\rho_1 := \ln\left[\frac{r_1^u a_2^l + c_1^u r_2^u}{a_1^l a_2^l - c_1^u c_2^u}\right], \quad \rho_2 := \ln\left[\frac{r_2^u + c_2^u e^{\rho_1}}{a_2^l}\right],$$

$$\rho_3 := \ln\left[\frac{h_1^l}{r_1^u + c_1^u e^{\rho_2}}\right], \quad \rho_4 := \ln\left[\frac{h_2^l}{r_2^u + c_2^u e^{\rho_1}}\right],$$

then system (1.2) *admits at least four positive almost periodic solutions.*

Proof: It is easy to see that if system (3.1) has one almost periodic solution $(\bar{x}_1, \bar{x}_2)^T$, then $(\bar{y}_1, \bar{y}_2)^T$=$(e^{\bar{x}_1}, e^{\bar{x}_2})^T$ is a positive almost periodic solution of system (1.2). Therefore, to complete the proof it suffices to show that system (3.1) has four almost periodic solutions.

In order to use the Mawhin's continuous theorem, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 3 and L, N, P, Q the same as those defined in Lemmas 4 and 5, respectively. It remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Corresponding to the operator equation $Lw = \lambda w$, $\lambda \in (0, 1)$, we have

$$\begin{cases} x_1'(t) = \lambda\big[r_1(t) - a_1(t)e^{x_1(t)} \\ \qquad -b_1(t)e^{\int_{-\mu_1}^0 k_1(s)x_1(t+s)\,\mathrm{d}s} \\ \qquad +c_1(t)e^{x_2(t-\nu_1(t))} - \frac{h_1(t)}{e^{x_1(t)}}\big], \\ x_2'(t) = \lambda\big[r_2(t) - a_2(t)e^{x_2(t)} \\ \qquad -b_2(t)e^{\int_{-\mu_2}^0 k_2(s)x_2(t+s)\,\mathrm{d}s} \\ \qquad +c_2(t)e^{x_1(t-\nu_2(t))} - \frac{h_2(t)}{e^{x_2(t)}}\big]. \end{cases} \tag{3.2}$$

Suppose that $(x_1, x_2)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.2) for some $\lambda \in (0, 1)$, where $\mathrm{Dom}L = \{x \in \mathbb{X} : x \in C^1(\mathbf{R}), x' \in C(\mathbf{R})\}$. By Lemma 3, for $\forall\epsilon \in (0, 1)$, there are two points $\xi_\epsilon^{(1)}, \xi_\epsilon^{(2)} \in [0, +\infty)$ such that

$$x_1'(\xi_\epsilon^{(1)}) = 0,\ x_1(\xi_\epsilon^{(1)}) \in [x_1^* - \epsilon, x_1^*];$$

$$x_2'(\xi_\epsilon^{(2)}) = 0,\ x_2(\xi_\epsilon^{(2)}) \in [x_2^* - \epsilon, x_2^*], \tag{3.3}$$

where $x_1^* = \sup_{s\in\mathbf{R}} x_1(s)$ and $x_2^* = \sup_{s\in\mathbf{R}} x_2(s)$.

Further, in view of (H_1), we may assume the above ϵ is small enough so that

$$a_1^l a_2^l > e^{2\epsilon} c_1^u c_2^u.$$

From system (3.2), it follows from (3.3) that

$$\begin{aligned} & a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} \\ & +b_1(\xi_\epsilon^{(1)})\int_{-\mu_1}^0 k_1(s)e^{x_1(\xi_\epsilon^{(1)}-s)}\,\mathrm{d}s + \frac{h_1(\xi_\epsilon^{(1)})}{e^{x_1(\xi_\epsilon^{(1)})}} \\ =\ & r_1(\xi_\epsilon^{(1)}) + c_1(\xi_\epsilon^{(1)})e^{x_2(\xi_\epsilon^{(1)}-\nu_1(\xi_\epsilon^{(1)}))}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & a_2(\xi_\epsilon^{(2)})e^{x_2(\xi_\epsilon^{(2)})} \\ & +b_2(\xi_\epsilon^{(2)})\int_{-\mu_2}^0 k_2(s)e^{x_2(\xi_\epsilon^{(1)}-s)}\,\mathrm{d}s + \frac{h_2(\xi_\epsilon^{(2)})}{e^{x_2(\xi_\epsilon^{(2)})}} \\ =\ & r_2(\xi_\epsilon^{(2)}) + c_2(\xi_\epsilon^{(2)})e^{x_1(\xi_\epsilon^{(2)}-\nu_2(\xi_\epsilon^{(2)}))}. \end{aligned} \tag{3.5}$$

In view of (3.4), we have from (3.3) that

$$\begin{aligned} a_1^l e^{x_1^*-\epsilon} &\le a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} \\ &\le r_1(\xi_\epsilon^{(1)}) + c_1(\xi_\epsilon^{(1)})e^{x_2(\xi_\epsilon^{(1)}-\nu_1(\xi_\epsilon^{(1)}))} \\ &\le r_1^u + c_1^u e^{x_2^*}. \end{aligned}$$

That is,

$$a_1^l e^{x_1^*} \le e^\epsilon r_1^u + e^\epsilon c_1^u e^{x_2^*}. \tag{3.6}$$

Similarly, we obtain from (3.5) that

$$a_2^l e^{x_2^*} \le e^\epsilon r_2^u + e^\epsilon c_2^u e^{x_1^*}. \tag{3.7}$$

Substituting (3.7) into (3.6) leads to

$$a_1^l a_2^l e^{x_1^*} \le e^\epsilon r_1^u a_2^l + e^\epsilon c_1^u[e^\epsilon r_2^u + e^\epsilon c_2^u e^{x_1^*}],$$

which implies that

$$\big[a_1^l a_2^l - e^{2\epsilon} c_1^u c_2^u\big]e^{x_1^*} \le e^\epsilon r_1^u a_2^l + e^{2\epsilon} c_1^u r_2^u$$

is equivalent to

$$x_1^* \le \ln\left[\frac{e^\epsilon r_1^u a_2^l + e^{2\epsilon} c_1^u r_2^u}{a_1^l a_2^l - e^{2\epsilon} c_1^u c_2^u}\right].$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$x_1^* \le \ln\left[\frac{r_1^u a_2^l + c_1^u r_2^u}{a_1^l a_2^l - c_1^u c_2^u}\right] := \rho_1. \tag{3.8}$$

Substituting (3.8) into (3.7) leads to

$$a_2^l e^{x_2^*} \le e^\epsilon r_2^u + e^\epsilon c_2^u e^{\rho_1}.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$x_2^* \le \ln\left[\frac{r_2^u + c_2^u e^{\rho_1}}{a_2^l}\right] := \rho_2. \tag{3.9}$$

Also, by Lemma 4, for $\forall\epsilon\in(0,1)$, there are two points $\eta_\epsilon^{(1)},\eta_\epsilon^{(2)}\in[0,+\infty)$ such that

$$x_1'(\eta_\epsilon^{(1)})=0,\ x_1(\eta_\epsilon^{(1)})\in[x_{1*},x_{1*}+\epsilon];$$

$$x_2'(\eta_\epsilon^{(2)})=0,\ x_2(\eta_\epsilon^{(2)})\in[x_{2*},x_{2*}+\epsilon], \tag{3.10}$$

where $x_{1*}=\inf_{s\in\mathbf{R}}x_1(s)$ and $x_{2*}=\inf_{s\in\mathbf{R}}x_2(s)$.

From system (3.2), it follows from (3.10) that

$$\begin{aligned}&a_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)})}\\&+b_1(\eta_\epsilon^{(1)})\int_{-\mu_1}^0 k_1(s)e^{x_1(\eta_\epsilon^{(1)}-s)}\,\mathrm{d}s+\frac{h_1(\eta_\epsilon^{(1)})}{e^{x_1(\eta_\epsilon^{(1)})}}\\=\ &r_1(\eta_\epsilon^{(1)})+c_1(\eta_\epsilon^{(1)})e^{x_2(\eta_\epsilon^{(1)}-\nu_1(\eta_\epsilon^{(1)}))},\end{aligned}\tag{3.11}$$

$$\begin{aligned}&a_2(\eta_\epsilon^{(2)})e^{x_2(\eta_\epsilon^{(2)})}\\&+b_2(\eta_\epsilon^{(2)})\int_{-\mu_2}^0 k_2(s)e^{x_2(\eta_\epsilon^{(1)}-s)}\,\mathrm{d}s+\frac{h_2(\eta_\epsilon^{(2)})}{e^{x_2(\eta_\epsilon^{(2)})}}\\=\ &r_2(\eta_\epsilon^{(2)})+c_2(\eta_\epsilon^{(2)})e^{x_1(\eta_\epsilon^{(2)}-\nu_2(\eta_\epsilon^{(2)}))}.\end{aligned}\tag{3.12}$$

In view of (3.11), we have from (3.8)-(3.10) that

$$\begin{aligned}\frac{h_1^l}{e^{x_{1*}+\epsilon}}&\le\frac{h_1(\eta_\epsilon^{(1)})}{e^{x_1(\eta_\epsilon^{(1)})}}\\&\le r_1(\eta_\epsilon^{(1)})+c_1(\eta_\epsilon^{(1)})e^{x_2(\eta_\epsilon^{(1)}-\nu_1(\eta_\epsilon^{(1)}))}\\&\le r_1^u+c_1^ue^{\rho_2}.\end{aligned}$$

That is,

$$e^{x_{1*}}\ge\frac{h_1^l}{e^\epsilon(r_1^u+c_1^ue^{\rho_2})}$$

is equivalent to

$$x_{1*}\ge\ln\left[\frac{h_1^l}{e^\epsilon(r_1^u+c_1^ue^{\rho_2})}\right].$$

Letting $\epsilon\to0$ in the above inequality leads to

$$x_{1*}\ge\ln\left[\frac{h_1^l}{r_1^u+c_1^ue^{\rho_2}}\right]:=\rho_3. \tag{3.13}$$

By a parallel argument as that in (3.13), we can obtain that

$$x_{2*}\ge\ln\left[\frac{h_2^l}{r_2^u+c_2^ue^{\rho_1}}\right]:=\rho_4. \tag{3.14}$$

In view of (3.4) and (3.11), we have that

$$a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})}+b_1(\xi_\epsilon^{(1)})\int_{-\mu_1}^0 k_1(s)e^{x_1(\xi_\epsilon^{(1)}-s)}\,\mathrm{d}s$$
$$+\frac{h_1(\xi_\epsilon^{(1)})}{e^{x_1(\xi_\epsilon^{(1)})}}\ge r_1^l+c_1^le^{\rho_4},$$
$$a_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)})}+b_1(\eta_\epsilon^{(1)})\int_{-\mu_1}^0 k_1(s)e^{x_1(\eta_\epsilon^{(1)}-s)}\,\mathrm{d}s$$
$$+\frac{h_1(\eta_\epsilon^{(1)})}{e^{x_1(\eta_\epsilon^{(1)})}}\ge r_1^l+c_1^le^{\rho_4},$$

which yield that

$$a_1^ue^{2x_1(\xi_\epsilon^{(1)})}-(r_1^l-b_1^ue^{\rho_1}+c_1^le^{\rho_4})e^{x_1(\xi_\epsilon^{(1)})}+h_1^u\ge0,$$

$$a_1^ue^{2x_1(\eta_\epsilon^{(1)})}-(r_1^l-b_1^ue^{\rho_1}+c_1^le^{\rho_4})e^{x_1(\eta_\epsilon^{(1)})}+h_1^u\ge0,$$

which imply from (H_2) that

$$x_1(\xi_\epsilon^{(1)})\ge\ln\kappa_+^1,\quad x_1(\xi_\epsilon^{(1)})\le\ln\kappa_-^1,$$
$$x_1(\eta_\epsilon^{(1)})\ge\ln\kappa_+^1,\quad x_1(\eta_\epsilon^{(1)})\le\ln\kappa_-^1,$$

where

$$\kappa_\pm^1:=\frac{r_1^l-b_1^ue^{\rho_1}+c_1^le^{\rho_4}\pm2\sqrt{a_1^uh_1^u}}{2a_1^u}.$$

Letting $\epsilon\to0$ in the above inequalities lead to

$$x_1^*\ge\ln\kappa_+^1\quad\text{or}\quad x_1^*\le\ln\kappa_-^1,$$

$$x_{1*}\ge\ln\kappa_+^1\quad\text{or}\quad x_{1*}\le\ln\kappa_-^1. \tag{3.15}$$

By a similar argument as that in (3.15), we obtain that

$$x_2^*\ge\ln\kappa_+^2\quad\text{or}\quad x_2^*\le\ln\kappa_-^2,$$

$$x_{2*}\ge\ln\kappa_+^2\quad\text{or}\quad x_{2*}\le\ln\kappa_-^2, \tag{3.16}$$

where

$$\kappa_\pm^2:=\frac{r_2^l-b_2^ue^{\rho_2}+c_2^le^{\rho_3}\pm2\sqrt{a_2^uh_2^u}}{2a_2^u}.$$

From (3.8)-(3.9) and (3.13)-(3.16), it follows that

$$\rho_3\le x_1(t)\le\ln\kappa_-^1\quad\text{or}\quad\ln\kappa_+^1\le x_1(t)\le\rho_1, \tag{3.17}$$

$$\rho_4\le x_2(t)\le\ln\kappa_-^2\quad\text{or}\quad\ln\kappa_+^2\le x_2(t)\le\rho_2. \tag{3.18}$$

Obviously, $\ln\kappa_\pm^1$, $\ln\kappa_\pm^2$, ρ_1, ρ_2, ρ_3 and ρ_4 are independent of λ. Let $\varepsilon_i=\frac{\ln\kappa_+^i-\ln\kappa_-^i}{4}$ $(i=1,2)$ and

$$\Omega_1=\Big\{w=(x_1,x_2)^T\in\mathbb{X}:1-\rho_3<x_1(t)<\ln\kappa_-^1+\varepsilon_1,$$
$$\rho_4-1<x_2(t)<\ln\kappa_-^2+\varepsilon_2,\forall t\in\mathbf{R}\Big\},$$

$$\Omega_2=\Big\{w=(x_1,x_2)^T\in\mathbb{X}:1-\rho_3<x_1(t)<\ln\kappa_-^1+\varepsilon_1,$$
$$\ln\kappa_+^2-\varepsilon_2<x_2(t)<\rho_2+1,\forall t\in\mathbf{R}\Big\},$$

$$\Omega_3=\Big\{w=(x_1,x_2)^T\in\mathbb{X}:\ln\kappa_+^1-\varepsilon_1<x_1(t)<\rho_1+1,$$
$$\rho_4-1<x_2(t)<\ln\kappa_-^2+\varepsilon_2,\forall t\in\mathbf{R}\Big\},$$

$$\Omega_4=\Big\{w=(x_1,x_2)^T\in\mathbb{X}:\ln\kappa_+^1-\varepsilon_1<x_1(t)<\rho_1+1,$$
$$\ln\kappa_+^2-\varepsilon_2<x_2(t)<\rho_2+1,\forall t\in\mathbf{R}\Big\}.$$

Then Ω_1, Ω_2, Ω_3 and Ω_4 are bounded open subsets of $\mathbb{X}$, $\Omega_i\cap\Omega_j=\emptyset$, $i\ne j$, $i,j=1,2,3,4$. Therefore, Ω_1, Ω_2, Ω_3 and Ω_4 satisfy condition (a) of Mawhin's continuous theorem.

Now we show that condition (b) of Mawhin's continuous theorem holds, i.e., we prove that $QNw \neq 0$ for all $w = (x_1, x_2)^T \in \partial\Omega_i \cap \text{Ker}L = \partial\Omega_i \cap \mathbf{R}^2$, $i = 1, 2, 3, 4$. If it is not true, then there exists at least one constant vector $w_0 = (x_1^0, x_2^0)^T \in \partial\Omega_i$ $(i = 1, 2, 3, 4)$ such that

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^{x_1^0} - \bar{b}_1 e^{x_1^0} + \bar{c}_1 e^{x_2^0} - \frac{\bar{h}_1}{e^{x_1^0}} = 0, \\ \bar{r}_2 - \bar{a}_2 e^{x_2^0} - \bar{b}_2 e^{x_2^0} + \bar{c}_2 e^{x_1^0} - \frac{\bar{h}_2}{e^{x_2^0}} = 0. \end{cases}$$

Similar to the argument as that in (3.17)-(3.18), it follows that

$$\rho_3 \leq x_1^0 \leq \ln \kappa_-^1 \quad \text{or} \quad \ln \kappa_+^1 \leq x_1^0 \leq \rho_1,$$
$$\rho_4 \leq x_2^0 \leq \ln \kappa_-^2 \quad \text{or} \quad \ln \kappa_+^2 \leq x_2^0 \leq \rho_2.$$

Then $w_0 \in \Omega_1 \cap \mathbf{R}^2$ or $w_0 \in \Omega_2 \cap \mathbf{R}^2$ or $w_0 \in \Omega_3 \cap \mathbf{R}^2$ or $w_0 \in \Omega_4 \cap \mathbf{R}^2$. This contradicts the fact that $w_0 \in \partial\Omega_i$ $(i = 1, 2, 3, 4)$. This proves that condition (b) of Mawhin's continuous theorem holds.

Finally, we will show that condition (c) of Mawhin's continuous theorem is satisfied. Let us consider the homotopy

$$H(\iota, w) = \iota QNw + (1 - \iota)\Phi w, \ \ (\iota, w) \in [0, 1] \times \mathbf{R}^2,$$

where

$$\Phi w = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \bar{r}_1 - \bar{a}_1 e^{x_1} + \bar{c}_1 e^{\rho_2} - \frac{\bar{h}_1}{e^{x_1}} \\ \bar{r}_2 - \bar{a}_2 e^{x_2} + \bar{c}_2 e^{\rho_1} - \frac{\bar{h}_2}{e^{x_2}} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, w) \neq 0$ on $\partial\Omega_i \cap \text{Ker}L$, $\forall \iota \in [0, 1]$, $i = 1, 2, 3, 4$. Further, $\Phi w = 0$ has four distinct solutions:

$$(x_1^*, x_2^*)^T = (\ln x_1^-, \ln x_2^-)^T, \ (y_1^*, y_2^*)^T = (\ln x_1^+, \ln x_2^+)^T,$$
$$(u_1^*, u_2^*)^T = (\ln x_1^+, \ln x_2^-)^T, \ (v_1^*, v_2^*)^T = (\ln x_1^-, \ln x_2^+)^T,$$

where

$$x_1^{\pm} = \frac{\bar{r}_1 + \bar{c}_1 e^{\rho_2} \pm \sqrt{(\bar{r}_1 + \bar{c}_1 e^{\rho_2})^2 - 4\bar{a}_1 \bar{h}_1}}{2\bar{a}_1},$$
$$x_2^{\pm} = \frac{\bar{r}_2 + \bar{c}_2 e^{\rho_1} \pm \sqrt{(\bar{r}_2 + \bar{c}_2 e^{\rho_1})^2 - 4\bar{a}_2 \bar{h}_2}}{2\bar{a}_2}.$$

It is easy to verify that

$$\rho_3 < \ln x_1^- < \ln \kappa_1^- < \ln \kappa_1^+ < \ln x_1^+ < \rho_1,$$
$$\rho_4 < \ln x_2^- < \ln \kappa_2^- < \ln \kappa_2^+ < \ln x_2^+ < \rho_2.$$

Therefore

$$(x_1^*, x_2^*)^T \in \Omega_1, \quad (v_1^*, v_2^*)^T \in \Omega_2,$$
$$(u_1^*, u_2^*)^T \in \Omega_3, \quad (y_1^*, y_2^*)^T \in \Omega_4.$$

A direct computation yields

$$\begin{aligned} &\deg\big(\Phi, \Omega_i \cap \text{Ker}L, 0\big) \\ &= \text{sign} \begin{vmatrix} -\bar{a}_1 e^{x_1^*} + \frac{\bar{h}_1}{e^{x_1^*}} & 0 \\ 0 & -\bar{a}_2 e^{x_2^*} + \frac{\bar{h}_2}{e^{x_2^*}} \end{vmatrix}_{\Phi(x_1^*, x_2^*)^T = 0} \\ &= \text{sign}\Bigg[\bigg(-\bar{a}_1 e^{x_1^*} + \frac{\bar{h}_1}{e^{x_1^*}}\bigg) \\ &\quad \times \bigg(-\bar{a}_2 e^{x_2^*} + \frac{\bar{h}_2}{e^{x_2^*}}\bigg)\Bigg]_{\Phi(x_1^*, x_2^*)^T = 0} \\ &= \text{sign}\Bigg[\bigg(\bar{r}_1 - 2\bar{a}_1 e^{x_1} + \bar{c}_1 e^{\rho_2}\bigg) \\ &\quad \times \bigg(\bar{r}_2 - 2\bar{a}_2 e^{x_2} + \bar{c}_2 e^{\rho_1}\bigg)\Bigg]_{\Phi(x_1^*, x_2^*)^T = 0}, \end{aligned}$$

where $i = 1, 2, 3, 4$. Thus

$$\deg\big(\Phi, \Omega_1 \cap \text{Ker}L, 0\big) = \text{sign}\Bigg[\bigg(\bar{r}_1 - 2\bar{a}_1 x_1^- + \bar{c}_1 e^{\rho_2}\bigg) \times \bigg(\bar{r}_2 - 2\bar{a}_2 x_2^- + \bar{c}_2 e^{\rho_1}\bigg)\Bigg] = 1,$$

$$\deg\big(\Phi, \Omega_2 \cap \text{Ker}L, 0\big) = \text{sign}\Bigg[\bigg(\bar{r}_1 - 2\bar{a}_1 x_1^- + \bar{c}_1 e^{\rho_2}\bigg) \times \bigg(\bar{r}_2 - 2\bar{a}_2 x_2^+ + \bar{c}_2 e^{\rho_1}\bigg)\Bigg] = -1,$$

$$\deg\big(\Phi, \Omega_3 \cap \text{Ker}L, 0\big) = \text{sign}\Bigg[\bigg(\bar{r}_1 - 2\bar{a}_1 x_1^+ + \bar{c}_1 e^{\rho_2}\bigg) \times \bigg(\bar{r}_2 - 2\bar{a}_2 x_2^- + \bar{c}_2 e^{\rho_1}\bigg)\Bigg] = -1,$$

$$\deg\big(\Phi, \Omega_4 \cap \text{Ker}L, 0\big) = \text{sign}\Bigg[\bigg(\bar{r}_1 - 2\bar{a}_1 x_1^+ + \bar{c}_1 e^{\rho_2}\bigg) \times \bigg(\bar{r}_2 - 2\bar{a}_2 x_2^+ + \bar{c}_2 e^{\rho_1}\bigg)\Bigg] = 1.$$

By the invariance property of homotopy, we have

$$\begin{aligned} \deg\big(JQN, \Omega_i \cap \text{Ker}L, 0\big) &= \deg\big(QN, \Omega_i \cap \text{Ker}L, 0\big) \\ &= \deg\big(\Phi, \Omega_i \cap \text{Ker}L, 0\big) \neq 0, \quad i = 1, 2, 3, 4, \end{aligned}$$

where $\deg(\cdot, \cdot, \cdot)$ is the Brouwer degree and J is the identity mapping since $\text{Im}Q = \text{Ker}L$. Obviously, all the conditions of Mawhin's continuous theorem are satisfied. Therefore, system (3.1) has four almost periodic solutions, that is, system (1.2) has at least four positive almost periodic solutions. This completes the proof. ■

From the proof of Theorem 3.1, we can show that

Corollary 1. *Assume that* (H_0)-(H_2) *hold. Suppose further that* r_i, a_i, b_i, c_i, ν_i *and* h_i *of system* (1.2) *are continuous nonnegative periodic functions with different periods,* $i = 1, 2$, *then system* (1.2) *admits at least four positive almost periodic solutions.*

Corollary 2. *Assume that* (H_0)-(H_2) *hold. Suppose further that* r_i, a_i, b_i, c_i, ν_i *and* h_i *of system* (1.2) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2$, *then system* (1.2) *admits at least four positive* ω*-periodic solutions.*

Remark 1. In system (1.2), when $b_i \equiv 0$ and $\nu_i \equiv 0$, $i = 1, 2$, Hu and Zhang [10] obtained Corollary 2, but they couldn't obtain Corollary 1. Therefore, the main result in this paper extends their work in article [10].

IV. LOCAL ASYMPTOTICAL STABILITY

In this section, we will construct some suitable Lyapunov functions to study the local asymptotical stability of system (1.2).

Theorem 2. *Assume that*

(H_3) *there exist two positive constants* y_1^* *and* y_2^* *such that* $\Theta = \min_{i=1,2}\{r_i^- - 2\bar{a}_i y_i^* - \bar{b}_i y_i^* - \bar{c}_i y_i^*\} > 0.$

Then system (1.2) *is locally asymptotically stable.*

Proof: Let $\mathbb{B} := \{y = (y_1, y_2)^T \in \mathbf{R}^2 : 0 \le y_i \le y_i^*, i = 1, 2\}$. Assume that $y(t) = (y_1(t), y_2(t))^T \in \mathbb{B}$ and $y^*(t) = (y_1^*(t), y_2^*(t))^T \in \mathbb{B}$ are any two solutions of system (1.2). In view of system (1.2), for $t \in \mathbf{R}^+$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
&(y_1(t) - y_1^*(t))' \\
= \ & r_1(t)\big[y_1(t) - y_1^*(t)\big] \\
& -a_1(t)\big[y_1(t) + y_1^*(t)\big]\big[y_1(t) - y_1^*(t)\big] \\
& -b_1(t)y_1(t)\int_{-\mu_1}^{0} k_1(s)\big[y_1(t+s) - y_1^*(t+s)\big]\,\mathrm{d}s \\
& -b_1(t)\int_{-\mu_1}^{0} k_1(s)y_1^*(t+s)\,\mathrm{d}s\big[y_1(t) - y_1^*(t)\big] \\
& +c_1(t)y_2^*(t-\nu_1)\big[y_1(t) - y_1^*(t)\big] \\
& +c_1(t)y_1^*(t)\big[y_2(t-\nu_1) - y_2^*(t-\nu_1)\big] \\
\ge \ & r_1(t)\big[y_1(t) - y_1^*(t)\big] \\
& -\bar{a}_1 2y_1^*\big|y_1(t) - y_1^*(t)\big| \\
& -\bar{b}_1 y_1^*\int_{-\mu_1}^{0} k_1(s)\big|y_1(t+s) - y_1^*(t+s)\big|\,\mathrm{d}s \\
& -\bar{b}_1 y_1^*\big|y_1(t) - y_1^*(t)\big| \\
& -\bar{c}_1(t)y_2^*\big|y_1(t) - y_1^*(t)\big| \\
& -\bar{c}_1 y_1^*\big|y_2(t-\nu_1) - y_2^*(t-\nu_1)\big|,
\end{aligned}$$

similarly,

$$\begin{aligned}
&(y_2(t) - y_2^*(t))' \\
\ge \ & r_2(t)\big[y_2(t) - y_2^*(t)\big] \\
& -\bar{a}_2 2y_2^*\big|y_2(t) - y_2^*(t)\big| \\
& -\bar{b}_2 y_2^*\int_{-\mu_2}^{0} k_2(s)\big|y_2(t+s) - y_2^*(t+s)\big|\,\mathrm{d}s \\
& -\bar{b}_2 y_2^*\big|y_2(t) - y_2^*(t)\big| \\
& -\bar{c}_2(t)y_1^*\big|y_2(t) - y_2^*(t)\big| \\
& -\bar{c}_2 y_2^*\big|y_1(t-\nu_2) - y_1^*(t-\nu_2)\big|.
\end{aligned}$$

Let
$$V(t) = V_1(t) - V_2(t) - V_3(t),$$
where
$$V_1(t) = \sum_{i=1}^{2}|y_i(t) - y_i^*(t)|,$$
$$V_2(t) = \sum_{i=1}^{2}\bar{b}_i y_i^*\int_{-\mu_i}^{0}\int_{t+s}^{t} k_i(l-s)\big|y_i(l) - y_i^*(l)\big|\,\mathrm{d}l\,\mathrm{d}s,$$
$$V_3(t) = \sum_{i=1}^{2}\bar{c}_i y_i^*\int_{t-\nu_i}^{t}\big|y_{3-i}(s) - y_{3-i}^*(s)\big|\,\mathrm{d}s.$$

Hence we can obtain from (H_7)-(H_9) that

$$\begin{aligned}
&D^+V(t) \\
\ge \ & \sum_{i=1}^{2}(r_i^- - 2\bar{a}_i y_i^* - \bar{b}_i y_i^* - \bar{c}_i y_i^*)|y_i(t) - y_i^*(t)| \\
= \ & \Theta\sum_{i=1}^{2}|y_i(t) - y_i^*(t)|.
\end{aligned}$$

Integrating the last inequality from T_0 to t leads to
$$V(T_0) + \Theta\sum_{i=1}^{2}\int_{T_0}^{t}|y_i(s) - y_i^*(s)|\,\Delta s \le V(t) < +\infty,$$
that is,
$$\sum_{i=1}^{2}\int_{T_0}^{+\infty}|y_i(s) - y_i^*(s)|\,\Delta s < +\infty,$$
which implies that
$$\sum_{i=1}^{2}\lim_{s\to+\infty}|y_i(s) - y_i^*(s)| = 0.$$
Thus, system (1.2) is locally asymptotically stable. This completes the proof. ■

Theorem 3. *Assume that (H_1)-(H_3) hold. Then the unique almost periodic solution of system (1.2) is locally asymptotically stable.*

V. An example

Example 1. Consider the following almost periodic model:

$$\begin{cases}
y_1'(t) = y_1(t)\big[1 - a_1(t)y_1(t) \\
\qquad -b_1(t)\int_{-\ln 2}^{0} 2e^s y_1(t+s)\,\mathrm{d}s \\
\qquad +y_2(t-2)\big] - 0.05, \\
y_2'(t) = y_2(t)\big[1 - a_2(t)y_2(t) \\
\qquad -b_2(t)\int_{-\ln 3}^{0} \frac{3}{2}e^s y_2(t+s)\,\mathrm{d}s \\
\qquad +y_1(t-\cos^2 t)\big] - 0.069,
\end{cases} \tag{5.1}$$

where
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 + 0.5|\cos\sqrt{3}t| \\ 2.3 \end{pmatrix},$$
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.02 + 0.01\cos^2(\sqrt{2}t) \end{pmatrix}.$$

Obviously, $r_1^l = r_1^u = r_2^l = r_2^u = 1$, $c_1^l = c_1^u = c_2^l = c_2^u = 1$, $a_1^l = 1$, $a_1^u = 1.5$, $a_2^l = a_2^u = 2.3$, $b_1^l = b_1^u = 0.02$, $b_2^l = 0.02$, $b_2^u = 0.03$, $h_1^l = h_1^u = 0.05$, $h_2^l = h_2^u = 0.069$. It is easy to see that
$$a_1^l a_2^l = 2.3 > 1 = c_1^u c_2^u.$$
So (H_1) holds. By a easy calculation, we can obtain
$$e^{\rho_1} \approx 3.31, \quad e^{\rho_2} \approx 1.87, \quad e^{\rho_3} \approx 0.02, \quad e^{\rho_4} \approx 0.02.$$
Then
$$r_1^l - b_1^u e^{\rho_1} + c_1^l e^{\rho_4} = 0.9538 > 2\sqrt{0.075} = 2\sqrt{a_1^u h_1^u},$$
$$r_2^l - b_2^u e^{\rho_2} + c_2^l e^{\rho_3} = 0.9639 > 2\sqrt{0.1587} = 2\sqrt{a_2^u h_2^u},$$
which imply that (H_2) holds. Therefore, all the conditions in Theorem 1 are satisfied. By Theorem 1, system (4.1) admits at least four positive almost periodic solutions.

Remark 2. Clearly, system (4.1) is with incommensurable periods. Through all the coefficients of system (4.1) are periodic functions, the positive periodic solutions could not possibly exist. However, by the work in this paper, the positive almost periodic solutions of system (4.1) exactly exist.

VI. Conclusions

By using a fixed point theorem of coincidence degree theory, some criterions for the multiplicity of positive almost periodic solution to a kind of two-species harvesting model of facultative mutualism with both discrete and distributed delays are obtained. Theorems 1 gives the sufficient conditions for the multiplicity of positive almost periodic solution of system (1.2). Theorems 2 gives the sufficient conditions for the local asymptotical stability of system (1.2). The results obtained in this paper are new and different from the results in [10, 12]. Therefore, The method used in this paper provides a possible method to study the multiplicity of positive almost periodic solution and the local asymptotical stability of the models in biological populations.

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