Numerical Solution of Fractional Partial Differential Equation of Parabolic Type Using Chebyshev Wavelets Method

Mulin Li and Lifeng Wang*

***Abstract*—Numerical solution of the fractional differential equation is almost an important topic in recent years. In this paper, in order to solve the numerical solution of a class of fractional partial differential equation of parabolic type, we present a collocation method of two-dimensional Chebyshev wavelets. Using the definition and property of Chebyshev wavelets, we give the definition of two-dimensional Chebyshev wavelets. We transform the initial problems into solving a system of nonlinear algebraic equations by applying the wavelets collocation method. Convergence analysis is investigated to show that the method is convergent. The numerical example shows the effectiveness of the approach.**

***Index Terms*—fractional derivative, fractional partial differential equation, Chebyshev wavelets, convergence analysis, numerical solution**

I. Introduction

FRACTIONAL differential equations are generalizations of differential equations that replace integral order derivatives by fractional order derivatives. In general, ordinary differential equations are applied on describing dynamic phenomena in various fields such as physics, biology and chemistry. However, for some complicated systems the common simple differential equations cannot provide agreeable results. Therefore, in order to obtain better models, fractional differential equations are employed instead of integer order ones [1-3]. On the other hand, the fractional differential equations are too complicated to solve by analytical methods and theoretical background for this problem is not well developed. Hence, in recent 10 years mathematicians have discovered new methods of numerical solution. There are several methods to solve fractional differential equations, such as variational iteration method [4, 5], Adomain decomposition method [6], fractional differential transformation method [7], fractional finite difference method [8], and wavelets method [9, 10].

Orthogonal functions and polynomials have been used by many authors for solving various functional equations. The main idea of using an orthogonal basis is that the problem under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncated series of orthogonal basis function for the solution of problem and using the operational matrices. In this paper, we introduce a method to approximate the solutions of fractional partial differential equations with given initial values. In this technique, the solution is approximated by Chebyshev wavelets vectors. The considered equations are as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -u\frac{\partial}{\partial x}\left(\frac{2u}{x} - x\right) + u\frac{\partial^2 u}{\partial x^2}, \quad t>0, x>0 \tag{1}$$

such as

$u(0,t)=0$ and $u(1,t)=E_\alpha(t^\alpha)$,

where E_α is Mittag-Leffler function, $u(x,t)$ is unknown function, defined in $u(x,t)\in L^2([0,1]\times[0,1])$. $\frac{\partial^\alpha u}{\partial t^\alpha}$ is fractional derivative of Caputo sense.

II. Fractional calculus

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$ of a function is defined as [11]

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha>0 \tag{2}$$

$$J^0 f(x) = f(x) \tag{3}$$

Definition 2. The fractional differential operator in Caputo sense is defined as

$$D^\alpha f(x) = \begin{cases} \dfrac{d^r f(x)}{dx^r}, & \alpha = r \in N; \\ \dfrac{1}{\Gamma(r-\alpha)}\displaystyle\int_0^x \frac{f^{(r)}(\tau)}{(x-\tau)^{\alpha-r+1}}d\tau, & 0\le r-1<\alpha<r. \end{cases} \tag{4}$$

The Caputo fractional derivatives of order α is also given by $D^\alpha f(x) = J^{r-\alpha}D^r f(x)$, where D^r is the usual integer differential operator of order r. The relation between the Caputo operator and Riemann-Liouville operator are given by:

$$D^\alpha J^\alpha f(x) = f(x) \tag{5}$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{r-1} f^{(k)}(0^+)\frac{x^k}{k!}, \qquad x>0 \tag{6}$$

III. The second kind Chebyshev wavelets

For the interval $[0,1)$, the second kind Chebyshev wavelets are defined as [12]

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$$\psi_{nm}(t)=\begin{cases}\frac{2}{\sqrt{\pi}}2^{\frac{k}{2}}U_m(2^k t-n), & t\in\left[\frac{n}{2^k},\frac{n+1}{2^k}\right)\\ 0, & otherwise\end{cases} \quad (7)$$

where $n=0,1,\ldots,2^k-1$, $m=0,1,\ldots,M-1$, k,M are fixed positive integer, $U_m(t)$ denote the shifted second kind Chebyshev polynomials, which are defined on the interval $[0,1)$ as follows

$$U_m(t)=\sum_{i=0}^{m}(-1)^{m-i}\frac{2^{2i}(m+i+1)!}{(m-i)!(2i+1)!}t^i, \quad m=0,1,\ldots \quad (8)$$

The second kind Chebyshev wavelets functions are orthogonal with respect to the weight function $\omega_n(t)$, as follows

$$\psi_{nm}(t)=\begin{cases}\sqrt{1-(2^{k+1}t-2n-1)^2}, & t\in\left[\frac{n}{2^k},\frac{n+1}{2^k}\right)\\ 0, & otherwise\end{cases} \quad (9)$$

Any function $u(t)\in L^2[0,1)$ can be expressed by the second kind Chebyshev wavelets

$$u(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(t) \quad (10)$$

where $c_{nm}=\langle u(t),\psi_{nm}(t)\rangle$, and $\langle\cdot,\cdot\rangle$ denotes the inner product.

By truncating the infinite series in Eq.(10), we can rewritten Eq.(10) as

$$u(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^T\Psi(t) \quad (11)$$

where C and $\Psi(t)$ are $\hat{m}=2^k M$ column vectors

$$\begin{aligned}C=[&c_{00},c_{01},\ldots,c_{0M-1},c_{10},c_{11},\ldots,\\ &c_{1M-1},\ldots,c_{2^k-10},c_{2^k-11},\ldots,c_{2^k-1M-1}]^T\end{aligned} \quad (12)$$

$$\begin{aligned}\Psi(t)=[&\psi_{00},\psi_{01},\ldots,\psi_{0M-1},\psi_{10},\psi_{11},\ldots,\\ &\psi_{1M-1},\ldots,\psi_{2^k-10},\psi_{2^k-11},\ldots,\psi_{2^k-1M-1}]^T\end{aligned} \quad (13)$$

For simplicity, Eq.(11) can be also written as

$$u(t)\cong\sum_{i=1}^{\hat{m}}c_i\psi_i(t)=C^T\Psi(t) \quad (14)$$

where $c_i=c_{nm}$, $\psi_i=\psi_{nm}$. The index i is determined by the relation $i=Mn+m+1$. Therefore, we can also write the vectors

$$C=[c_1,c_2,\ldots,c_{\hat{m}}]^T \quad (15)$$

$$\Psi(t)=[\psi_1,\psi_2,\ldots,\psi_{\hat{m}}]^T \quad (16)$$

Similarly, for the function $u(x,t)$ over $[0,1)\times[0,1)$ can be expressed as follows

$$u(x,t)=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t) \quad (17)$$

Eq.(17) can be rewritten as

$$u(x,t)\cong\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}\psi_i(x)\psi_j(t)=\Psi^T(x)U\Psi(t) \quad (18)$$

where $U=[u_{ij}]$ and $u_{ij}=\langle\psi_i(x),\langle u(x,t),\psi_j(t)\rangle\rangle$ are the second kind Chebyshev wavelets coefficients.

IV. SOLUTION OF THE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

In this section, the second kind Chebyshev wavelets is applied to solve a class of fractional partial differential equations Eq.(1). For solving this problem, we assume that

$$u(x,t)=\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t) \quad (19)$$

where $n=1,\ldots,2^{k_1-1}$, $i=0,\ldots,M_1-1$, $l=1,\ldots,2^{k_2-1}$, $j=0,\ldots,M_2-1$.

Integrating Eq.(1) by using Riemann-Liouville fractional integral operator, we have

$$u(x,t)-u(x,0)=J_t^{\alpha}\left[-u\frac{\partial}{\partial x}\left(\frac{2u}{x}-x\right)+u\frac{\partial^2 u}{\partial x^2}\right] \quad (20)$$

Substituting Eq.(19) into Eq.(20), we get

$$\begin{aligned}&u(x,t)-u(x,0)\\ &=J_t^{\alpha}\left[\begin{aligned}&-\left(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)\right)\times\\ &\frac{\partial}{\partial x}\left(\frac{2}{x}\left(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)\right)-x\right)\\ &+\left(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)\right)\times\\ &\frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{2^{k_1-1}}\sum_{i=0}^{M_1-1}\sum_{l=1}^{2^{k_2-1}}\sum_{j=0}^{M_2-1}c_{n,i,l,j}\psi_{n,i,l,j}(x,t)\right)\end{aligned}\right]\end{aligned} \quad (21)$$

Dispersing Eq.(21) by points $x_l=\frac{l-0.5}{2^{k_1-1}M_1}$ and $t_r=\frac{r-0.5}{2^{k_2-1}M_2}$, $l=1,2,\ldots,2^{k_1-1}M_1$, $r=1,2,\ldots,2^{k_2-1}M_2$, then we can obtain a nonlinear algebraic equations which contains $\left(2^{k_1-1}M_1\right)\left(2^{k_2-1}M_2\right)$ equations. The equations can be solved easily by using Matlab.

V. CONVERGENCE ANALYSIS

Theorem 1 The Chebyshev wavelets numerical solutions $u(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(x)$ converge to the exact solution $u(x)$.

Proof. For $k=1$, let $u(x)=\sum_{i=0}^{M-1}c_{1i}\psi_{1i}(x)$,

where $c_{1i}=\langle u(x),\psi_{1i}(x)\rangle$, then we have

$$u(x)=\sum_{i=1}^{n}\langle u(x),\psi_{1i}(x)\rangle\psi_{1i}(x).$$

Let $\psi_{1i}(x)$ is $\psi(x)$, $\gamma_j=\langle u(x),\psi(x)\rangle$.

$\{S_n\}$ is the sum of sequence $\gamma_j\psi(x_j)$, S_n and S_m are the arbitrary partial sum, $n \ge m$. Then we will prove $\{S_n\}$ is Cauchy convergence.

Let $S_n = \sum_{j=1}^{n} \gamma_j \psi(x_j)$, we get

$$\begin{aligned}\langle u(x), S_n\rangle &= \left\langle u(x), \sum_{j=1}^{n}\gamma_j\psi(x_j)\right\rangle \\ &= \sum_{j=1}^{n}\overline{\gamma_j}\langle u(x),\psi(x_j)\rangle \\ &= \sum_{j=1}^{n}\overline{\gamma_j}\gamma_j \\ &= \sum_{j=1}^{n}|\gamma_j|^2.\end{aligned}$$

For $n > m$, we obtain $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n}|\gamma_j|^2$, thus

$$\begin{aligned}\left\|\sum_{j=m+1}^{n}\gamma_j\psi(x_j)\right\|^2 &= \left\langle \sum_{i=m+1}^{n}\gamma_i\psi(x_i), \sum_{j=m+1}^{n}\gamma_j\psi(x_j)\right\rangle \\ &= \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}\gamma_i\overline{\gamma_j}\langle\psi(x_i),\psi(x_j)\rangle \\ &= \sum_{j=m+1}^{n}\gamma_j\overline{\gamma_j} \\ &= \sum_{j=m+1}^{n}|\gamma_j|^2,\end{aligned}$$

therefore

$$\|S_n - S_m\|^2 = \sum_{j=m+1}^{n}|\gamma_j|^2, \; n > m.$$

Using Bessel inequality, we know that $\sum_{j=1}^{\infty}|\gamma_j|^2$ is convergent. Namely, $\|S_n - S_m\|^2 \to 0$ when $n, m \to \infty$. So $\{S_n\}$ is Cauchy convergence, set $\{S_n\} \to S$, then we get $u(x) = S$.

In fact

$$\begin{aligned}\langle S - u(x), \psi(x_j)\rangle = \langle S,\psi(x_j)\rangle - \langle u(x),\psi(x_j)\rangle &= \left\langle \lim_{n\to\infty} S_n, \psi(x_j)\right\rangle - \gamma_j \\ &= \lim_{n\to\infty}\langle S_n,\psi(x_j)\rangle - \gamma_j \\ &= \gamma_j - \gamma_j \\ &= 0.\end{aligned}$$

So $u(x) = S$ and $\sum_{j=1}^{n}\gamma_j\psi(x_j) \to u(x)$.

This completes the proof.

VI. Numerical Examples

Applying the wavelets collocation method, for convenience, take $k_1 = k_2 = k$ and $M_1 = M_2 = M$, we can acquire the numerical solutions of Eq.(1). Fig. 1-4 show the numerical solutions for different k, $M = 2$ and $\alpha = 1$.

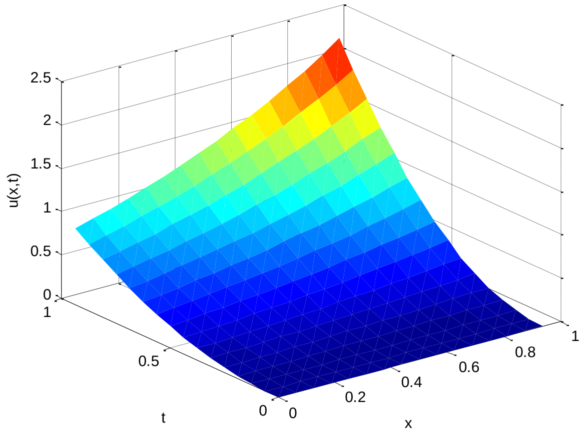

Fig.1 Numerical solution for $k = 4$.

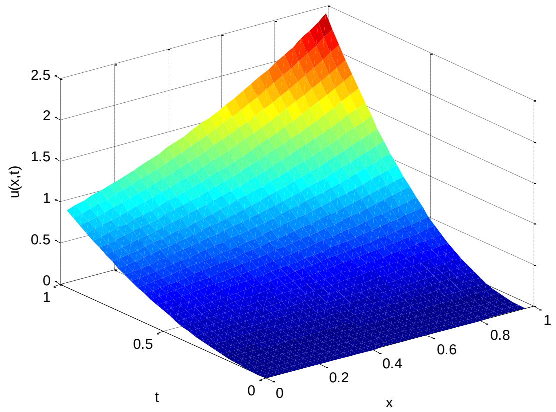

Fig.2 Numerical solution for $k = 5$.

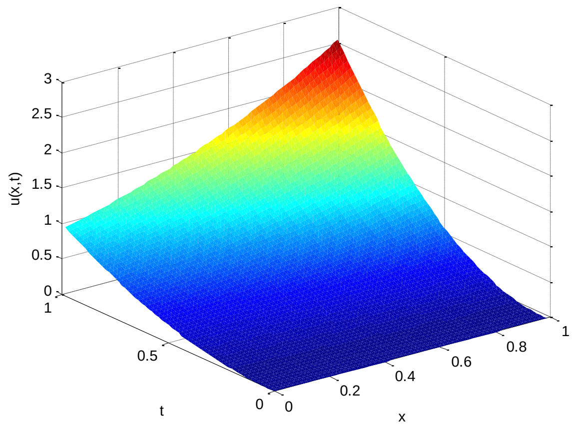

Fig.3 Numerical solution for $k = 6$.

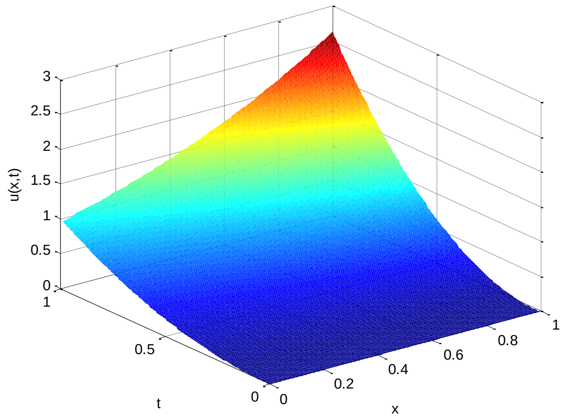

Fig.4 Numerical solution for $k = 7$.

TABLE I
THE ABSOLUTE ERRORS OF NUMERICAL SOLUTIONS AND EXACT SOLUTIONS FOR $\alpha = 0.5$.

x	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$
0	0	0	0	0
0.1	4.4561e-005	3.5633e-005	5.5387e-005	6.1754e-005
0.2	1.7411e-004	1.4214e-004	4.4171e-004	6.7617e-004
0.3	4.1487e-004	3.0017e-004	4.8102e-004	7.4154e-004
0.4	6.8432e-004	5.7657e-004	5.7341e-004	8.1123e-004
0.5	7.4154e-004	6.1761e-004	6.2118e-004	1.0245e-003
0.6	1.5114e-003	1.6379e-003	1.8841e-003	1.9478e-003
0.7	1.8013e-003	1.9341e-003	2.0148e-003	2.3884e-003
0.8	2.2857e-003	2.5322e-003	2.6347e-003	3.7147e-003
0.9	2.5388e-003	2.9847e-003	3.1012e-003	5.3576e-003
1	3.0141e-003	3.5411e-003	4.1557e-003	6.7430e-003

TABLE II
THE ABSOLUTE ERRORS OF NUMERICAL SOLUTIONS AND EXACT SOLUTIONS FOR $\alpha = 1$.

x	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$
0	0	0	0	0
0.1	3.9244e-008	1.0931e-007	1.3362e-007	1.2711e-007
0.2	1.3771e-007	4.3717e-007	5.2583e-007	5.1368e-007
0.3	3.2441e-007	9.0129e-007	1.2368e-006	1.1457e-006
0.4	6.7271e-007	1.7678e-006	2.3684e-006	2.3645e-006
0.5	9.1172e-007	2.8633e-006	3.3546e-006	3.1889e-006
0.6	1.2781e-006	4.0416e-006	4.6571e-006	4.3686e-006
0.7	1.9375e-006	5.1121e-006	6.3102e-006	6.6223e-006
0.8	2.5378e-006	7.5406e-006	8.3554e-006	9.4663e-006
0.9	3.1102e-006	8.8841e-006	1.0024e-005	1.0785e-005
1	5.6981e-006	2.4790e-005	4.3243e-005	8.8989e-005

TABLE III
THE COMPARISONS FOR DIFFERENT METHOD $\alpha = 1$

x	$t=0.2$		$t=0.4$	
	$u_{Chebyshev}$	u_{HPM}	$u_{Chebyshev}$	u_{HPM}
0	0	0	0	0
0.1	0.01802	0.01799	0.02438	0.02430
0.2	0.07211	0.07196	0.09738	0.09720
0.3	0.16235	0.16191	0.21907	0.21870
0.4	0.28857	0.28784	0.38939	0.38881
0.5	0.45087	0.44975	0.60844	0.60751
0.6	0.64951	0.64765	0.87609	0.87482
0.7	0.88342	0.88152	1.19248	1.19072
0.8	1.15441	1.15137	1.55752	1.55523
0.9	1.46088	1.45720	1.97126	1.96833
1	1.80359	1.79902	2.43008	2.43004

The absolute errors for different x, t and α are established in Table I and II. The comparisons between the homotopy perturbation method (HPM) [13] and our results are given in Table III.

Through Table I~III, we can also see that the absolute errors are vary small, and the errors based on our method are less than the errors obtained by HPM.

VII. CONCLUSION

The objective of this paper is to demonstrate numerical solutions of fractional partial differential equations. The technique employs Chebyshev wavelets series approximations. Moreover, a convergence analysis is also proved. Furthermore, this method converts initial value problems into nonlinear systems of algebraic equations. The method is computationally very easy and provides a structured approach to numerical approximate solutions.

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