Balance Analysis of Two Layers Coupled Public Traffic Network with Dual Time-varying Delays

Wenju Du, Jiangang Zhang, Yinzhen Li and Jianning Yu

Abstract—**The paper established the transfer network and bus line network by respectively using space P and space R model method. Then regard these two networks as the sub-networks, this paper presented a new two layers coupled public traffic network through the connection between the bus stops and bus lines, and this model can well reflect the connection between the passengers and bus operating vehicles. Based on the synchronization theory of coupled network with dual time-varying delays and taking "Lorenz system" as the network node, the paper studied the stability of two layers coupled public traffic network. Finally, the numerical results are given to show the impact of public traffic dispatching, delayed departure, the number of common bus stops between bus lines and the accessibility between bus stops to the two layers coupled public traffic network balance.**

Index Terms—**Coupled complex network synchronization; Dual time-varying delays; Two layers coupled public traffic network; Network Balance**

I. Introduction

In real life, there exist multifarious complex systems and these systems can be abstracted as the complex network. Many scholars have studied the complex network in the past few years, and the synchronization of complex network is a very important research content [1-8]. However, these subjects just focused on synchronization of single network, but the studies of synchronization between two different networks are not many. Li et al. [9] investigated two unidirectionally coupled networks and derive analytically a criterion for the synchronization of these two networks. Tang et al. [10] designed the effective adaptive controllers and addresses the theoretical analysis of synchronization between two complex networks with nonidentical topological structures. Chen et al. [11] presented a general network model for two complex networks with time-varying delay coupling and derived a synchronization criterion by using adaptive controllers. Sun et al. [12] investigated the linear generalized synchronization between two complex networks. Wang et al. [13] designed an adaptive controller to achieve synchronization between two different complex networks with time-varying delay coupling. Sun et al. [14] studied the outer synchronization between two complex networks with discontinuous coupling and obtained the sufficient conditions for complete outer synchronization and generalized outer synchronization. Due to the existence of time delay is inevitable in real life and many synchronization phenomenon affected by the time delay, so the study of synchronization problem of the coupled network with time-varying delay is particularly important.

Complex network can describe the communication network, power network, traffic network and social network, etc., and the study of complex network has penetrated the various fields. At present, the researches on complex network are mostly limited to the single network, but the majority of complex network are not exist in isolation in reality. The single network is only a sub-network of the whole complex network, and the sub-networks with different structure and function constitute a coupled network through some coupling effect. In recent years, many scholars have begun to study the two layer network. Gu et al. [15] proposed a model of traffic dynamics and revealed a transition at the onset of cooperation between layered networks, and Li et al. [16] presented the study on the cooperation onset in five real world two layers networks. Wang et al. [17] established an optimization model of urban public transit skeleton-network, and designed the genetic algorithm and the tabu search algorithm. Based on betweenness analytical method, Shen et al. [18] investigated the cascading failure for double layer complex networks. Chen et al. [19] presented some of works in the complex network modeling, analyzing and optimization which are related with spatially embedded networks, multiple-layer coupled networks and public bus transportation networks. Luo et al. [20] establishes a compound network of subway and bus transport networks in the Space L and Space P, and the topology characteristics of the compound network are compared with its sub-networks. Du et al. [21] proposed a new multi-coupling-links scale-free coupled network model, and the cascading failure of multi-coupling-links three-dimensional coupled network based on coupled map lattices model is investigated.

At present, there are a series of problems in the development of urban traffic in China, such as passenger travel difficulty and the increase of traffic time cost. Traffic congestion in big cities is becoming more and more serious, and priority to the development of urban public transportation can improve road utilization and solve urban congestion. It is not only the key to improve urban traffic, but also directly related to the sustainable development of the city. Therefore, priority to develop the public transport has become the main means to ease urban traffic congestion and improve the efficiency of urban travel [22]. Urban public

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traffic network is a typical complex network, and as one of important research tool, complex network have been widely applied to urban traffic system [23-25]. However, most of the studies only focus on the static statistical characteristics of the public traffic network, such as the study of the topological properties, reliability, robustness and structural optimization of the network. There are few articles to investigate the dynamic characteristics of urban public traffic network. Due to the urban public traffic network have its own characteristics, so it is necessary to investigate its dynamic characteristics. Therefore, the paper focus on a coupled complex network with dual time-varying delays, and design an adaptive controller to make the two networks achieve synchronization based on LaSalle invariable principle. And combined with the characteristics of public traffic network, the paper constructed a new two layers coupled public traffic network with dual time-varying delay. The balance problem of this network is investigated by using the synchronization theory of coupled complex network with dual time-varying delays.

The paper organizes as follows. The synchronization theory of two different networks with dual time-varying delays is presented in section 2. In section 3, a new two layers coupled public traffic network model is established. The simulation results are given to show the impact of various factors in urban public traffic network to the two layers coupled public traffic network balance in section 4. In section 5, we conclude the paper.

II. Synchronization between two different networks with dual time-varying delays

Consider two networks with time-varying delays, and they both consisting of same nodes can be described by

$$\begin{aligned}\dot{x}_i(t) = {} & f\left(x_i(t)\right)+\varepsilon_1\sum_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}_1 x_j\left(t-\tau_1(t)\right)+ \\ & \mu\sum_{j=1}^{N_2}c_{ij}y_j\left(t-\tau_2(t)\right),\quad i=1,2,\ldots,N_1\end{aligned}\tag{1}$$

$$\begin{aligned}\dot{y}_i(t) = {} & g\left(y_i(t)\right)+\varepsilon_2\sum_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}_2 y_j\left(t-\tau_2(t)\right)+ \\ & \mu\sum_{j=1}^{N_1}d_{ij}x_j\left(t-\tau_1(t)\right)+u_i,\quad i=1,2,\ldots,N_2\end{aligned}\tag{2}$$

where $x_i=\left(x_{i1},x_{i2},\ldots,x_{in}\right)^{\mathrm{T}}\in R^n, y_i=\left(y_{i1},y_{i2},\ldots,y_{in}\right)^{\mathrm{T}}\in R^n$ are the state vectors of the i th node of networks (1) and (2), $\dot{x}_i(t),\dot{y}_i(t)$ are the dynamic equation for a single node, $\tau_1(t),\tau_2(t)$ are two different time-varying coupling delays, $f(\cdot),g(\cdot):R^n\to R^n$ are the nonlinear continuous differentiable vector functions, N_1,N_2 are the number of nodes of networks (1) and (2), $\boldsymbol{\Gamma}_1,\ \boldsymbol{\Gamma}_2\in R^{n\times n}$ are the constant matrix of linking the coupled variables. The constant $\varepsilon_1,\varepsilon_2$ denotes the inner coupling strength of networks (1) and (2), respectively; the constant μ denotes the outer coupling strength. Coupling matrices $A=(a_{ij})\in R^{N_1\times N_1}, B=(b_{ij})\in R^{N_2\times N_2}$ are respectively the inner connection matrices of the driving network (1) and response network (2), where $a_{ii}=-\sum_{j=1,i\neq j}^{N_1}a_{ij}, b_{ii}=-\sum_{j=1,i\neq j}^{N_2}b_{ij}$ and $a_{ij}(b_{ij})$ are defined as follows: if there is a connection from node j to node i $(i\neq j)$, then $a_{ij}(b_{ij})>0(i\neq j)$; otherwise $a_{ij}(b_{ij})=0(i\neq j)$. The matrices $C=(c_{ij})\in R^{N_1\times N_2}, D=(d_{ij})\in R^{N_2\times N_1}$ are the coupling matrixes between two networks, where c_{ij},d_{ij} are defined as follows: if there is a connection from node i (belongs to network (1)) to node j (belongs to network (2)), then $c_{ij}>0$; otherwise $c_{ij}=0$; if there is a connection from node i (belongs to network (2)) to node j (belongs to network (1)), then $d_{ij}>0$; otherwise $d_{ij}=0$, $u_i(t)$ is the controller of node i to be designed according to the specific network structures A and B . Without loss of generality, assuming that $N_1>N_2$, that is networks (1) and (2) has the different number of nodes.

Definition 1. Let $x_i(t,X_0)(i=1,2,\ldots,N_1)$ and $y_i(t,Y_0,u_i)$ $(i=1,2,\ldots,N_2)$ be the solutions of the networks (1) and (2), where $X_0=\left(x_1^0,x_2^0,\ldots,x_{N_1}^0\right)^{\mathrm{T}}\in R^{nN_1}, Y_0=\left(y_1^0,y_2^0,\ldots,y_{N_2}^0\right)^{\mathrm{T}}\in R^{nN_2}$, and $f,g:\Omega\to R^n$ are the continuously differentiable mappings with $\Omega\subseteq R^n$. If there is a nonempty open subset $\Lambda\subseteq\Omega$, with $x_i^0,y_i^0\in\Lambda$, so when $t\geq 0$, such that $x_i(t,X_0)(1\leq i\leq N_1)$, $y_i(t,Y_0,u_i)(1\leq i\leq N_2)\in\Omega$, and

$$\lim_{t\to\infty}\left\|y_i(t,Y_0,u_i)-x_i(t,X_0)\right\|=0,\quad (i=1,2,\cdots,N_2)\tag{3}$$

then the complex networks (1) and (2) are said to realize synchronization.

Assumption 1. For function $f(x)$ there exists a positive constant L such that

$$\left\|f\left(y(t)\right)-f\left(x(t)\right)\right\|\leq L\left\|y(t)-x(t)\right\|,\tag{4}$$

where $\forall x(t),y(t)\in R^n$.

Assumption 2. $\tau_1(t),\tau_2(t)$ are the differential functions with $0\leq\dot{\tau}_1(t)\leq\xi_1<1,\ 0\leq\dot{\tau}_2(t)\leq\xi_2<1.$ Obviously, this assumption includes constant time delay as a special case.

Lemma 1. For arbitrary $x,y\in R^n,\ \eta>0,\ 2x^{\mathrm{T}}y\leq\eta x^{\mathrm{T}}x+\frac{1}{\eta}y^{\mathrm{T}}y$ is established.

Theorem 1. Suppose that Assumptions 1, 2 holds. We select the controllers as follows:

$$\begin{aligned}u_i = {} & f\left(y_i(t)\right)-g\left(y_i(t)\right)+\varepsilon_1\sum_{j=1}^{N_2}a_{ij}\boldsymbol{\Gamma}_1 y_j\left(t-\tau_1(t)\right) \\ & -\varepsilon_2\sum_{j=1}^{N_1}b_{ij}\boldsymbol{\Gamma}_2 x_j\left(t-\tau_2(t)\right)+\mu\sum_{j=1}^{N_1}c_{ij}x_j\left(t-\tau_2(t)\right) \\ & -\mu\sum_{j=1}^{N_2}d_{ij}y_j\left(t-\tau_1(t)\right)+\varepsilon_1\sum_{j=N_2+1}^{N_1}a_{ij}\boldsymbol{\Gamma}_1 x_j\left(t-\tau_1(t)\right) \\ & +\varepsilon_2\sum_{j=N_2+1}^{N_1}b_{ij}\boldsymbol{\Gamma}_2 x_j\left(t-\tau_2(t)\right)-\mu\sum_{j=N_2+1}^{N_1}c_{ij}x_j\left(t-\tau_2(t)\right) \\ & -\mu\sum_{j=N_2+1}^{N_1}d_{ij}x_j\left(t-\tau_1(t)\right)-g_i e_i(t),\end{aligned}\tag{5}$$

then the driving network (1) and the response network (2) can realize synchronization under the controllers (5), where $\dot{g}_i = k_i \|e_i\|^2$, k_i is a positive constant, $i = 1, 2, \ldots, N_2$.

Proof. Define the errors vector by $e_i(t) = y_i(t) - x_i(t)$, $i = 1, 2, \ldots, N_2$, and the error systems can be described by:

$$
\begin{aligned}
\dot{e}_i(t) = {} & f(y_i(t)) - f(x_i(t)) + \varepsilon_1 \sum_{j=1}^{N_2} a_{ij} \boldsymbol{\Gamma}_1 e_j(t-\tau_1(t)) \\
& + \varepsilon_2 \sum_{j=1}^{N_2} b_{ij} \boldsymbol{\Gamma}_2 e_j(t-\tau_2(t)) - \mu \sum_{j=1}^{N_2} c_{ij} e_j(t-\tau_2(t)) \\
& - \mu \sum_{j=1}^{N_2} d_{ij} e_j(t-\tau_1(t)) - g_i e_i(t),
\end{aligned} \tag{6}
$$

Choose the Lyapunov-Krasovskii candidate as follows:

$$
\begin{aligned}
V(t) = {} & \frac{1}{2}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) + \frac{1}{2}\sum_{i=1}^{N_2} \frac{1}{k_i}(g_i - \bar{g})^2 \\
& + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} \int_{t-\tau_1(t)}^{t} e_i^{\mathrm{T}}(\alpha) e_i(\alpha)\, d\alpha \\
& + \frac{1}{2(1-\xi_2)} \sum_{i=1}^{N_2} \int_{t-\tau_2(t)}^{t} e_i^{\mathrm{T}}(\beta) e_i(\beta)\, d\beta,
\end{aligned} \tag{7}
$$

where $\bar{g}$ is a sufficiently larger positive constant which is to be determined. Using Assumptions 1, 2 and Lemma 1, we can get the following formula by derivation of Eq. (7):

$$
\begin{aligned}
\dot{V}(t) = {} & \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) \dot{e}_i(t) + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) - \\
& \frac{1-\dot{\tau}_1(t)}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau_1(t)) e_i(t-\tau_1(t)) + \frac{1}{2(1-\xi_2)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) \\
& - \frac{1-\dot{\tau}_2(t)}{2(1-\xi_2)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau_2(t)) e_i(t-\tau_2(t)) + \sum_{i=1}^{N_2} \frac{1}{k_i}(g_i - \bar{g})\dot{g}_i \\
= {} & \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) \left[f(y_i(t)) - f(x_i(t)) - g_i e_i(t) \right] + \sum_{i=1}^{N_2} (g_i - \bar{g}) \|e_i(t)\|^2 + \\
& \varepsilon_1 \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) a_{ij} \boldsymbol{\Gamma}_1 e_j(t-\tau_1(t)) + \varepsilon_2 \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) b_{ij} \boldsymbol{\Gamma}_2 e_j(t-\tau_2(t)) \\
& - \mu \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) c_{ij} e_j(t-\tau_2(t)) - \mu \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) d_{ij} e_j(t-\tau_1(t)) \\
& + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) - \frac{1-\dot{\tau}_1(t)}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau_1(t)) e_i(t-\tau_1(t)) \\
\le {} & \sum_{i=1}^{N_2} (L - \bar{g}) \|e_i(t)\|^2 + \varepsilon_1 \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) a_{ij} \boldsymbol{\Gamma}_1 e_j(t-\tau_1(t)) \\
& + \varepsilon_2 \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) b_{ij} \boldsymbol{\Gamma}_2 e_j(t-\tau_2(t)) - \mu \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) c_{ij} e_j(t-\tau_2(t)) \\
& - \mu \sum_{i=1}^{N_2} \sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t) d_{ij} e_j(t-\tau_1(t)) + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) \\
& - \frac{1-\dot{\tau}_1(t)}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau_1(t)) e_i(t-\tau_1(t)) + \frac{1}{2(1-\xi_2)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) \\
& - \frac{1-\dot{\tau}_2(t)}{2(1-\xi_2)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t-\tau_2(t)) e_i(t-\tau_2(t)) \\
= {} & \sum_{i=1}^{N_2} (L - \bar{g}) \|e_i(t)\|^2 + \varepsilon_1 \sum_{j=1}^{n} e_j^{\mathrm{T}}(t) \gamma_{1j} A' e_j(t-\tau_1(t)) \\
& + \varepsilon_2 \sum_{j=1}^{n} e_j^{\mathrm{T}}(t) \gamma_{2j} B e_j(t-\tau_2(t)) - \mu \sum_{j=1}^{n} e_j^{\mathrm{T}}(t) C' e_j(t-\tau_2(t)) \\
& - \mu \sum_{j=1}^{n} e_j^{\mathrm{T}}(t) D' e_j(t-\tau_1(t)) + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) - \\
& \frac{1-\dot{\tau}_1(t)}{2(1-\xi_1)} \sum_{j=1}^{n} e_j^{\mathrm{T}}(t-\tau_1(t)) e_j(t-\tau_1(t)) + \frac{1}{2(1-\xi_2)} \times \\
& \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) - \frac{1-\dot{\tau}_2(t)}{2(1-\xi_2)} \sum_{j=1}^{n} e_j^{\mathrm{T}}(t-\tau_2(t)) e_j(t-\tau_2(t)) \\
\le {} & \sum_{i=1}^{N_2} (L - \bar{g}) \|e_i(t)\|^2 + \sum_{j=1}^{n} \frac{\gamma_{1j}^2}{2} e_j^{\mathrm{T}}(t) (\varepsilon_1 A')(\varepsilon_1 A')^{\mathrm{T}} e_j(t) + \\
& \sum_{j=1}^{n} \frac{\gamma_{2j}^2}{2} e_j^{\mathrm{T}}(t) (\varepsilon_2 B)(\varepsilon_2 B)^{\mathrm{T}} e_j(t) - \sum_{j=1}^{n} \frac{1}{2} e_j^{\mathrm{T}}(t) (\mu C')(\mu C')^{\mathrm{T}} e_j(t) \\
& - \sum_{j=1}^{n} \frac{1}{2} e_j^{\mathrm{T}}(t) (\mu D')(\mu D')^{\mathrm{T}} e_j(t) + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) \\
& + \left(\frac{1}{2} - \frac{1-\dot{\tau}_1(t)}{2(1-\xi_1)} \right) \sum_{j=1}^{n} e_j^{\mathrm{T}}(t-\tau_1(t)) e_j(t-\tau_1(t)) + \frac{1}{2(1-\xi_2)} \\
& \times \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) + \left(\frac{1}{2} - \frac{1-\dot{\tau}_2(t)}{2(1-\xi_2)} \right) \sum_{j=1}^{n} e_j^{\mathrm{T}}(t-\tau_2(t)) e_j(t-\tau_2(t)) \\
\le {} & \sum_{i=1}^{N_2} (L - \bar{g}) \|e_i(t)\|^2 + \sum_{j=1}^{n} \frac{\gamma_{1j}^2}{2} e_j^{\mathrm{T}}(t) (\varepsilon_1 A')(\varepsilon_1 A')^{\mathrm{T}} e_j(t) \\
& + \sum_{j=1}^{n} \frac{\gamma_{2j}^2}{2} e_j^{\mathrm{T}}(t) (\varepsilon_2 B)(\varepsilon_2 B)^{\mathrm{T}} e_j(t) - \sum_{j=1}^{n} \frac{1}{2} e_j^{\mathrm{T}}(t) (\mu C')(\mu C')^{\mathrm{T}} e_j(t) \\
& - \sum_{j=1}^{n} \frac{1}{2} e_j^{\mathrm{T}}(t) (\mu D')(\mu D')^{\mathrm{T}} e_j(t) + \frac{1}{2(1-\xi_1)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) \\
& + \frac{1}{2(1-\xi_2)} \sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t) e_i(t) \\
\le {} & \sum_{i=1}^{N_2} \left(L - \bar{g} + \frac{1}{2(1-\xi_1)} + \frac{1}{2(1-\xi_2)} \right) \|e_i(t)\|^2 + \gamma_j \sum_{j=1}^{n} e_j^{\mathrm{T}}(t) P e_j(t) \\
\le {} & e^{\mathrm{T}}(t) \left\{ \left[L - \bar{g} + \frac{1}{2(1-\xi_1)} + \frac{1}{2(1-\xi_2)} + \max_{1 \le j \le n}(\gamma_j) \lambda_{\max}(P) \right] I_{nN_2} \right\} \\
& \times e(t) = \boldsymbol{e}^{\mathrm{T}}(t) Q \boldsymbol{e}(t),
\end{aligned}
$$

where $\boldsymbol{e}(t) = \left(e_1(t), e_2(t), \ldots, e_{N_2}(t)\right)^{\mathrm{T}} \in R^{nN_2}$, γ_{1j}, γ_{2j} are the j th diagonal elements of $\boldsymbol{\Gamma_1}$, $\boldsymbol{\Gamma_2}$, respectively. And

$$
Q = \left[L - \bar{g} + \frac{1}{2(1-\xi_1)} + \frac{1}{2(1-\xi_2)} + \max_{1 \le j \le n}(\gamma_j) \lambda_{\max}(P) \right] I_{nN_2},
$$

$\gamma_j = \max\left\{ \dfrac{\gamma_{1j}^2}{2}, \dfrac{\gamma_{2j}^2}{2}, \dfrac{1}{2} \right\}$, $P = (\varepsilon_1 A')(\varepsilon_1 A')^{\mathrm{T}} + (\varepsilon_2 B)(\varepsilon_2 B)^{\mathrm{T}} + (\mu C')(\mu C')^{\mathrm{T}} + (\mu D')(\mu D')^{\mathrm{T}}$, $\lambda_{\max}(P)$ is the largest eigenvalue of matrix P, I_{nN_2} is the unit $n \times N_2$ matrix, A', C', D' are the N_2 order principal minor determinant of matrixes A, C, D, respectively. Obviously, there exist a sufficiently large positive constant $\bar{g}$ such that the symmetry matrix Q is negative definite, that is $\dot{V}(t) < 0$. Here, the largest invariant set contained in set $E = \{\dot{V}(t) = 0\} = \{\mathbf{e}(t) = 0, i = 1, 2, \ldots, N_2\}$ can be described as

$$
M = \left\{ (\mathbf{e}, \mathbf{g}) \in R^{nN_2} \times R^{N_2} : \mathbf{e} = 0, \dot{\mathbf{g}} = 0 \right\},
$$

where $\mathbf{g} = \left(g_1, g_2, \ldots, g_{N_2}\right)^{\mathrm{T}}$. According to the LaSalle invariance principle [26], starting with arbitrary initial values, the trajectory asymptotically converges to the largest invariant M which means $\lim_{t \to \infty} e_i(t) = 0, i = 1, 2, \ldots, N_2$, so the networks (1) and (2) realized synchronization.

III. A NEW TWO LAYERS COUPLED PUBLIC TRAFFIC NETWORK MODEL

Urban public traffic network is a complex network which composed of different bus stops and lines, and there are mainly three kinds of modeling methods to construct the urban public traffic network: space L modeling method, space P modeling method, and space R modeling method [27, 28]. Using space L modeling method to construct the public traffic stops network model, and taking the bus stop as the network's nodes, if they are adjacent in a bus line then the two bus stops have edge. Using space P modeling method to construct the public traffic transfer network model, and also taking the bus stop as the network's nodes, if there are direct bus lines between two bus stops then this two bus stops have edge. Using space R modeling method to construct the public traffic roads network model, and taking the bus lines as the network's nodes, if there are the same bus stops between two bus lines then this two bus lines have edge.

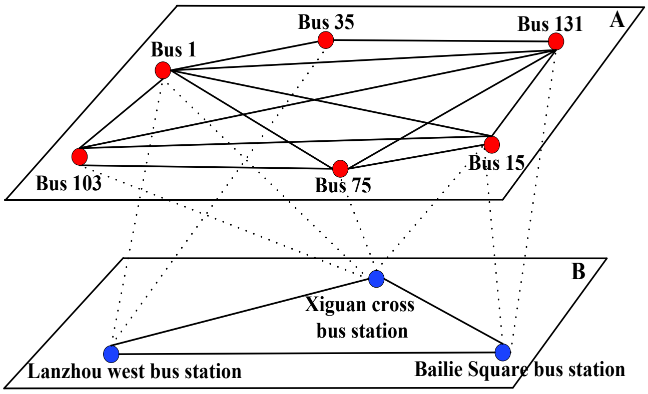

Fig. 1. The topology map of two layers coupled public traffic network

This paper presented a new two layers coupled public traffic network model, and the specific modeling method described as follows:

(1) Firstly, taking bus stops as the network's node, then establish the transfer network B based on the space P modeling method. It mainly reflects the accessibility and convenience of the urban public traffic network. Then select the bus lines which pass the stops of network B and taking these bus lines as the network's node, and then establish the bus lines network A based on the space R modeling method. The weight of edge of network B defined as the number of direct buses between two stops, it reflects the connectivity between this two stops. And the weight of edge of network A defined as the number of common bus stops between bus lines.

(2) If one bus line of network A passing a bus stop of network B, we link their corresponding nodes and constitute the coupling edges of two layers coupled public traffic network. The coupling edges reflects the connection between bus lines and bus stops, namely, a coupling edge represent one bus line can park in a bus stop. The bus lines network A, transfer network B and its coupling edges formed the two layers coupled public traffic network.

Without loss of generality, taking three public traffic hub stops (Lanzhou west bus station, Xiguan cross bus station and Bailie Square bus station) and six bus lines (bus no.1, 15, 35, 75, 103, 131) at Lanzhou as the network's nodes, we established a new two layers coupled public traffic network model as show in Fig. 1.

IV. BALANCE ANALYSIS OF TWO LAYERS COUPLED PUBLIC TRAFFIC NETWORK

For the two layers coupled public traffic network, we taking the bus lines network A and transfer network B as the two sub-networks, then investigate the balance problem of two layers coupled public traffic network by using the above synchronization theory. According to references [29], we know that the passenger flow of urban public traffic fulfills the nonlinear behavior. And through the analysis of global public traffic network, we can get the urban public traffic network has the characteristics of BA scale-free networks. Suppose that the passenger flow fulfills the Lorenz chaotic system, that is, the nodes dynamical equations can be described as follows:

$$\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{bmatrix}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -8/3\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{bmatrix}+\begin{bmatrix}0\\ -x_{i1}x_{i3}\\ x_{i1}x_{i2}\end{bmatrix}, \tag{8}$$

$$\begin{bmatrix}\dot{y}_{i1}\\ \dot{y}_{i2}\\ \dot{y}_{i3}\end{bmatrix}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -8/3\end{bmatrix}\begin{bmatrix}y_{i1}\\ y_{i2}\\ y_{i3}\end{bmatrix}+\begin{bmatrix}0\\ -y_{i1}y_{i3}\\ y_{i1}y_{i2}\end{bmatrix}, \tag{9}$$

Assuming that $\boldsymbol{\Gamma}_1=\boldsymbol{\Gamma}_2=\mathrm{diag}\{1,1,1\}$ and the bus lines network A and transfer network B have the same node dynamical, and then the controllers can be designed as follows:

$$\begin{aligned}u_i={}&\varepsilon_1\sum_{j=1}^{3}a_{ij}y_j\left(t-\tau_1(t)\right)-\varepsilon_2\sum_{j=1}^{6}b_{ij}x_j\left(t-\tau_2(t)\right)\\&+\mu\sum_{j=1}^{6}c_{ij}x_j\left(t-\tau_2(t)\right)-\mu\sum_{j=1}^{3}d_{ij}y_j\left(t-\tau_1(t)\right)\\&+\varepsilon_1\sum_{j=4}^{6}a_{ij}x_j\left(t-\tau_1(t)\right)+\varepsilon_2\sum_{j=4}^{6}b_{ij}x_j\left(t-\tau_2(t)\right)\\&-\mu\sum_{j=4}^{6}c_{ij}x_j\left(t-\tau_2(t)\right)-\mu\sum_{j=4}^{6}d_{ij}x_j\left(t-\tau_1(t)\right)\\&-g_ie_i(t),\end{aligned} \tag{10}$$

where $\dot{g}_i=k_i\left\|e_i\right\|^2$, k_i is a positive constant, $i=1,2,3$, and

$$\begin{aligned}&a_{12}=3,\ a_{13}=1,\ a_{14}=2,\ a_{15}=2,\ a_{16}=4,\ a_{23}=0,\\&a_{24}=1,\ a_{25}=2, a_{26}=4, a_{34}=0,\ a_{35}=0,\ a_{36}=6,\\&a_{45}=1,\ a_{46}=1,\ a_{56}=6,\ a_{ji}=a_{ij}(i\neq j,i,j=1,2,\\&\ldots,6),\ a_{ii}=-\sum_{i=1,i\neq j}^{6}a_{ij}(i,j=1,2,\ldots,6),\end{aligned} \tag{11}$$

$$\begin{aligned}&b_{11}=-3,\ b_{12}=2,\ b_{13}=1,\ b_{21}=2,\\&b_{22}=-8,\ b_{23}=6,\ b_{31}=1,\ b_{32}=6,\ b_{33}=-7,\end{aligned} \tag{12}$$

$$\begin{aligned}&c_{11}=1,\ c_{12}=1,\ c_{13}=0,\ c_{21}=1,\ c_{22}=0,\ c_{23}=1,\\&c_{31}=0,\ c_{32}=1,\ c_{33}=0,\ c_{41}=1,\ c_{42}=0,\ c_{43}=0,\\&c_{51}=1,\ c_{52}=0,\ c_{53}=0,\ c_{61}=0,\ c_{62}=0,\ c_{63}=1,\\&d_{ji}=c_{ij}(i=1,2,\ldots,6,\ j=1,2,3)\end{aligned} \tag{13}$$

According to Eq. (1), the dynamical equations of bus lines network A for each node $i(1\le i\le 6)$ can be described by

$$\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{bmatrix}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -8/3\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{bmatrix}+\begin{bmatrix}0\\ -x_{i1}x_{i3}\\ x_{i1}x_{i2}\end{bmatrix}+\begin{bmatrix}M_{i1}\\ M_{i2}\\ M_{i3}\end{bmatrix}, \tag{14}$$

$$M_{1j}=\varepsilon_1\big(a_{11}x_{1j}(t-\tau_1(t))+a_{12}x_{2j}(t-\tau_1(t))+a_{13}x_{3j}(t-\tau_1(t))+a_{14}x_{4j}(t-\tau_1(t))+a_{15}x_{5j}(t-\tau_1(t))+a_{16}x_{6j}(t-\tau_1(t))\big)+\mu\big(c_{11}y_{1j}(t-\tau_2(t))+c_{12}y_{2j}(t-\tau_2(t))+c_{13}y_{3j}(t-\tau_2(t))\big),\ j=1,2,3$$

$$M_{2j}=\varepsilon_1\big(a_{21}x_{1j}(t-\tau_1(t))+a_{22}x_{2j}(t-\tau_1(t))+a_{23}x_{3j}(t-\tau_1(t))+a_{24}x_{4j}(t-\tau_1(t))+a_{25}x_{5j}(t-\tau_1(t))+a_{26}x_{6j}(t-\tau_1(t))\big)+\mu\big(c_{21}y_{1j}(t-\tau_2(t))+c_{22}y_{2j}(t-\tau_2(t))+c_{23}y_{3j}(t-\tau_2(t))\big),\ j=1,2,3$$

$$M_{3j}=\varepsilon_1\big(a_{31}x_{1j}(t-\tau_1(t))+a_{32}x_{2j}(t-\tau_1(t))+a_{33}x_{3j}(t-\tau_1(t))+a_{34}x_{4j}(t-\tau_1(t))+a_{35}x_{5j}(t-\tau_1(t))+a_{36}x_{6j}(t-\tau_1(t))\big)+\mu\big(c_{31}y_{1j}(t-\tau_2(t))+c_{32}y_{2j}(t-\tau_2(t))+c_{33}y_{3j}(t-\tau_2(t))\big),\ j=1,2,3$$

$$M_{4j}=\varepsilon_1\big(a_{41}x_{1j}(t-\tau_1(t))+a_{42}x_{2j}(t-\tau_1(t))+a_{43}x_{3j}(t-\tau_1(t))+a_{44}x_{4j}(t-\tau_1(t))+a_{45}x_{5j}(t-\tau_1(t))+a_{46}x_{6j}(t-\tau_1(t))\big)+\mu\big(c_{41}y_{1j}(t-\tau_2(t))+c_{42}y_{2j}(t-\tau_2(t))+c_{43}y_{3j}(t-\tau_2(t))\big),\ j=1,2,3$$

$$M_{5j}=\varepsilon_1\big(a_{51}x_{1j}(t-\tau_1(t))+a_{52}x_{2j}(t-\tau_1(t))+a_{53}x_{3j}(t-\tau_1(t))+a_{54}x_{4j}(t-\tau_1(t))+a_{55}x_{5j}(t-\tau_1(t))+a_{56}x_{6j}(t-\tau_1(t))\big)+\mu\big(c_{51}y_{1j}(t-\tau_2(t))+c_{52}y_{2j}(t-\tau_2(t))+c_{53}y_{3j}(t-\tau_2(t))\big),\ j=1,2,3$$

$$M_{6j}=\varepsilon_1\big(a_{61}x_{1j}(t-\tau_1(t))+a_{62}x_{2j}(t-\tau_1(t))+a_{63}x_{3j}(t-\tau_1(t))+a_{64}x_{4j}(t-\tau_1(t))+a_{65}x_{5j}(t-\tau_1(t))+a_{66}x_{6j}(t-\tau_1(t))\big)+\mu\big(c_{61}y_{1j}(t-\tau_2(t))+c_{62}y_{2j}(t-\tau_2(t))+c_{63}y_{3j}(t-\tau_2(t))\big),\ j=1,2,3$$

And based on Eq. (2), the dynamical equations of transfer network B for each node $i(1\le i\le 3)$, can be described by

$$\begin{bmatrix}\dot{y}_{i1}\\ \dot{y}_{i2}\\ \dot{y}_{i3}\end{bmatrix}=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -8/3\end{bmatrix}\begin{bmatrix}y_{i1}\\ y_{i2}\\ y_{i3}\end{bmatrix}+\begin{bmatrix}0\\ -y_{i1}y_{i3}\\ y_{i1}y_{i2}\end{bmatrix}+\begin{bmatrix}N_{i1}\\ N_{i2}\\ N_{i3}\end{bmatrix}+u_i,\quad(15)$$

$$N_{1j}=\varepsilon_2\big(b_{11}y_{1j}(t-\tau_2(t))+b_{12}y_{2j}(t-\tau_2(t))+b_{13}y_{3j}(t-\tau_2(t))\big)+\mu\big(d_{11}x_{1j}(t-\tau_1(t))+d_{12}x_{2j}(t-\tau_1(t))+d_{13}x_{3j}(t-\tau_1(t))+d_{14}x_{4j}(t-\tau_1(t))+d_{15}x_{5j}(t-\tau_1(t))+d_{16}x_{6j}(t-\tau_1(t))\big),\ j=1,2,3$$

$$N_{2j}=\varepsilon_2\big(b_{21}y_{1j}(t-\tau_2(t))+b_{22}y_{2j}(t-\tau_2(t))+b_{23}y_{3j}(t-\tau_2(t))\big)+\mu\big(d_{21}x_{1j}(t-\tau_1(t))+d_{22}x_{2j}(t-\tau_1(t))+d_{23}x_{3j}(t-\tau_1(t))+d_{24}x_{4j}(t-\tau_1(t))+d_{25}x_{5j}(t-\tau_1(t))+d_{26}x_{6j}(t-\tau_1(t))\big),\ j=1,2,3$$

$$N_{3j}=\varepsilon_2\big(b_{31}y_{1j}(t-\tau_2(t))+b_{32}y_{2j}(t-\tau_2(t))+b_{33}y_{3j}(t-\tau_2(t))\big)+\mu\big(d_{31}x_{1j}(t-\tau_1(t))+d_{32}x_{2j}(t-\tau_1(t))+d_{33}x_{3j}(t-\tau_1(t))+d_{34}x_{4j}(t-\tau_1(t))+d_{35}x_{5j}(t-\tau_1(t))+d_{36}x_{6j}(t-\tau_1(t))\big),\ j=1,2,3$$

For any vectors x_i and y_i of the Lorenz chaotic system, there exists a positive constant R such that $\|x_{im}\|\le R, \|y_{im}\|\le R\,(m=1,2,3)$, since the Lorenz chaotic system is bounded in a certain region. So, we have

$$\begin{aligned}\|f(y_i)-f(x_i)\|&=\sqrt{(-y_{i1}y_{i3}-(-x_{i1}x_{i3}))^2+(y_{i1}y_{i2}-x_{i1}x_{i2})^2}\\&=\sqrt{(-y_{i3}(y_{i1}-x_{i1})-x_{i1}(y_{i3}-x_{i3}))^2+(y_{i2}(y_{i1}-x_{i1})+x_{i1}(y_{i2}-x_{i2}))^2}\\&\le\sqrt{2}R\|y_i-x_i\|,\end{aligned}\quad(16)$$

namely, Assumption 1 is satisfied. If we select the proper $\tau_1(t),\tau_2(t)$ such that Assumption 2 is satisfied, and according to Theorem 1, the bus lines network A and transfer network B achieved synchronization under the controllers (10), namely, the whole two layers coupled public traffic network reached a balance state.

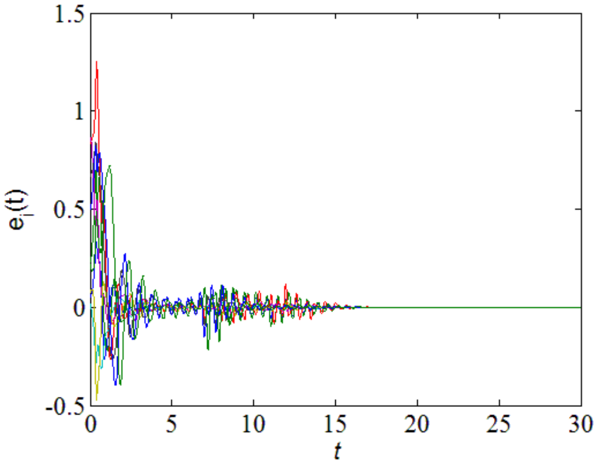

Fig. 2. Synchronization errors for two layers coupled public traffic network with $\varepsilon_1=\varepsilon_2=0.2, \mu=0.3$

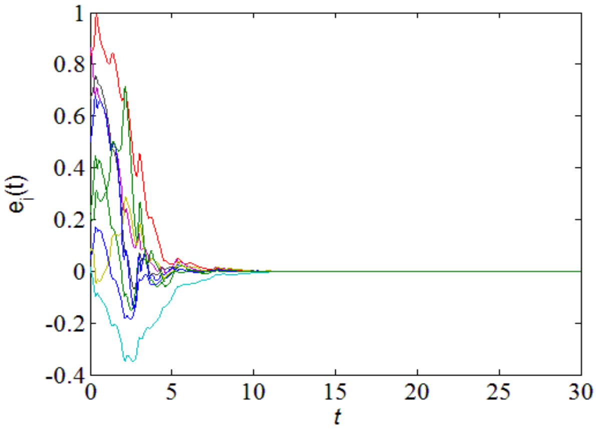

Fig. 3. Synchronization errors for two layers coupled public traffic network with $\varepsilon_1=\varepsilon_2=0.3, \mu=0.4$

The balance of urban public traffic network is refers to a dynamic balance between operating vehicles and passengers, namely, the operating time of public traffic vehicles is most close to the preset time (the traffic delay is shortest), at the same time the residence time for passengers at the traffic station is shortest. In this section, we will investigate the impact of public traffic dispatching, delayed departure, the number of common bus stops between bus lines and the accessibility between bus stops, namely, the coupling strength $\varepsilon_1, \varepsilon_2, \mu$, time-varying delays $\tau_1(t), \tau_2(t)$, the size of the edge weight of network A and the size of the edge weights of network B on the balance of whole two layers coupled public traffic network balance. We let $k_i = 10 (1 \le i \le 3)$ in the all numerical simulation process, and randomly chose the initial values of the network's node from $(0,1)$, then use Matlab to the simulation.

Fixed $\tau_1(t) = \tau_2(t) = 0.03$, and plot the synchronization errors for the two layers coupled public traffic network with different value of the coupling strength, are show in Fig. 2 and Fig. 3. From the simulation results, the two layers coupled public traffic network achieves balance in 17 time units when $\varepsilon_1 = \varepsilon_2 = 0.2, \mu = 0.3$, and the two layers coupled public traffic network achieves balance in 11 time units when $\varepsilon_1 = \varepsilon_2 = 0.3, \mu = 0.4$. Compared with Fig. 2 the synchronization time of Fig. 3 reduced 6 time units. Namely, the greater the value of coupling strengths, the shorter the time required to balance the two layers coupled public traffic network. And this shows that increase the artificial scheduling (appropriate adjust the departing frequency and time and optimize transfer facilities) can speed up the whole network's synchronization, and the two layers coupled public traffic network can reach steady state faster.

Fixed $\varepsilon_1 = \varepsilon_2 = 0.2, \mu = 0.3$, and the synchronization errors for the two layers coupled public traffic network with $\tau_1(t) = \tau_2(t) = 0.05$, is show in Fig. 4. The Fig. 4 shows that the two layers coupled public traffic network achieves balance in 25 time units, and compared with Fig. 2 the synchronization time increased 8 time units. This suggests that the bus delays or traffic jams caused by weather or man-made factors can make the passengers stranded time extended, so the time for the network reaches balanced have delayed.

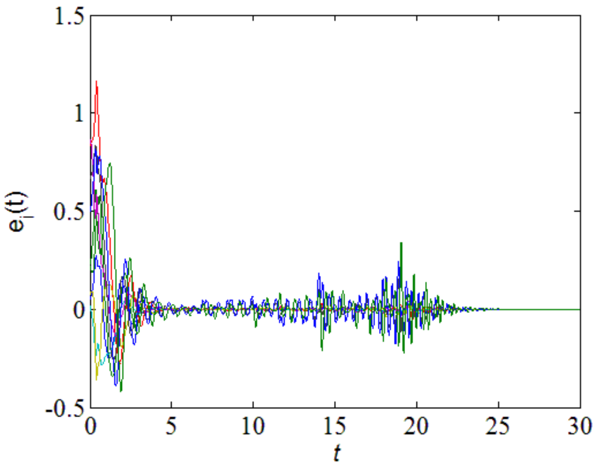

Fig. 4. Synchronization errors for two layers coupled public traffic network with $\tau_1(t) = \tau_2(t) = 0.05$

Fixed $\varepsilon_1 = \varepsilon_2 = 0.2, \mu = 0.3, \tau_1(t) = \tau_2(t) = 0.03$, and if the edge weights of bus lines network A are changed to

$$\begin{aligned} & a_{12} = 4,\ a_{13} = 2,\ a_{14} = 3,\ a_{15} = 3,\ a_{16} = 5,\ a_{23} = 0, \\ & a_{24} = 2,\ a_{25} = 3,\ a_{26} = 5,\ a_{34} = 0,\ a_{35} = 0,\ a_{36} = 7, \\ & a_{45} = 2,\ a_{46} = 2,\ a_{56} = 7,\ a_{ji} = a_{ij} (i \ne j, i, j = 1, 2, \\ & \ldots, 6),\ a_{ii} = -\sum_{i=1, i\ne j}^{6} a_{ij} (i, j = 1, 2, \ldots, 6), \end{aligned} \tag{17}$$

namely, increase the common bus stops between some bus lines, the synchronization errors is show in Fig. 5. Clearly, the two layers public traffic coupled network achieves balance in 15 time units, and compared with Fig. 2 the synchronization time shortens 2 time units. This shows that the more public bus stops between bus lines the more vehicles for the passengers to transfer, the passengers stranded time have shortened and the faster the whole network reach balance.

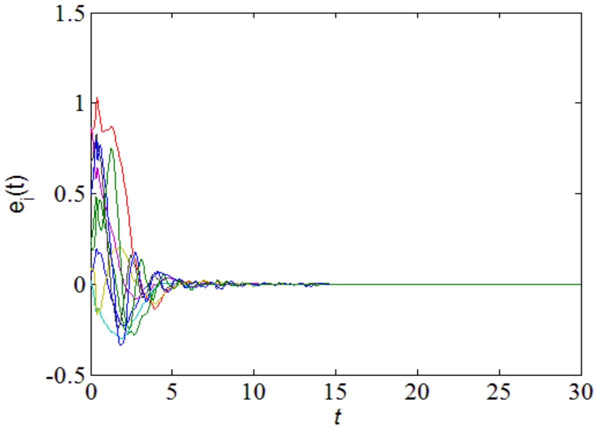

Fig. 5. Synchronization errors for two layers coupled public traffic network after increase the common bus stops between two bus lines

Fixed $\varepsilon_1 = \varepsilon_2 = 0.2, \mu = 0.3, \tau_1(t) = \tau_2(t) = 0.03$, and if the edge weights of bus transfer network B are changed to

$$\begin{aligned} & b_{11} = -5,\ b_{12} = 3,\ b_{13} = 2,\ b_{21} = 3, \\ & b_{22} = -10,\ b_{23} = 7,\ b_{31} = 2,\ b_{32} = 7,\ b_{33} = -9, \end{aligned} \tag{18}$$

namely, increase the direct buses between two stops, the synchronization errors is show in Fig. 6. The results show that the two layers public traffic coupled network achieves balance in 12 time units, and compared with Fig. 2 the synchronization time shortens 5 time units. This indicated that the more direct buses between two stops, the better the accessibility between bus stops, and the whole networks can reach balance faster.

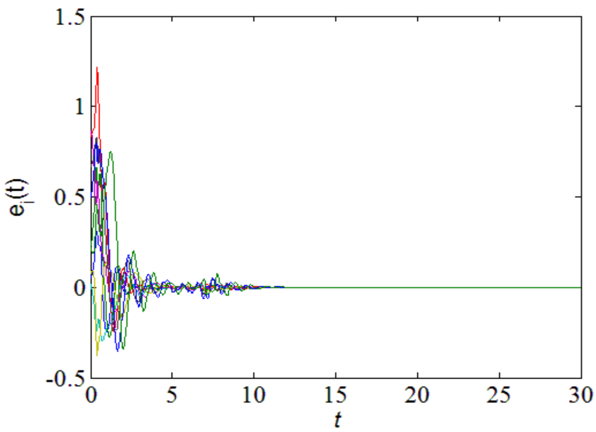

Fig. 6. Synchronization errors for two layers coupled public traffic network after increase the direct bus between two stops

V. Conclusions

The paper first designed an adaptive controller for the coupled complex network with dual time-varying delays, and which makes the two different networks achieved synchronization. Then using space P and space R modeling methods, the paper constructed a new two layers coupled public traffic network and then investigated its balance problem. Through the numerical simulations the impact of public traffic dispatching, delayed departure, the number of common bus stops between bus lines and the accessibility between bus stops to the two layers coupled public traffic network balance problem are analyzed. The results show that the reasonable artificial scheduling, such as appropriately adjust the departure time and frequency, optimize transfer facilities and provide guidance for the passengers to reduce the delays caused by congestion can speed up the whole networks reach balance. In addition, increase the common bus stops between some bus lines and the direct bus between two stops both can help the two layers coupled public traffic network reach balance faster, and shorten the passengers stranded time.

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