Asymptotic Synchronization of Complex Dynamical Networks with Time-Varying Delays on Time Scales

Guanghui Liu, Yiping Luo, and Li Shu

Abstract—**In this paper, the problem on the asymptotic synchronization of complex dynamical networks on time scales is investigated. By constructing appropriate Lyapunov-Krasovskii functionals and using some effective mathematical techniques, several novel synchronization criteria for the considered complex networks are obtained. Finally, a numerical example is given to illustrate the effectiveness of theoretical results.**

Index Terms—**synchronization, complex dynamical networks, Lyapunov-Krasovskii functionals, time scales**

I. Introduction

Complex dynamical networks(CDNS) have recently attracted increasing attention from various fields of science and engineering[1-8]. The synchronization phenomena are common and important in real-world networks, such as synchronization on the Internet, synchronization transfer of digital or analog signals in communication networks and synchronization related to biological neural networks [10-32]. Therefore, the synchronization analysis of complex networks is very important both in theory and practice. As a typical collective behavior in complex networks, synchronization or consensus problem has been widely studied in the past decades[5-11].

However, the above mentioned complex dynamical networks are either continuous-time CDNS or discrete-time CDNS, it is troublesome and is not necessary to study synchronization in two kinds of models. In real-world systems, the interaction among agents can happen at any time, may be some continuous time intervals accompanying some discrete moments. So, it is necessary and meaningful to consider both continuous-time and discrete-time cases at the same time in network systems. Empirical results show that the theory of time scales is not only a pure theoretical field of mathematics but also a useful tool to deal with many practical problems. The field of dynamic equations on time scales contains links and extends the classical theory of differential and difference equations. Recently, the theory of time scale calculus has been applied in neural networks and complex networks [38-43]. Since it provides a powerful tool to generalize the discussion of these systems, the theory is undergoing a rapid development. In [38,39], the authors studied the global exponential synchronization of CDNS on time scales. In [40,42], the authors discussed the global stability of complex-valued neural networks on time scales. In fact, the problem of asymptotic synchronization of CDNS on time scales is also to be important and challenging. As an attempt, in this paper, we will combine continuous-time and discrete-time cases together and design the consensus/synchronization protocols under a unified framework, and then some novel asymptotic synchronization criteria for the CDNS on time scales will be established by constructing appropriate Lyapunov-Krasovskii functionals and using linear matrix inequality(LMI[36]) techniques.

The remainder of this paper is organized as follows. In section 2, some preliminaries and notations are given. In section 3, several novel criteria are derived to guarantee the complex networks to be synchronized. In section 4, some numerical results are given to support our conclusions.

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II. Preliminaries and Notations

In order to obtain the main results, some elementary notations and lemmas in the theory of time scales are presented as follows.

In 1980s, Stefan Hilger initiated the theory of time scale calculus. Bohner and Peterson developed and consummated it[33-35]. This novel and fascinating type of mathematics is more general and versatile than the traditional theories of differential and difference equations, as it can mathematically describe continuous and discrete hybrid processes under one framework, hence it is the optimal way for accurate and malleable mathematical modeling.

Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}$ denote the positive integer collection and integer collection, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all n×m real matrices, respectively. $P > 0$ means that matrix P is real, symmetric and positive definite. The superscript T stands for a matrix transposition. For square matrices M and N, the notation $M>(\geq,<,\leq)N$ denotes $M-N$ is a positive-definite (positive-semi-definite, negative, negative-semi-definite) matrix. I and o denote the identity matrix and the zero matrix with compatible dimensions, respectively, diag$\{\ldots\}$ stands for a block-diagonal matrix. The Kronecker product of matrices $U\in\ \mathbb{R}^{n\times m}$ and $R\in\ \mathbb{R}^{p\times q}$ is a matrix in $\mathbb{R}^{np\times mq}$ and denoted as $U\otimes R$. Let $\omega\geq 0$ and $C([-\omega,0]_{\mathbb{T}};\mathbb{R}^n)$ denote the family of continuous functions ϕ from $[-\omega,0]_{\mathbb{T}}$ to $\mathbb{R}^n$ with the norm $\parallel\phi\parallel= sup_{-\omega\leq\theta\leq 0}\parallel \phi(\theta)\parallel$, where $\parallel\cdot\parallel$ is the Euclidean norm in $\mathbb{R}^n$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from

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$\mathbb{R}$. Set $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T}, a \leq t \leq b\}$. $\mathbb{T}^+ := \{t \in \mathbb{T}, t \geq 0\}$. Assume that $0 \in \mathbb{T}$, $\mathbb{T}$ is unbounded above, that is, $sup\mathbb{T} = \infty$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}$; $\rho(t) := \sup\{s < t : s \in \mathbb{T}\}$, respectively, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu(t) := \sigma(t) - t$.

We put $inf\varnothing := sup\mathbb{T}$ and $sup\varnothing := inf\mathbb{T}$, where $\varnothing$ denotes the empty set.

A point $t \in \mathbb{T}$, $t > \inf\mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, left- scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^k$ to be $\mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous provided it is continuous at right-dense point of $\mathbb{T}$ and the left side limit exists (finite) at left-dense point of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is defined by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

A point $t \in \mathbb{T},\ t > \inf\mathbb{T}$ is called regressive provided $1 + \mu(t)f(t) \neq 0, \forall t \in \mathbb{T}$.

Definition 2.1. (Bohner and Peterson [33]). For a function $f : \mathbb{T} \to \mathbb{R}, t \in \mathbb{T}^k$, the delta derivative of $f(t), f^\Delta(t)$, is the number (if it exists) with the property that, for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\mid [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \mid < \varepsilon \mid \sigma(t) - s \mid,$$

for all $s \in U$.

For all $t \in \mathbb{T}^k$, one can get

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

If f and g are two differentiable functions, then the product rule for the derivative of product $f \cdot g$ is that

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma.$$

Definition 2.2. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta = f$ holds for all $t \in \mathbb{T}^k$. In this case, the integral of f is defined by

$$\int_a^t f(s)\Delta s = F(t) - F(a),$$

for $t \in \mathbb{T}$. Then we have

$$(\int_a^t f(s)\Delta s)^\Delta = f(t),$$

for $t \in \mathbb{T}^k$.

Let $A = (a_{ij})$ be an m×n-matrix-valued function on $\mathbb{T}$. We say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$. In this case, we put

$$A^\Delta = (a_{ij}^\Delta).$$

Similarly, we denote that $A^\sigma = (a_{ij}^\sigma)$.

Lemma 2.1. (Bohner and Peterson [33]) Suppose Φ and Ψ are differentiable n×n-matrix-valued functions. Then
(i) $(\Phi + \Psi)^\Delta = \Phi^\Delta + \Psi^\Delta$;
(ii) $(a\Phi)^\Delta = a\Phi^\Delta$ if a is a constant;
(iii) $(\Phi\Psi)^\Delta = \Phi^\Delta\Psi^\sigma + \Phi\Psi^\Delta$.

Lemma 2.2. ([37])For any given constant c, and any matrices P, Q, R, S with appropriate dimensions, the Kronecker product has the following properties:
(i) $(cP) \otimes Q = P \otimes (cQ)$;
(ii) $(P + Q) \otimes R = P \otimes R + Q \otimes R$;
(iii) $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$;
(iv) $(P \otimes Q)^T = P^T \otimes Q^T$.

In this paper, the asymptotic synchronization problem is investigated for a class of complex networks with varying time delays which is described by the following dynamic equation on time scale $\mathbb{T}$:

$$\begin{aligned} x_i^\Delta(t) = &-Dx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\ &+ \Sigma_{j=1}^N C_{ij}\Gamma x_j(t) + J(t),\ i = 1, 2, \ldots, N, \quad (1) \end{aligned}$$

where $t \in \mathbb{T}, x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the ith network at time t; D denotes a known connection matrix; A and B denote the connection weight matrices; $\Gamma \in \mathbb{R}^{n\times n}$ is the matrix describing the inner-coupling between the subsystems at time t; $C = (C_{ij})_{N\times N}$ is the outer-coupling configuration matrix representing the coupling strength and the topological structure of the complex networks. $J(t)$ is the external inputs. The $\tau(t)$ stands for the time delay. $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T$ is an unknown but sector-bounded nonlinear function.

Let $s(t)$ is a solution of an isolated node satisfying

$$s^\Delta(t) = -Ds(t) + Af(s(t)) + Bf(s(t-\tau(t))).$$

Defining the synchronization errors as $e_i(t) = x_i(t) - s(t)$, then the system (1) can be showed as:

$$\begin{aligned} e_i^\Delta(t) = &-De_i(t) + A\bar{f}(e_i(t)) + B\bar{f}(e_i(t-\tau(t))) \\ &+ \Sigma_{j=1}^N C_{ij}\Gamma e_j(t) + J(t),\ i = 1, 2, \ldots, N, \quad (2) \end{aligned}$$

where $\bar{f}(e_i(t)) = f(x_i(t)) - f(s(t)),\ i = 1, 2, \ldots, N.$

The initial conditions associated with system (1) are given by

$$x_i(s) = \varphi_i(s) \in C_{rd}([-\tau, 0]_{\mathbb{T}}, \mathbb{R}^n), \quad i = 1, 2, \ldots, N,$$

where $\varphi_i(s)$ is rd-continuous, and the corresponding state trajectory is denoted as $x_i(t, \varphi_i)$.

Let

$$\begin{aligned} e(t) &= [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T, \\ \jmath &= [J^T(t), J^T(t), \ldots, J^T(t)]^T, \\ F(e(t)) &= [\bar{f}^T(e_1)(t), \bar{f}^T(e_2)(t), \ldots, \bar{f}^T(e_N)(t)]^T, \end{aligned}$$

together with the Kronecker product for matrices, system (2) can be recast into

$$\begin{aligned} e^\Delta(t) = &-(I_N \otimes D)e(t) + (I_N \otimes A)F(e(t)) \\ &+ (I_N \otimes B)F(e(t-\tau(t))) + (C \otimes \Gamma)e(t) + \jmath. \quad (3) \end{aligned}$$

Throughout this paper, the following assumptions are needed.

Assumption 2.1. There exist a constant $k > 0$ and a semi-positive define matrix Γ, such that

$$F^T(e(t))F(e(t)) \leq ke^T(t)\Xi e(t).$$

Assumption 2.2. The outer-coupling configuration matrix of the complex networks (1) satisfies

$$C_{ij} = C_{ji} \geq 0 (i \neq j), C_{ii} = -\Sigma_{j=1, j\neq i}^N C_{ij}\ (i, j = 1, 2, \ldots, N).$$

Assumption 2.3. The time-varying delay is a delta defferentiable function with $\tau^{\Delta}(t) \leq 1$.

Remark 2.1. System (1) is a general model of a class of complex networks. Its one special case with continuous time system is the following:

$$\frac{dx_i(t)}{dt} = -Dx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) + \Sigma_{j=1}^{N} C_{ij}\Gamma x_j(t) + J(t), \quad i = 1, 2, \ldots, N, \quad (4)$$

for $t \in [t_0, +\infty)$. If $\mathbb{T} = \mathbb{Z}$, then its another special case with discrete time system is the following:

$$\Delta x_i(k) = -Dx_i(k) + Af(x_i(k)) + Bf(x_i(k-\tau(k))) + \Sigma_{j=1}^{N} C_{ij}\Gamma x_j(k) + J(k), \; i = 1, 2, \ldots, N, \quad (5)$$

for $t = k \in \mathbb{Z}$, where $\tau(k) \in \mathbb{N}, \Delta x_i(k) = x_i(k+1) - x_i(k)$ is the forward difference operator.

III. Main results

The continuous-time system(4) and the discrete-time system(5) are unified to the system(1). The main objective of this paper is to study the synchronization problem of system(1) under the same framework.

Theorem 3.1. Suppose that the assumptions holds, there exist $n \times n$ positive-define matrices R, Φ, Ψ, such that the following LMI is satisfied:

$$\theta = \begin{bmatrix} \theta_{11} & 0 & \theta_{13} & \theta_{14} \\ \circledast & \theta_{22} & 0 & 0 \\ \circledast & 0 & \theta_{33} & \theta_{34} \\ \circledast & 0 & \theta_{43} & \theta_{44} \end{bmatrix} < 0,$$

where

$$\begin{aligned}
\theta_{11} &= 2(U \otimes R)\Lambda + u\Lambda^T(U \otimes R)\Lambda + U \otimes \Phi + k\Xi, \\
\theta_{13} &= (U \otimes R)(I_N \otimes A) + \frac{1}{2}u[\Lambda^T(U \otimes R)(I_N \otimes A) \\
&\quad + (I_N \otimes A)^T(U \otimes R)\Lambda], \\
\theta_{14} &= (U \otimes R)(I_N \otimes B) + \frac{1}{2}u[\Lambda^T(U \otimes R)(I_N \otimes B) \\
&\quad + (I_N \otimes B)^T(U \otimes R)\Lambda], \\
\theta_{22} &= -U \otimes \Phi(1 - \tau_m), \\
\theta_{33} &= u(I_N \otimes A)^T(U \otimes R)(I_N \otimes A) + U \otimes \Psi - I, \\
\theta_{34} &= \frac{1}{2}u(I_N \otimes A)^T(U \otimes R)(I_N \otimes B) \\
&\quad + \frac{1}{2}u(I_N \otimes B)^T(U \otimes R)(I_N \otimes A), \\
\theta_{44} &= u(I_N \otimes B)^T(U \otimes R)(I_N \otimes B) - (U \otimes \Psi)(1 - \tau_m),
\end{aligned}$$

where

$$\Lambda = C \otimes \Gamma - I_N \otimes D,$$

$$U = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix},$$

then system(1) can reach asymptotic synchronization.

Proof. Construct the following Lyapunov-Krasovskii functional:

$$V(t) = V_1 + V_2 + V_3,$$

where

$$\begin{aligned}
V_1 &= e^T(t)(U \otimes R)e(t), \\
V_2 &= \int_{t-\tau(t)}^{t} e^T(s)(U \otimes \Phi)e(s)\Delta s, \\
V_3 &= \int_{t-\tau(t)}^{t} F^T(e(s))(U \otimes \Psi)F(e(s))\Delta s.
\end{aligned}$$

Taking the derivative of $V_i(t)(i = 1, 2, 3)$ along the trajectories of (2), we have

$$\begin{aligned}
V_1^{\Delta}(t) &= e^T(t)(U \otimes R)e^{\Delta}(t) + (e^{\Delta})^T(t)(U \otimes R)e^{\sigma}(t) \\
&= e^T(t)(U \otimes R)e^{\Delta}(t) + (e^{\Delta})^T(t)(U \otimes R)[e(t) + e^{\Delta}(t)u] \\
&= 2e^T(t)(U \otimes R)[\Lambda e(t) + (I_N \otimes A)\overline{F}(e(t)) + I_N \otimes B \\
&\quad \cdot F(e(t-\tau(t))) + J] + u[\Lambda e(t) + (I_N \otimes A)F(e(t)) \\
&\quad + I_N \otimes BF(e(t-\tau(t))) + J]^T \cdot U \otimes R[\Lambda e(t) \\
&\quad + (I_N \otimes A)F(e(t)) + I_N \otimes BF(e(t-\tau(t))) + J] \\
&= 2e^T(t)(U \otimes R)\Lambda e(t) + 2e^T(t)(U \otimes R)(I_N \otimes A)F(e(t)) \\
&\quad + 2e^T(t)(U \otimes R)(I_N \otimes B)F(e(t-\tau(t))) \\
&\quad + 2e^T(t)(U \otimes R)J + u[e^T(t)\Lambda^T + F^T(e(t))(I_N \otimes A)^T \\
&\quad + F^T(e(t-\tau(t)))(I_N \otimes B)^T + J^T]U \otimes R[\Lambda e(t) \\
&\quad + (I_N \otimes A)F(e(t)) + I_N \otimes BF(e(t-\tau(t))) + J] \\
&= 2e^T(t)(U \otimes R)\Lambda e(t) + 2e^T(t)(U \otimes R)(I_N \otimes A)F(e(t)) \\
&\quad + 2e^T(t)(U \otimes R)(I_N \otimes B)F(e(t-\tau(t))) \\
&\quad + 2e^T(t)(U \otimes R)J + ue^T(t)\Lambda^T(U \otimes R)\Lambda e(t) \\
&\quad + ue^T(t)\Lambda^T(U \otimes R)(I_N \otimes A)F(e(t)) \\
&\quad + ue^T(t)\Lambda^T(U \otimes R)(I_N \otimes B)F(e(t-\tau(t))) \\
&\quad + uF^T(e(t))(I_N \otimes A)^T(U \otimes R)\Lambda e(t) \\
&\quad + uF^T(e(t))(I_N \otimes A)^T(U \otimes R)(I_N \otimes A)F(e(t)) \\
&\quad + uF^T(e(t))(I_N \otimes A)^T(U \otimes R)(I_N \otimes B)F(e(t-\tau(t))) \\
&\quad + uF^T(e(t-\tau(t)))(I_N \otimes B)^T(U \otimes R)\Lambda e(t) \\
&\quad + uF^T(e(t-\tau(t)))(I_N \otimes B)^T(U \otimes R)(I_N \otimes A) \\
&\quad\; F(e(t)) + uF^T(e(t-\tau(t)))(I_N \otimes B)^T(U \otimes R) \\
&\quad\; \cdot (I_N \otimes B)F(e(t-\tau(t))). \qquad (6)
\end{aligned}$$

$$\begin{aligned}
V_2^{\Delta}(t) &= e^T(t)(U \otimes \Phi)e(t) \\
&\quad - e^T(t-\tau(t))(U \otimes \Phi)e(t-\tau(t))(1 - \tau^{\Delta}(t). \qquad (7)
\end{aligned}$$

$$\begin{aligned}
V_3^{\Delta}(t) &= F^T(e(t))(U \otimes \Psi)F(e(t)) - F^T(e(t-\tau(t))) \\
&\quad (U \otimes \Psi)F(e(t-\tau(t)))(1 - \tau^{\Delta}(t)). \qquad (8)
\end{aligned}$$

Considering assumption 1, we can deduce

$$\begin{aligned}
V_3^{\Delta}(t) &\leq F^T(e(t))(U \otimes \Psi - I)F(e(t)) + ke^T(t)\Xi e(t) \\
&\quad - (1 - \tau_m)F^T(e(t-\tau(t)))(U \otimes \Psi)F(e(t-\tau(t))). \quad (9)
\end{aligned}$$

Then combing with the terms in $(6) - (9)$, we can get

$$V^{\Delta}(t) < \xi^T(t)\theta\xi(t) < 0,$$

where

$$\xi^T(t) = [e^T(t), e^T(t-\tau(t)), F^T(e(t)), F^T(e(t-\tau(t)))].$$

Based on the theorem of Lyapunov-Krasovskii stability theorem, the system (1) can achieve the desired synchronization and the proof is completed.

Remark 3.1. If $\mathbb{T} = \mathbb{R}$, then $u(t) = 0$, the system (1) can be rewritten as the continuous system (4). From Theorem 3.1, we can immediately derive the following result.

Theorem 3.2. Suppose that the assumptions hold,there exist $n \times n$ positive-define matrices R, Φ, Ψ, such that the following LMI is satisfied:

$$\theta = \begin{bmatrix} \theta_{11} & 0 & \theta_{13} & \theta_{14} \\ \circledast & \theta_{22} & 0 & 0 \\ \circledast & 0 & \theta_{33} & \theta_{34} \\ \circledast & 0 & \theta_{43} & \theta_{44} \end{bmatrix} < 0,$$

where

$$\begin{aligned} \theta_{11} &= 2(U \otimes R)\Lambda + U \otimes \Phi + k\Xi, \\ \theta_{13} &= (U \otimes R)(I_N \otimes A) + (I_N \otimes A)^T (U \otimes R)\Lambda], \\ \theta_{14} &= (U \otimes R)(I_N \otimes B) + (I_N \otimes B)^T (U \otimes R)\Lambda], \\ \theta_{22} &= -(U \otimes \Phi)(1 - \tau_m), \\ \theta_{33} &= U \otimes \Psi - I, \\ \theta_{34} &= 0, \\ \theta_{44} &= -(U \otimes \Psi)(1 - \tau_m). \end{aligned}$$

Then system (4) can reach asymptotic synchronization.

Remark 3.2. If $\mathbb{T} = \mathbb{N}$, then $u(t) = 1$, the system (1) can be rewritten as the discrete system (5), from Theorem 3.1, we can immediately derive the following result.

Theorem 3.3. Suppose that the assumptions holds, there exist $n \times n$ positive-define matrices R, Φ, Ψ, such that the following LMI is satisfied:

$$\theta = \begin{bmatrix} \theta_{11} & 0 & \theta_{13} & \theta_{14} \\ \circledast & \theta_{22} & 0 & 0 \\ \circledast & 0 & \theta_{33} & \theta_{34} \\ \circledast & 0 & \theta_{43} & \theta_{44} \end{bmatrix} < 0,$$

where

$$\begin{aligned} \theta_{11} &= 2(U \otimes R)\Lambda + \Lambda^T (U \otimes R)\Lambda + U \otimes \Phi + k\Xi, \\ \theta_{13} &= (U \otimes R)(I_N \otimes A) + \frac{1}{2}[\Lambda^T (U \otimes R)(I_N \otimes A) \\ &\quad + (I_N \otimes A)^T (U \otimes R)\Lambda], \\ \theta_{14} &= (U \otimes R)(I_N \otimes B) + \frac{1}{2}[\Lambda^T (U \otimes R)(I_N \otimes B) \\ &\quad + (I_N \otimes B)^T (U \otimes R)\Lambda], \\ \theta_{22} &= -U \otimes \Phi(1 - \tau_m), \\ \theta_{33} &= (I_N \otimes A)^T (U \otimes R)(I_N \otimes A) + U \otimes \Psi - I, \\ \theta_{34} &= \frac{1}{2}(I_N \otimes A)^T (U \otimes R)(I_N \otimes B) \\ &\quad + \frac{1}{2}(I_N \otimes B)^T (U \otimes R)(I_N \otimes A), \\ \theta_{44} &= (I_N \otimes B)^T (U \otimes R)(I_N \otimes B) - (U \otimes \Psi)(1 - \tau_m). \end{aligned}$$

Then system (5) can reach asymptotic synchronization .

Remark 3.3. If $\tau(t) = 0$, that is equivalent to $B = 0$ in the system(1), then we have :

Corollary 3.1. Suppose that the assumptions hold, there exist $n \times n$ positive-define matrices R, Φ, Ψ, such that the following LMI is satisfied:

$$\theta = \begin{bmatrix} \theta_{11} & 0 & \theta_{13} \\ \circledast & \theta_{22} & 0 \\ \circledast & 0 & \theta_{33} \end{bmatrix} < 0,$$

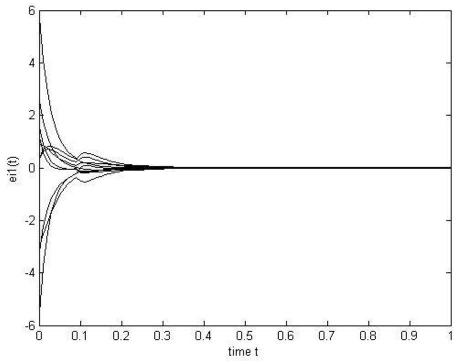

Fig. 1. synchronization error $e_{i1}(t)$ of the system (10).

where

$$\begin{aligned} \theta_{11} &= 2(U \otimes R)\Lambda + u\Lambda^T (U \otimes R)\Lambda + U \otimes \Phi + k\Xi, \\ \theta_{13} &= (U \otimes R)(I_N \otimes A) + \frac{1}{2}u[\Lambda^T (U \otimes R)(I_N \otimes A) \\ &\quad + (I_N \otimes A)^T (U \otimes R)\Lambda], \\ \theta_{22} &= -U \otimes \Phi(1 - \tau_m), \\ \theta_{33} &= u(I_N \otimes A)^T (U \otimes R)(I_N \otimes A) + U \otimes \Psi - I. \end{aligned}$$

Then system (1) can reach asymptotic synchronization.

IV. Illustrative Example

In this section, a numerical example is given to verify the theoretical result.

Example. Consider the two-dimensional delayed dynamical network

$$\begin{aligned} x_i^{\Delta}(t) &= -Dx_i(t) + Af(x_i(t)) + Bf(x_i(t - \tau(t))) \\ &\quad + \Sigma_{j=1}^{10} C_{ij} \Gamma x_j(t) + J(t), \end{aligned} \quad (10)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T, (i = 1, 2, \ldots, 10)$ is the state vector of the ith subsystem, the activation function is $f(x_i(t)) = (f_1(x_{i1}(t), f_2(x_{i2}(t)))^T$ with $f_j(x_{ij}(t)) = tanh(x_{ij}(t)), (j = 1, 2)$. The time scale is chosen as $\mathbb{T} = \cup_{i=0}^{\infty}[t_i, t_i + 0.1], t_0 = 0, t_{i+1} = t_i + 0.1, \tau(t) = 1 + |sint|$. and the parameters:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1.5 & -0.15 \\ 5.15 & 2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -0.5 \\ -2.5 & 1.5 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad J(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad U = \begin{bmatrix} \frac{-9}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{-9}{2} & \cdots & \frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & \frac{-9}{2} \end{bmatrix}$$

It is easy to verify that the system (10) can achieve synchronization. Fig.1 and Fig.2 show the simulation results of synchronization of system (10), where $e_{i1}(t)$ and $e_{i2}(t)$ are the synchronization errors.

V. Conclusion

In this paper, both continuous-time and discrete-time synchronization have been discussed under a unified framework, some new synchronization conditions have been proposed. Compared with the existing results, the computational complexity and control cost are reduced greatly. However, the paper only considers the systems converge asymptotically to the equilibrium as the time goes to infinity. Therefore, how to design a finite time-controller for the systems is naturally regarded as another interesting research topic.

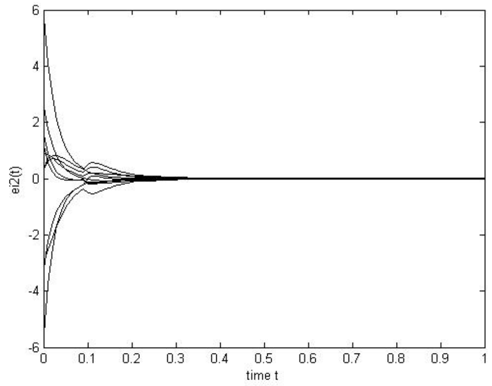

Fig. 2. synchronization error $e_{i2}(t)$ of the system (10).

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