Analytical Solution of the Mechanical Problem on Additive Thickening of Aging Viscoelastic Tapers Under Nonstationary Longitudinal End Forces

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***Abstract*—The piecewise-continuous processes of additive forming sufficiently long in the axial direction solids of conical shape under simultaneous action of end loads that are statically equivalent to the axial tension–compression with a time-varying force are studied. The being formed solids exhibit properties of deformation heredity and aging. On the basis of the approaches of mechanics of growing solids a nonclassical boundary value problem of the linear theory of viscoelasticity of the homogeneously aging isotropic media to describe the modeled process with the integral satisfaction of the force condition on the end surface of the formed solid is stated. A lemma about the possibility to carry in terms of the work objectives the product of the operator of differentiation with respect to time and the integral operator of viscoelasticity with a limit of time integration depending on solid point through the sign of integral over an arbitrary, expanding due to the growth, surface inside or on the boundary of the growing solid is proved. With its help a closed analytical solution of the stated problem of growing solids mechanics is built. This solution allows to retrace the evolution of the stress-strain state of the solid under consideration during and after the process of its additive formation.**

***Index Terms*—additive manufacturing, growing solid, longitudinal force, taper, viscoelasticity.**

I. Introduction

THE additive formation of solids is realized in a wide variety of natural and technological processes. Many of these processes should be considered as continuous growing processes, such that during the formation of a solid an infinitely thin layer of additional material joins to its surface each infinitely small period of time. In the course of additive processes different factors influence on solids being formed and cause their deformation. The development of stress-strain state of such solids is impossible to describe within the framework of classical concepts of continuum mechanics in principle. This is due to the lack of any configuration of the continuously growing solid which could be associated with introduction of the strain measures. An adequate description of mechanical behavior of solids deforming in processes of their continuous growing can be given on the basis of approaches and methods of mechanics of growing solids being actively developed nowadays [1]–[6]. Statements and solutions of various problems on growing solids deformation can be found, for example, in [7]–[20]. One can find there specific examples of solutions of such problems and discussing a variety of new mechanical effects discovered thanks to the obtained solutions. Note that among a great number of papers on additive manufacturing (AM) technologies written by technologists, chemists, and physicists (see, e.g., [21]–[43]) a rare one is devoted to mechanical aspects of analyses of AM fabricated parts and similar problematics.

The present work is devoted to studying additive processes of formation of the relatively long in the axial direction conical solids. It is assumed that in the process of formation of the solid its end surfaces are acted with loading statically equivalent to the axial tension–compression force which can in general case change over time [44]. Forming the solid under consideration is carried out by means of its thickening in the radial direction due to the influx of additional material to the conic side surface. This process is piecewise-continuous, i.e. consists of arbitrary number of stages of continuous accreating alternating with arbitrary long pauses during which the influx of the material does not take place.

In the proposed study we consider the situation when the solid being formed exhibits the properties of deformation heredity (viscoelasticity) and aging (weakening the deformation properties over time regardless stresses existing in the solid), and therefore, during pauses in the growing process as well as after the final cessation of growth the solid continues to change its stress-strain state. This situation is quite difficult to simulate as rheological manifestations in the deformation response of the material continuously interact with mechanical reactions of the solid on the developing in time process of adding new material elements to it [45]–[47].

The problem is solved in quasistatic statement in the approximation of small strains. The latter let us consider the radii of the growing solid ends expanding due to the influx of additional material to be known functions of time, which are prescribed by a specific simulated growing process. The process itself is considered to be those that the additional material influx to the surface of the formed solid does not acquire nonzero stresses near this surface at the time moment of the material inclusion in the composition of this solid. The difference between the radius of one of the solid end and the radius of the other one may change arbitrarily during the process of the solid growing both in size and in sign.

II. Constitutive Relations for Aging Viscoelastic Accreted Solids

We will consider homogeneous isotropic linearly viscoelastic aging material described by the following equation

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of state [48], [49]:

$$\mathbf{T}(\mathbf{r},t)=\mathcal{H}^{-1}_{\tau_0(\mathbf{r})}\big[2\mathbf{E}(\mathbf{r},t)+(\varkappa-1)\mathbf{1}\,\mathrm{tr}\,\mathbf{E}(\mathbf{r},t)\big]. \qquad (1)$$

Here $\tau_0(\mathbf{r})$ is the time of occurrence of stresses at the point of the solid with the radius-vector $\mathbf{r}$; $\mathbf{T}$ and $\mathbf{E}$ are the stress and small strain tensors, $\mathbf{1}$ is the unit tensor of the second rank; $\varkappa=(1-2\nu)^{-1}$, where $\nu=\mathrm{const}$ is the Poisson's ratio. The timing t is counted from the moment of the material nucleation. The linear operator $\mathcal{H}_s^{-1}=G(t)\big(\mathcal{I}+\mathcal{N}_s\big)$ is inverse to the linear integral operator $\mathcal{H}_s=\big(\mathcal{I}-\mathcal{L}_s\big)G(t)^{-1}$ with the real parameter $s>0$, where $G(t)$ is the elastic shear modulus, $\mathcal{I}$ is the identity operator, and

$$\begin{Bmatrix}\mathcal{L}_s\\ \mathcal{N}_s\end{Bmatrix} f(t)=\int_s^t f(\tau)\begin{Bmatrix}K\\ R\end{Bmatrix}(t,\tau)\,d\tau,$$

$$K(t,\tau)=G(\tau)\,\frac{\partial\Delta(t,\tau)}{\partial\tau},\qquad \Delta(t,\tau)=\frac{1}{G(\tau)}+\omega(t,\tau).$$

$K(t,\tau)$ and $R(t,\tau)$ are the kernels of creep and relaxation, $\Delta(t,\tau)$ and $\omega(t,\tau)$ are the specific strain function and the creep measure for pure shear ($t\geqslant\tau\geqslant 0$). It is accepted by definition that $\omega(\tau,\tau)\equiv 0$. With this in mind we have the identity $\mathcal{H}_\tau^{-1}\Delta(t,\tau)\equiv 1$.

The state equation (1) is used in the present work to describe the mechanical behaviour of *growing solids* which are built up additively by attaching the additional material to the current solid surface. We suppose that the additional material is being loaded in such accreting process simultaneously with its attaching to the solid. In this case, the function $\tau_0(\mathbf{r})$ in (1) is to be determined in the following way.

The *original part* V_0 of the growing solid was initially formed stress-free and loaded then before the accreting process start. In the original part the function $\tau_0(\mathbf{r})$ should be identically equal to the time moment t_0 of loading of this part. In the formed due to accreting process *additional part* V_A of the solid the function $\tau_0(\mathbf{r})$ should coincide with the distribution $\tau_*(\mathbf{r})$ of moments of attaching particles $\mathbf{r}$ of the additional material to the solid. So,

$$\tau_0(\mathbf{r})=\begin{cases}t_0, & \mathbf{r}\in V_0,\\ \tau_*(\mathbf{r}), & \mathbf{r}\in V_\mathrm{A}.\end{cases} \qquad (2)$$

If the accreting process starts at the time instant $t=t_1$ then, obviously, $\tau_*(\mathbf{r})\geqslant t_1\geqslant t_0$.

We assume that the considered process of adding the material to the solid can be adequately modelled as a process of *piecewise continuous accretion*. This means that the process consists of N *stages of continuous accretion* $t\in[t_{2k-1},t_{2k})$ $(k=\overline{1,N})$ during which an infinitely thin layer of the additional material adheres to the growing solid surface every infinitely small time period. Before the first stage of continuous accretion when $t\in[t_0,t_1)$, between the stages when $t\in[t_{2k},t_{2k+1})$ $(k=\overline{1,N-1})$ and after the last stage when $t\in[t_{2N},+\infty)$, the additional material influx is absent, the solid does not grow. The part $\Sigma(t)$ of the growing solid surface to which the additional material continuously inflows at the current time instant t during any stage of continuous accretion is named the *(current) growth surface*. It is clear that the growth surface $\Sigma(t)$ is the t-level surface of the function $\tau_*(\mathbf{r})$.

We investigate mechanical problems for growing solids in quasistatic statement and in the approximation of small strains and displacements. The latter let us consider the time-variable space domain $V(t)=V_0\cup V_*(t)$ occupied with the whole growing solid to the current time instant t to be known at any time instant and prescribed by the specific simulated growing process; here the domain $V_*(t)\subseteq V_\mathrm{A}$ is the piece of the additional part already formed to the time instant t. So the growth surface $\Sigma(t)$ of the growing solid moves in the space in a known manner, and its motion forms the domain V_A.

We denote for the notation conciseness

$$g^\circ(\mathbf{r},t)=\mathcal{H}_{\tau_0(\mathbf{r})}\,g(\mathbf{r},t) \qquad (3)$$

for arbitrary function $g(\mathbf{r},t)$ of solid point $\mathbf{r}$ and time t, and

$$h^\circ(t)=\mathcal{H}_{t_0}\,h(t) \qquad (4)$$

for arbitrary function of time $h(t)$ which is not associated with specific points of considered solid.

Note that similar to (1) constitutive relations are widely used to describe the mechanical behavior of various natural and artificial stone (in particular, concrete), polymers, soil, ice, wood. Typical experimental curves representing the evolution with time t of the specific longitudinal strain

$$\frac{\varepsilon(t,\tau)}{\sigma_0}=\frac{[\mathcal{H}_\tau\sigma_0](t)}{2(1+\nu)}\Big/\sigma_0=\frac{\Delta(t,\tau)}{2(1+\nu)}$$

of such material at its uniaxial tension with the constant stress σ_0 applied at the time moment τ can be borrowed, for example, from [50].

III. Problem Description

Let there be a conical solid of rotation which length l significantly exceeds its transverse dimensions. It is made from isotropic homogeneous aging linearly viscoelastic material subordinated to the constitutive equation (1).

At the moment $t=t_0$ a load is applied to the ends of the existing solid. We believe that at every moment of time $t\geqslant t_0$ it is statically equivalent to axial forces acting in the central points of the ends and varying with time following the law $P(t)$. We will consider positive the magnitude of tensile end force.

Some time after the application loading at the time $t=t_1$ we start the process of gradual axisymmetric thickening of the considered conical solid by adding the additional material to its lateral initially free from stresses surface. Thickening occurs in such a way that in each time moment the accreted solid maintains the shape of a right circular truncated cone of length l. This process is piecewise continuous in time, i.e. it consists of N consecutive phases of continuous accreting $t\in[t_{2k-1},t_{2k})$ $(k=\overline{1,N})$, separated by pauses of arbitrary duration. At the stages of continuous accreting an infinitely thin layer of material attaches to the solid each infinitely small period of time. The added material is supposed identical to the original one. In pauses the influx of additional material to the solid does not take place and its lateral surface is free from stresses. In the process of piecewise continuous accreting and after its completion time-varying central axial forces $P(t)$ continue to act to the end surfaces of the cone.

Let us investigate the evolution of stress-strain state of the considered conical solid under specified conditions of loading before the start, during and upon the completion of the described process of accreting. The process of deformation is assumed quasi-static, and strains developing — small.

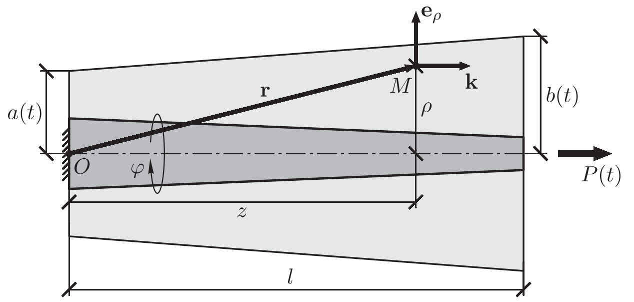

Fig. 1. A scheme of the simulated process

Changing the geometry of the considered conical solid due to its piecewise-continuous accreting is completely defined obviously by defining laws of increasing the radii of its ends in time. Denote them by $a(t)$ and $b(t)$ for any $t \geqslant t_0$. These functions are continuous and non-decreasing. They are constant outside the time spans $[t_{2k-1}, t_{2k})$.

Superpose the reference plane of a cylindrical polar coordinate system with that end of the taper in question which radius was denoted by $a(t)$. Place the beginning of coordinates O in the center of this end and extend coordinate axis Oz perpendicular to it inside the cone. Denote the polar radius and angle as ρ and φ. If $\{\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{k}\}$ is the normalized local basis of the introduced cylindrical coordinate system (ρ, φ, z), then the radius-vector of an arbitrary point of the solid can be represented in the form $\mathbf{r} = \mathbf{e}_\rho(\varphi)\,\rho + \mathbf{k}\,z$.

A scheme of the simulated process with the introduced geometrical parameters is illustrated in Fig. 1.

Let the lateral surface of the cone under consideration moving in space due to the influx of additional material (accreting) be described by the equation $\rho = \Lambda(z,t)$, where $\Lambda(z,t) = a(t)\cdot(1-z/l) + b(t)\cdot z/l$. The trace of its passing in space forms an additional part of the considered solid. At all time moments $t \in [t_{2k-1}, t_{2k})$ $(k = \overline{1,N})$ the lateral surface represents the actual growing surface of the considered accreted conical solid, i.e. the t-level surface of the function $\tau_*(\mathbf{r})$. Unit vectors of the external (directed from the axis of the cone) normal line to this surface form a vector field $\mathbf{n}(\mathbf{r}) = \mathbf{e}_\rho(\varphi)\cos\alpha\big(\tau_*(\mathbf{r})\big) - \mathbf{k}\sin\alpha\big(\tau_*(\mathbf{r})\big)$, in the additional part of the solid, where $\alpha(t) = \arctan\{[b(t)-a(t)]/l\}$ is the current shape opening angle of the growing taper.

IV. Boundary Value Problem on the Stage Before the Start of Accretion

Before the start of accretion the stress-strain state of the considered conical solid can be determined on the basis of the theory of viscoelasticity of homogeneously aging isotropic solids [49] and the Saint-Venant principle — from the solution of the following classical mechanical boundary value problem with integral force condition on the cone end surface, for $t_0 \leqslant t \leqslant t_1$:

$$\begin{gathered}
\nabla\cdot\mathsf{T} = \mathbf{0}, \quad 0 \leqslant \rho < \Lambda(z,t_0), \quad 0 \leqslant \varphi < 2\pi;\\
\mathcal{H}_{t_0}\mathsf{T} = 2\mathsf{E} + (\varkappa-1)\mathbf{1}\operatorname{tr}\mathsf{E}, \quad \mathsf{E} = (\nabla\mathbf{u}^{\mathrm{T}} + \nabla\mathbf{u})/2;\\
\mathbf{n}\cdot\mathsf{T} = \mathbf{0}, \quad \rho = \Lambda(z,t_0);\\
\int_{\{z=l\}} \left\| \begin{matrix} \mathbf{k}\cdot\mathsf{T} \\ \mathbf{e}_\rho\,\rho\times(\mathbf{k}\cdot\mathsf{T}) \end{matrix} \right\| dS = \left\| \begin{matrix} \mathbf{k}\,P(t) \\ \mathbf{0} \end{matrix} \right\|;\\
\mathbf{u} = \mathbf{0}, \quad \nabla\times\mathbf{u} = \mathbf{0}, \quad \rho = 0, \quad z = 0.
\end{gathered} \tag{5}$$

Here $\mathbf{u}(\mathbf{r},t)$ is the vector field of displacements. To exclude displacement components not causing deformation of the solid we imposed conditions of fixing the neighborhood of the center point of one of its end surfaces. We require these conditions to be satisfied after the start of the process of the considered solid accretion as well.

Using the notation (3) the boundary value problem (5) can be reformulated for values $\mathbf{u}$, E, T°:

$$\begin{gathered}
\nabla\cdot\mathsf{T}^\circ = \mathbf{0}, \quad 0 \leqslant \rho < \Lambda(z,t_0), \quad 0 \leqslant \varphi < 2\pi;\\
\mathsf{T}^\circ = 2\mathsf{E} + (\varkappa-1)\mathbf{1}\operatorname{tr}\mathsf{E}, \quad \mathsf{E} = (\nabla\mathbf{u}^{\mathrm{T}} + \nabla\mathbf{u})/2;\\
\mathbf{n}\cdot\mathsf{T}^\circ = \mathbf{0}, \quad \rho = \Lambda(z,t_0);\\
\int_{\{z=l\}} \left\| \begin{matrix} \mathbf{k}\cdot\mathsf{T}^\circ \\ \mathbf{e}_\rho\,\rho\times(\mathbf{k}\cdot\mathsf{T}^\circ) \end{matrix} \right\| dS = \left\| \begin{matrix} \mathbf{k}\,P(t) \\ \mathbf{0} \end{matrix} \right\|;\\
\mathbf{u} = \mathbf{0}, \quad \nabla\times\mathbf{u} = \mathbf{0}, \quad \rho = 0, \quad z = 0;
\end{gathered} \tag{6}$$

for $t_0 \leqslant t \leqslant t_1$. In the boundary value problem (6) time t is not a significant variable but acts only as a parameter.

We will call the tensor T° the operator stress tensor.

V. Boundary Value Problem on the Stage of Piecewise-Continuous Accretion

A. Transition to the Rate-Characteristics of the Deformation Process for the Accreted Solid

Due to the objective lack of natural (unstressed) configuration in the growing solid the kinematic description of the process of its deformation that is traditional in the mechanics of deformable solids is not suitable for this solid. However, it is clear that the particles of the new material after the attaching to the surface of growth continue to move as a part of continuous, even though growing, solid. This means that in the region of space occupied by the whole growing solid at this time, the enough smooth velocity field $\mathbf{v}(\mathbf{r},t)$ of the motion of its particles is uniquely determined. Therefore, the problem of such a solid deformation can be put in terms of velocity. In this case the strain velocity tensor $\mathsf{D}(\mathbf{r},t) = (\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})/2$ may play a part of the deforming process characteristic in the formulation of the defining relations of the material. The adopted equation of state (1) can be rewritten by using this tensor in the form [45]:

$$\mathsf{S} = 2\mathsf{D} + (\varkappa-1)\mathbf{1}\operatorname{tr}\mathsf{D}, \tag{7}$$

where we have introduced the so-called operator stress velocity tensor $\mathsf{S}(\mathbf{r},t) = \partial\mathsf{T}^\circ/\partial t$.

The approach requires knowing the whole story of changing the state of additional material elements up to their inclusion in the composition of the solid considered. In the studied in the present work process of accreting the additional material is supposed to be initially free of stresses (see Section III). In other words, we believe that the additional material begins to deform directly in the time of its attaching to the formed solid, and the attaching layers of additional material to the surface of the solid does not cause the appearance nonzero stresses in the formed solid near the surface of its growth:

$$\mathsf{T} = \mathbf{0}, \quad \rho = \Lambda(z,t), \quad t \in [t_{2k-1}, t_{2k}) \ \ (k = \overline{1,N}). \tag{8}$$

Note that condition (8) provides the equality to zero of the stress vector $\mathbf{n}\cdot\mathsf{T}$ at the current growth surface, i.e. absence of any loading onto this surface.

The condition of instantaneous local equilibrium in the growing solid has obviously the same form as in the classical solid of permanent composition. In the considered case of mass forces absence this condition is expressed by the standard equation

$$\nabla \cdot \mathsf{T} = \mathbf{0}. \tag{9}$$

It is possible to show [45] that for the simulated growth process (in the absence of load on the future and the actual growth surface of the solid during the whole process of its deformation) this equation generates similar differential equations for the tensors T° and S:

$$\nabla \cdot \mathsf{T}^\circ = \mathbf{0}, \quad \nabla \cdot \mathsf{S} = \mathbf{0}. \tag{10}$$

Equations (10) are fair at every moment of time $t > t_1$ in the region of space occupied by the whole growing solid at this moment. It should be emphasized that these equations are not a trivial consequence of the equilibrium equation (9), as in the case of accreting the solid the integral operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ and the operator of divergence ($\nabla \cdot$) do not commute in general because of the principal dependence of time τ_0 of the occurrence of stresses in the growing solid on the point of this solid $\mathbf{r}$.

One can also show, following [45], that the specific boundary condition (8) on the moving growth surface $\rho = \Lambda(z, t)$ implies the condition on the components of the tensor S, for every k-th step of continuous accreting, which is similar in its appearance to the standard in solid mechanics boundary condition for the stresses:

$$\mathbf{n} \cdot \mathsf{S} = \mathbf{0}, \quad \rho = \Lambda(z, t), \quad t \in [t_{2k-1}, t_{2k}). \tag{11}$$

Indeed, the set of conditions (8) given for all moments t in the time span $[t_{2k-1}, t_{2k})$ can be written in the form of the peculiar initial condition in that part of the solid, which is formed on the k-th stage of its continuous growth:

$$\begin{gathered} \mathsf{T}(\mathbf{r}, t) = \mathbf{0}, \quad t = \tau_*(\mathbf{r}), \\ \Lambda(z, t_{2k-1}) \leqslant \rho < \Lambda(z, t_{2k}). \end{gathered} \tag{12}$$

According to the definition of the operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ the condition (12) is equivalent to identity

$$\mathsf{T}^\circ\big(\mathbf{r}, \tau_*(\mathbf{r})\big) \equiv \mathbf{0} \tag{13}$$

in the specified part of the solid. Acting on the identity (13) with the operator of divergence we get

$$\mathbf{0} \equiv \big[\nabla \cdot \mathsf{T}^\circ(\mathbf{r}, t)\big]\big|_{t=\tau_*(\mathbf{r})} + \nabla\tau_*(\mathbf{r}) \cdot \mathsf{S}\big(\mathbf{r}, \tau_*(\mathbf{r})\big).$$

Attracting the first equation (10) and the geometric identity $\mathbf{n} = \nabla\tau_*/|\nabla\tau_*|$ (see Sections II, III) we get from the latter relation the condition (11).

In the pauses between stages of continuous growth and after the completion of growing the non-traditional condition (8) on the lateral surface of the considered taper should be replaced by the classical condition of equality to zero of the stress vector on this surface: $\mathbf{n} \cdot \mathsf{T} = \mathbf{0}$. Acting on this condition with the operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ and differentiating the result by time t, we see that the boundary condition (11) saves force even out of the time spans $[t_{2k-1}, t_{2k})$. However, it has a completely different mechanical nature in this case.

B. Transformation of the Integral Force Conditions at the Taper End Surface

On the end surface of the cone under consideration after the start of its piecewise continuous accreting it is necessary to use the same integral force conditions as in the boundary value problem (5) stated before the accretion. However, after the accretion process starts the region of integration $\{z = l\}$ begins to depend on time t, and we are to solve an especial mathematical problem to perform the needed transition from the original integral conditions for the components of the tensor T to the integral conditions for the components of the tensor S using in the corresponding problem statement (see Subsection V-A). The solution of this mathematical problem we obtain on the basis of the following supporting statement which has, as one can see, a general nature.

Lemma. *Let Ω_0 and Ω_{A} be two arbitrary surfaces situated inside or on the boundary of an aging viscoelastic solid subordinated to the state equation* (1) *and formed in a process of piecewise continuous accretion in N stages of continuous growth $t \in [t_{2k-1}, t_{2k})$ ($k = \overline{1, N}$) with arbitrary long pauses $t_{2k+1} - t_{2k}$ ($k = \overline{1, N-1}$) between the stages. The surface Ω_0 lies entirely within the original (existing before accreting) part V_0 of the solid considered. The surface Ω_{A} lies entirely in the additional (formed in the accreting process) part V_{A} of the solid and is obtained by motion in space of an arbitrary curve $\Gamma(t)$, $t \in [t_1, +\infty)$, which belongs to the current growth surface $\Sigma(t)$ of the solid at every moment t of its continuous accreting and is fixed in the pauses between the stages of continuous accretion and after the accreting process end, i.e. in the time periods $t \in [t_{2k}, t_{2k+1})$ ($k = \overline{1, N}$) where $t_{2N+1} = +\infty$: $\Omega_{\mathrm{A}} = \big\{\Gamma(t) \subset \Sigma(t) \,\big|\, t_1 \leqslant t < +\infty\big\}$. Let $g(\mathbf{r}, t)$ be an arbitrary function defined at the points $\mathbf{r}$ of the both surfaces Ω_0 and Ω_{A} for any $t \geqslant \tau_0(\mathbf{r})$. Assume that*

$$g\big(\mathbf{r}, \tau_*(\mathbf{r})\big) \equiv 0, \quad \mathbf{r} \in \Omega_{\mathrm{A}}. \tag{14}$$

Then it will be fair the formula

$$\frac{d}{dt}\left[\int_{\Omega(t)} g(\mathbf{r}, t)\, dS\right]^\circ = \int_{\Omega(t)} \frac{\partial g^\circ(\mathbf{r}, t)}{\partial t}\, dS \tag{15}$$

for $t > t_1$. Here the surface $\Omega(t)$, expanding in time and being disconnected in general, consists from the surface Ω_0 and that part of the surface Ω_{A} which has already been formed by the time $t \geqslant t_0$:

$$\begin{gathered} \Omega(t) = \begin{cases} \Omega_0, & t \in [t_0, t_1], \\ \Omega_0 \cup \Omega_*(t), & t \in (t_1, +\infty), \end{cases} \\ \Omega_*(t) = \big\{\Gamma(\tau) \,\big|\, t_1 \leqslant \tau \leqslant t\big\} \subseteq \Omega_{\mathrm{A}}. \end{gathered} \tag{16}$$

The proof of Lemma we give in the next Subsection. We are to emphasize that the surfaces Ω_0 and Ω_{A} considered in the Lemma may have arbitrary curvatures. Meanwhile the boundaries of these surfaces may not have common points. It is also possible that $\Omega_0 = \varnothing$. Forming a surface Ω_{A} curves $\Gamma(t)$ can be unclosed or closed. In the latter case the surface Ω_{A} may “circle” the original part of the solid or form a “tube” enveloping only the material of the additional part of the having been finally formed solid.

As a surface $\Omega(t)$ from the above formulated Lemma in the being stated in the current Section problem on accreting a conical solid it is necessary to consider the flat surface constituting one end side of the growing cone $z = l$ on every $t \geqslant t_0$. In this case, the surface Ω_0 is the circle $0 \leqslant \rho \leqslant b_0$. The surface $\Omega_{\rm A}$ is annular, and its forming curves $\Gamma(t)$ are the concentric circumference $\rho = b(t)$.

Then by the Lemma because of the condition (12) we have

$$\frac{\partial}{\partial t}\left[\int_{\{z=l\}}\left\|\begin{matrix}\mathbf{k}\cdot\mathbf{T}\\ \mathbf{e}_\rho\,\rho\times(\mathbf{k}\cdot\mathbf{T})\end{matrix}\right\|dS\right]^\circ = \\ = \int_{\{z=l\}}\left\|\begin{matrix}\mathbf{k}\cdot\mathbf{S}\\ \mathbf{e}_\rho\,\rho\times(\mathbf{k}\cdot\mathbf{S})\end{matrix}\right\|dS, \quad t > t_1.$$

C. Proof of the Lemma

Let us introduce on the surface $\Omega_{\rm A}$ the parameterization $\mathbf{r} = \mathbf{r}(\xi,\eta)$ induced by the ongoing growth program, in the following way. Let us select a certain general parameter $\eta \in [A, B]$, where A and B are some constants, for all the curves $\Gamma(t)$, $t \geqslant t_1$, the surface $\Omega_{\rm A}$ is composed from. This means that curves $\Gamma(t)$ form a family of η-lines on the surface $\Omega_{\rm A}$. The choice of the parameter η for all the curves $\Gamma(t)$ is to ensure that through each point of the surface $\Omega_{\rm A}$ it goes one and only one line from the second family — the family of lines consisting of all those points on different curves $\Gamma(t)$, which correspond to the same value η.

Let the geometric position of the curve $\Gamma(t)$ on the surface $\Omega_{\rm A}$ at a particular value of time t be in one-to-one correspondance to a particular value of some quantity ξ, namely the value $\xi = \Xi(t)$: if at different t' and t'' the curves $\Gamma(t')$ and $\Gamma(t'')$ geometrically coincide, then $\Xi(t') = \Xi(t'')$; and vice versa, if at different t' and t'' the curves $\Gamma(t')$ and $\Gamma(t'')$ do not geometrically coincide, then $\Xi(t') \neq \Xi(t'')$. The function $\Xi(t)$ we consider continuous for all $t \geqslant t_1$ and monotonically non-decreasing on the time intervals $t \in [t_{2k-1}, t_{2k})$ $(k = \overline{1,N})$ corresponding to the stages of the continuous growth of the solid. Beyond these intervals, that is, when $t \in [t_{2k}, t_{2k+1})$, where $t_{2N+1} = +\infty$, when the influx of additional material to the solid is absent and therefore the movement of the curve $\Gamma(t)$ over the surface $\Omega_{\rm A}$ is temporarily (when $k = \overline{1, N-1}$) or ultimately (when $k = N$) completed the function $\Xi(t)$ takes obviously constant values:

$$\Xi(t) \equiv \xi_k, \quad t \in [t_{2k}, t_{2k+1}),$$

where $\xi_k = \Xi(t_{2k})$. Due to continuity of the function $\Xi(t)$ the value ξ_k is the value of the parameter ξ corresponding to the position of the curve $\Gamma(t)$ at the end of the k-th stage of continuous growth.

If the value $\xi_0 = \Xi(t_1)$ corresponds to the position of the curve $\Gamma(t)$ at the initial time instant of the process of growing the solid, then due to monotonicity of the function $\Xi(t)$ we will have

$$\xi_0 \leqslant \xi_1 \leqslant \ldots \leqslant \xi_{N-1} \leqslant \xi_N, \tag{17}$$

and the parameter ξ over the surface $\Omega_{\rm A}$ will vary on the interval $\xi \in [\xi_0, \xi_N]$.

Note that a possible non-strict monotonicity (non-decreasing) of the function $\Xi(t)$ in the spans of continuous growth $t \in [t_{2k-1}, t_{2k})$ and, consequently, possible non-strict signs in chain (17), arise from the possibility of pursuing such a variant of accreting the solid when a region of space that it occupies at the stage of continuous growth is constantly expanding due to the influx of additional material to the part $\Sigma(t)$ of boundary surface of the growing solid — the current surface of growth, — but herewith the trace $\Gamma(t)$ of the moving in space growth surface $\Sigma(t)$ on the surface $\Omega_{\rm A}$ selected by us inside or on the boundary of the growing solid remains for some time stationary.

The family of lines consisting of points corresponding to the same value of the parameter η on different curves $\Gamma(t)$ which was discussed above is obviously the family of ξ-lines on the surface $\Omega_{\rm A}$.

The couple of parameters (ξ, η) will be considered as curvilinear coordinates on the surface $\Omega_{\rm A}$. The convenience of introducing such coordinates is explained for us by the following key fact. Since at each value t the curve $\Gamma(t)$ lies entirely on the current growth surface $\Sigma(t)$, then at the coordinates (ξ, η) the time instant $\tau_0(\mathbf{r})$ of occurrence of stresses at the points $\mathbf{r}$ of the surface $\Omega_{\rm A}$ that coincides with time instant $\tau_*(\mathbf{r})$ of the inclusion of these points in the composition of the growing solid (see (2)) depends only on the coordinate ξ:

$$\tau_0(\mathbf{r}) \equiv \tau_0(\xi), \tag{18}$$

moreover, in accordance with the definition of the function $\Xi(t)$ it is true for all $\xi \in [\xi_0, \xi_N]$ the identity

$$\Xi\big(\tau_0(\xi)\big) \equiv \xi. \tag{19}$$

By using the curvilinear coordinates (ξ, η) the integral over the surface $\Omega_*(t) \subseteq \Omega_{\rm A}$ of the arbitrary function $f(\mathbf{r}, t)$ of the point and time is written as follows:

$$\int_{\Omega_*(t)} f(\mathbf{r}, t)\, dS = \\ = \int_A^B d\eta \int_{\xi_0}^{\Xi(t)} f(\xi, \eta, t)\, J(\xi, \eta)\, d\xi. \tag{20}$$

Here the value $J(\xi, \eta)$ defines an element of the area of the surface $\Omega_{\rm A}$ in the curvilinear coordinates (ξ, η) and equals to

$$J(\xi, \eta) = \left|\frac{\partial \mathbf{r}}{\partial \xi} \times \frac{\partial \mathbf{r}}{\partial \eta}\right|.$$

By choosing $f(\mathbf{r}, t) = \partial g^\circ(\mathbf{r}, t)/\partial t$ and using the rule of differentiation of the integral by parameter, we can transform the inner integral in (20):

$$\int_{\xi_0}^{\Xi(t)} \frac{\partial g^\circ(\xi, \eta, t)}{\partial t}\, J(\xi, \eta)\, d\xi = \\ = \frac{\partial}{\partial t}\int_{\xi_0}^{\Xi(t)} g^\circ(\xi, \eta, t)\, J(\xi, \eta)\, d\xi - \\ - g^\circ\big(\Xi(t), \eta, t\big)\, J\big(\Xi(t), \eta\big)\, \Xi'(t). \tag{21}$$

Let us reveal the symbolic notation $(\,)^\circ$ in the integral standing on the right in (21) (see definition (3) and formula (18)):

$$\int_{\xi_0}^{\Xi(t)} g^\circ(\xi, \eta, t)\, J(\xi, \eta)\, d\xi = \\ = \int_{\xi_0}^{\Xi(t)} \frac{g(\xi, \eta, t)}{G(t)}\, J(\xi, \eta)\, d\xi - \\ - \int_{\xi_0}^{\Xi(t)} J(\xi, \eta)\, d\xi \int_{\tau_0(\xi)}^{t} \frac{g(\xi, \eta, \tau)}{G(\tau)}\, K(t, \tau)\, d\tau. \tag{22}$$

Let us change in (22) the order of integration in the repeated integral (see identity (19)):

$$\int_{\xi_0}^{\Xi(t)} J(\xi,\eta)\,d\xi \int_{\tau_0(\xi)}^{t} \frac{g(\xi,\eta,\tau)}{G(\tau)}\,K(t,\tau)\,d\tau = \\ = \int_{t_1}^{t} K(t,\tau)\,d\tau \int_{\xi_0}^{\Xi(\tau)} \frac{g(\xi,\eta,\tau)}{G(\tau)}\,J(\xi,\eta)\,d\xi. \quad (23)$$

We consider now the term beyond the integrals in (21). We have (see definition (3) and formula (18))

$$g^\circ\big(\Xi(t),\eta,t\big) = \frac{g\big(\Xi(t),\eta,t\big)}{G(t)} - \\ - \int_{\tau_0(\Xi(t))}^{t} \frac{g\big(\Xi(t),\eta,\tau\big)}{G(\tau)}\,K(t,\tau)\,d\tau.$$

On all pieces of the strict monotonicity of the function $\Xi(t)$ this function has a reverse one, which is the function $\tau_0(\xi)$ in accordance with identity (19). Therefore at the lower limit in the last integral it will be $\tau_0\big(\Xi(t)\big) \equiv t$ and this integral will be identically equal to zero. Herewith we also have

$$g\big(\Xi(t),\eta,t\big) = g(\xi,\eta,t)\big|_{\xi=\Xi(t)} = \\ = g(\xi,\eta,t)\big|_{t=\tau_0(\xi)} = g(\mathbf{r},t)\big|_{t=\tau_0(\mathbf{r})} \equiv 0$$

in accordance with condition (14) and representation (2).

Thus, on pieces of the strict monotonicity of the function $\Xi(t)$ the term beyond the integrals in (21) becomes zero. On pieces of the function $\Xi(t)$ constancy (in particular, in the pauses between the stages of continuous accretion of the solid and after its ultimate growth completing) it will be $\Xi'(t) \equiv 0$, therefore this term vanishes as well.

So, from (21)–(23) we get

$$\int_{\xi_0}^{\Xi(t)} \frac{\partial g^\circ(\xi,\eta,t)}{\partial t}\,J(\xi,\eta)\,d\xi = \\ = \frac{\partial}{\partial t}\left[\int_{\xi_0}^{\Xi(t)} \frac{g(\xi,\eta,t)}{G(t)}\,J(\xi,\eta)\,d\xi - \right. \\ \left. - \int_{t_1}^{t} K(t,\tau)\,d\tau \int_{\xi_0}^{\Xi(\tau)} \frac{g(\xi,\eta,\tau)}{G(\tau)}\,J(\xi,\eta)\,d\xi\right].$$

Hence, due to representation (20) we have

$$\int_{\Omega_*(t)} \frac{\partial g^\circ(\mathbf{r},t)}{\partial t}\,dS = \frac{d}{dt}\left[\int_{\Omega_*(t)} \frac{g(\mathbf{r},t)}{G(t)}\,dS - \right. \\ \left. - \int_{t_1}^{t} K(t,\tau)\,d\tau \int_{\Omega_*(\tau)} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,dS\right]. \quad (24)$$

To complete the proof of formula (15) it remains to present the integral of the function $f(\mathbf{r},t) = \partial g^\circ(\mathbf{r},t)/\partial t$ over the surface Ω_0 supplementing the surface $\Omega_*(t)$ up to $\Omega(t)$ in a form similar to (24). As the surface Ω_0 is unchanged in time, then the time derivative can be taken outside the sign of the integral over this surface:

$$\int_{\Omega_0} \frac{\partial g^\circ(\mathbf{r},t)}{\partial t}\,dS = \frac{d}{dt}\int_{\Omega_0} g^\circ(\mathbf{r},t)\,dS. \quad (25)$$

We then reveal the symbolic notation $(\,)^\circ$ in the integral on the right (see definition (3)) by changing the order of integration over time and over the surface Ω_0 that does not dependent on time:

$$\int_{\Omega_0} g^\circ(\mathbf{r},t)\,dS = \int_{\Omega_0} \frac{g(\mathbf{r},t)}{G(t)}\,dS - \\ - \int_{\Omega_0} dS \int_{t_0}^{t} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,K(t,\tau)\,d\tau = \\ = \int_{\Omega_0} \frac{g(\mathbf{r},t)}{G(t)}\,dS - \int_{t_0}^{t} K(t,\tau)\,d\tau \int_{\Omega_0} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,dS. \quad (26)$$

It is taken here into account that $\tau_0(\mathbf{r}) \equiv t_0$ for $\mathbf{r} \in \Omega_0$ as the surface Ω_0 lies entirely in the originally existing part of the solid, stresses at all points of which arose at the same moment of time $t = t_0$ (see (2)).

When $t > t_1$ we can split the time integral in (26) into two integrals — from t_0 to t_1 and from t_1 to t, — taking into account that $\Omega(t) \equiv \Omega_0$ for $t \in [t_0, t_1]$ (see (16)):

$$\int_{\Omega_0} g^\circ(\mathbf{r},t)\,dS = \int_{\Omega_0} \frac{g(\mathbf{r},t)}{G(t)}\,dS - \\ - \int_{t_0}^{t_1} K(t,\tau)\,d\tau \int_{\Omega(\tau)} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,dS - \\ - \int_{t_1}^{t} K(t,\tau)\,d\tau \int_{\Omega_0} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,dS. \quad (27)$$

As a result, due to (25) we will have

$$\int_{\Omega_0} \frac{\partial g^\circ(\mathbf{r},t)}{\partial t}\,dS = \frac{d}{dt}\left[\int_{\Omega_0} \frac{g(\mathbf{r},t)}{G(t)}\,dS - \right. \\ - \int_{t_1}^{t} K(t,\tau)\,d\tau \int_{\Omega_0} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,dS - \\ \left. - \int_{t_0}^{t_1} K(t,\tau)\,d\tau \int_{\Omega(\tau)} \frac{g(\mathbf{r},\tau)}{G(\tau)}\,dS\right]$$

that together with (24) with regard to (16) and (4) gives the formula (15) being proved.

D. Summarized Formulation of the Boundary Value Problem

Thus, collecting together all the above-formulated relations for the quantities $\mathbf{v}$, $\mathbf{D}$, $\mathbf{S}$ we can supply the following boundary value problem describing the process of deforming the considered conical solid on all the temporary beam after the beginning of its accreting, for $t > t_1$:

$$\begin{gathered} \nabla\cdot\mathbf{S} = \mathbf{0}, \quad 0 \leqslant \rho < \Lambda(z,t), \quad 0 \leqslant \varphi < 2\pi; \\ \mathbf{S} = 2\mathbf{D} + (\varkappa - 1)\mathbf{1}\,\mathrm{tr}\,\mathbf{D}, \quad \mathbf{D} = (\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})/2; \\ \mathbf{n}\cdot\mathbf{S} = \mathbf{0}, \quad \rho = \Lambda(z,t); \\ \int_{\{z=l\}} \left\| \begin{matrix} \mathbf{k}\cdot\mathbf{S} \\ \mathbf{e}_\rho\,\rho \times (\mathbf{k}\cdot\mathbf{S}) \end{matrix} \right\| dS = \left\| \begin{matrix} \mathbf{k}\,\partial P^\circ(t)/\partial t \\ \mathbf{0} \end{matrix} \right\|; \\ \mathbf{v} = \mathbf{0}, \quad \nabla\times\mathbf{v} = \mathbf{0}, \quad \rho = 0, \quad z = 0. \end{gathered} \quad (28)$$

Given in (28) conditions for the vector field of velocities $\mathbf{v}(\mathbf{r},t)$ in the neibourhood of the coordinates origin O provide a rigid fixing this neighbourhood throughout the whole process of deformation of considered growing solid.

VI. Solution of the Auxiliary Problem on the Tension–Compression of a Truncated Cone

As we can see, the problem (6) and the problem (28) turned out to be mathematically equivalent to the same classical mechanical problem of the equilibrium of a linearly elastic truncated circular cone of permanent composition with free lateral surface $\rho = \Lambda(z,t)$, $z \in [0,l]$, rigidly fixed in the coordinates origin and being under the action of axial forces centrally applied to its ends. The radii of the ends of the cone and the value of forces acting on it depend on a real parameter t. This formal coincidence is gained by substituting in the problems (6) and (28) the values P° and $\partial P^\circ/\partial t$ to the value of tensile force related to the shear modulus, the tensors $\mathbf{T}^\circ$ and $\mathbf{S}$ to the stress tensor related to the shear modulus, and in the problem (28) — also the tensor $\mathbf{D}$ to the small strain tensor and the vector $\mathbf{v}$ to the displacement vector as well. Let us construct the analytical solution of the described classical problem of the theory of elasticity.

Consider a non-growing elastic truncated cone of length l, to that ends of radii a and b the central tensile axial forces of magnitude P are applied. We introduce the polar cylindrical coordinate system (ρ,φ,z) in the region busy by the cone in the way we did it in Section III for accreted conical solid. A cone is considered sufficiently long in the axial direction compared with its transverse dimensions. In this case, the specific distribution of acting on the ends forces does not influence the stress-strain state of the greater part of the cone, and this condition can be determined on the basis of the Saint-Venant principle. To do it we can use the known solution of the problem of tensioning an infinitely long pointed cone with an axial force P applied to its vertex [51]. Let us introduce an additional spherical coordinate system (R,Θ,Φ) with the center at the cone vertex, where R is the length of radius-vector, Φ is the longitudinal angle counted around the axis of symmetry of the cone, Θ is the pole angle counted from the axis of symmetry inside the solid. In this coordinate system the mentioned solution has the form:

$$\mathbf{u} = \mathbf{e}_R\,u_R + \mathbf{e}_\Theta\,u_\Theta,$$

$$\begin{Vmatrix} u_R \\ u_\Theta \end{Vmatrix} = \frac{P}{4\pi G R\, Q(\cos\Theta_0)} \times$$
$$\times \begin{Vmatrix} 2(\varkappa+1)\cos\Theta - (1+\cos\Theta_0) \\ \big[(1+\cos\Theta_0)/(1+\cos\Theta) - (\varkappa+2)\big]\sin\Theta \end{Vmatrix};$$

$$\mathbf{T} = \mathbf{e}_R\mathbf{e}_R\,\sigma_R + \mathbf{e}_\Theta\mathbf{e}_\Theta\,\sigma_\Theta + \mathbf{e}_\Phi\mathbf{e}_\Phi\,\sigma_\Phi + (\mathbf{e}_R\mathbf{e}_\Theta + \mathbf{e}_\Theta\mathbf{e}_R)\,\tau_{R\Theta},$$

$$\begin{Vmatrix} \sigma_R \\ \sigma_\Theta \\ \sigma_\Phi \\ \tau_{R\Theta} \end{Vmatrix} = \frac{P}{2\pi R^2\, Q(\cos\Theta_0)} \times$$
$$\times \begin{Vmatrix} 1+\cos\Theta_0 - (3\varkappa+1)\cos\Theta \\ \big[1-(1+\cos\Theta_0)/(1+\cos\Theta)\big]\cos\Theta \\ \cos\Theta - (1+\cos\Theta_0)/(1+\cos\Theta) \\ \big[1-(1+\cos\Theta_0)/(1+\cos\Theta)\big]\sin\Theta \end{Vmatrix}.$$

Here $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ is the normalized local basis of the spherical coordinate system, Θ_0 is the angle of the cone opening, and $Q(\xi) = \varkappa\xi^3 - \xi^2 + \xi - \varkappa$.

To apply the written solutions to the considered in this section classical problem of theory of elasticity it is necessary to extend the lateral surface of the considered truncated cone of length l in both sides in the axial direction so as to obtain infinitely long cone with a vertex. Denote this vertex as O'. After this it is necessary to analyze separately the cases $a < b$ and $a > b$.

In the case $a < b$ (or $a > b$) the reference end $z = 0$ of the truncated cone lies closer among its two ends to (or lies further from) the vertex O' of a pointed cone. Therefore, the introduced in Section III vector $\mathbf{k}$ is codirected (or oppositely directed) to the vector defining the direction $\Theta = 0$, and the vector $\mathbf{e}_\varphi$ is codirected (or oppositely directed) to the vector $\mathbf{e}_\Phi$. Thus, the transition from the additionally introduced spherical coordinate system to the original cylindrical one is maintained by means of the following transformation of the local bases:

$$\begin{Vmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_\Phi \end{Vmatrix} = \begin{Vmatrix} \mathbf{e}_\rho & \mathbf{e}_\varphi & \mathbf{k} \end{Vmatrix} \cdot \begin{Vmatrix} \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & \pm 1 \\ \pm\cos\Theta & \mp\sin\Theta & 0 \end{Vmatrix}.$$

The upper signs correspond to the case $a < b$, the lower ones — to the case $a > b$. Meanwhile, we also need to put $\cos\Theta = \pm(z+d)/R$, $\sin\Theta = \rho/R$, $R = \sqrt{\rho^2 + (z+d)^2}$, $d = la/(b-a)$, $\Theta_0 = \pm\alpha$, $\alpha = \arctan\{(b-a)/l\} = \arctan\{a/d\}$. The value of d is, accurate to sign, the distance from the reference end $z = 0$ of the truncated cone to the vertex O' of a pointed cone. The value of α is, accurate to sign, the cone opening angle.

Perform the specified transformations, ensuring rigid fixing of a neighborhood of the coordinates origin O by adding a proper constant to the axial displacement. We find:

$$\mathbf{u} = \mathbf{e}_\rho\,u_\rho + \mathbf{k}\,u_z,$$

$$\begin{Vmatrix} u_\rho \\ u_z + c \end{Vmatrix} = \frac{P}{4\pi G\, R(\rho,z)\, Q(\cos\alpha)} \times$$
$$\times \begin{Vmatrix} \left[\dfrac{\pm\varkappa(z+d)}{R(\rho,z)} - \dfrac{1+\cos\alpha}{1 \pm (z+d)/R(\rho,z)}\right]\dfrac{\rho}{R(\rho,z)} \\ \pm\Big[\varkappa(z+d)^2/R^2(\rho,z) + \varkappa + 1 - \cos\alpha\Big] \end{Vmatrix},$$

$$\mathbf{T} = \mathbf{e}_\rho\mathbf{e}_\rho\,\sigma_\rho + \mathbf{e}_\varphi\mathbf{e}_\varphi\,\sigma_\varphi + \mathbf{k}\mathbf{k}\,\sigma_z + (\mathbf{e}_\rho\mathbf{k} + \mathbf{k}\mathbf{e}_\rho)\,\tau_{\rho z},$$

$$\begin{Vmatrix} \sigma_\rho \\ \sigma_\varphi \\ \sigma_z \\ \tau_{\rho z} \end{Vmatrix} = \frac{P}{2\pi\, R^2(\rho,z)\, Q(\cos\alpha)} \times$$
$$\times \begin{Vmatrix} \dfrac{1+\cos\alpha}{1 \pm (z+d)/R(\rho,z)} \mp \left[\cos\alpha + \dfrac{3\varkappa\rho^2}{R^2(\rho,z)}\right]\dfrac{z+d}{R(\rho,z)} \\ \pm\dfrac{z+d}{R(\rho,z)} - \dfrac{1+\cos\alpha}{1 \pm (z+d)/R(\rho,z)} \\ \pm\big[\cos\alpha - 3\varkappa(z+d)^2/R^2(\rho,z)\big]\,(z+d)/R(\rho,z) \\ \pm\big[\cos\alpha - 3\varkappa(z+d)^2/R^2(\rho,z)\big]\,\rho/R(\rho,z) \end{Vmatrix}.$$

Above, the following constant is used:

$$c = \frac{(2\varkappa + 1 - \cos\alpha)P\tan\alpha}{4\pi G a\, Q(\cos\alpha)}.$$

The expressions received for the displacements and stresses can be written simplier with the function $\zeta(\rho,z) = \pm(z+d)/R(\rho,z) = \cos\Theta$, if we also enter the function of the shape of the cone lateral surface $\Lambda(z) = a\cdot(1-z/l)+b\cdot z/l$ and note that $z+d = \Lambda(z)/\tan\alpha$. Indeed, then we have $\pm R(\rho,z) = (z+d)/\cos\Theta = \Lambda(z)/[\zeta(\rho,z)\tan\alpha]$, moreover, the function $\zeta(\rho,z)$ can be calculated by the formula

$$\zeta(\rho,z) = \left[\rho^2\tan^2\alpha/\Lambda^2(z)+1\right]^{-1/2}, \tag{29}$$

as $\zeta = (\tan^2\Theta+1)^{-1/2}$ and $\tan\Theta = \sin\Theta/\cos\Theta = \pm\rho/(z+d) = \pm\rho\tan\alpha/\Lambda(z)$.

In result

$$\left\|\begin{matrix} u_\rho \\ u_z + c\end{matrix}\right\| = \frac{P\,\zeta(\rho,z)\tan\alpha}{4\pi G\,\Lambda(z)\,Q(\cos\alpha)}\times$$
$$\times\left\|\begin{matrix}\left[\varkappa\zeta(\rho,z) - \dfrac{1+\cos\alpha}{1+\zeta(\rho,z)}\right]\dfrac{\zeta(\rho,z)\,\rho\tan\alpha}{\Lambda(z)} \\ \varkappa\zeta^2(\rho,z)+\varkappa+1-\cos\alpha\end{matrix}\right\|,$$

$$\left\|\begin{matrix}\sigma_\rho\\ \sigma_\varphi\\ \sigma_z\\ \tau_{\rho z}\end{matrix}\right\| = \frac{P\,\zeta^2(\rho,z)\tan^2\alpha}{2\pi\Lambda^2(z)\,Q(\cos\alpha)}\times \tag{30}$$
$$\times\left\|\begin{matrix}\dfrac{1+\cos\alpha}{1+\zeta(\rho,z)} - \big[\cos\alpha - 3\varkappa\zeta^2(\rho,z) + \\ \qquad\qquad + 3\varkappa\big]\,\zeta(\rho,z) \\ \zeta(\rho,z) - \big[1+\cos\alpha\big]/\big[1+\zeta(\rho,z)\big] \\ \big[\cos\alpha - 3\varkappa\zeta^2(\rho,z)\big]\,\zeta(\rho,z) \\ \big[\cos\alpha - 3\varkappa\zeta^2(\rho,z)\big]\,\zeta(\rho,z)\,\rho\tan\alpha/\Lambda(z)\end{matrix}\right\|.$$

Note that all stresses are proportional to the value of $P/\big[\pi\Lambda^2(z)\big]$, which is, obviously, the average normal stress acting at any cross-section $z = \text{const}$ of the cone.

It is easy to make sure that the expression (30) remain in force even in the special case of cylindrical solid, which was so far excluded from our consideration. Indeed, if the parameter α tends to zero at fixed values of other geometric parameters a and l of the taper and at arbitrary fixed values of the variables $\rho\in[0,a)$ and $z\in(0,l)$, then given the representation $\Lambda(z) = a + z\tan\alpha$ we have

$$u_\rho \to -\frac{P}{\pi a^2}\cdot\frac{(\varkappa-1)\rho}{2(3\varkappa-1)G},\quad u_z\to\frac{P}{\pi a^2}\cdot\frac{\varkappa z}{(3\varkappa-1)G},$$
$$\sigma_z\to\frac{P}{\pi a^2},\quad \sigma_{\rho,\varphi},\tau_{\rho z}\to 0.$$

Obtained form (30) with $\alpha\to 0$ the limit values of displacements and stresses correspond, obviously, to the solution of the Saint-Venant problem on the uniaxial tension with a force P of a circular cylinder with the fixed end $z=0$.

VII. The Construction of the Solution of the Accreting Problem

In Section VI the solution of the classical problem on tension–compression of a non-growing elastic conical solid with an arbitrary correlation of radii of its fixed and loaded end surfaces is constructed. As indicated in Section VI, after a suitable replacement of variables contained in the solution it is possible to obtain the solutions of boundary value problems (6) and (28). These solutions will contain the introduced in Section III functions $\alpha(t)$ and $\Lambda(z,t)$ as well as the function

$$\zeta(\rho,z,t) = \left[\rho^2\tan^2\alpha(t)/\Lambda^2(z,t)+1\right]^{-1/2}$$

introduced by analogy with (29).

As a result, in each point $\mathbf{r}$ of the considered piecewise continuously accreted aging viscoelastic conical solid we will know the evolution of the velocity vector $\mathbf{v}$ and the operator stress velocity tensor $\mathbf{S}$ on the time beam

$$t > \tau_1(\mathbf{r}) = \begin{cases} t_1, & 0\leqslant\rho<\Lambda(z,t_0),\\ \tau_*(\mathbf{r}), & \Lambda(z,t_0)\leqslant\rho<\Lambda(z,t_{2N}),\end{cases}$$

which covers the entire deformation history of the neighborhood of a given point $\mathbf{r}$ in the composition of the formed solid from the beginning of the process of its accretion. And at the points of the original part of this solid we will also know the evolution of the displacement vector $\mathbf{u}$ and the operator stress tensor $\mathbf{T}^\circ$ on the time segment $t\in[t_0,t_1]$ before the beginning of the accretion process. After that, the evolution of the operator stress tensor $\mathbf{T}^\circ$ at any point $\mathbf{r}$ of the solid for all $t\geqslant\tau_1(\mathbf{r})$ can be reconstructed by using the integration procedure:

$$\mathbf{T}^\circ(\mathbf{r},t) = \mathbf{T}^\circ\big(\mathbf{r},\tau_1(\mathbf{r})\big) + \int_{\tau_1(\mathbf{r})}^{t}\mathbf{S}(\mathbf{r},\tau)\,d\tau.$$

Here we have $\mathbf{T}^\circ\big(\mathbf{r},\tau_1(\mathbf{r})\big) = \mathbf{0}$ in the additional part of the solid according to the initial condition (13).

When in a point $\mathbf{r}$ of the considered additionally formed solid we have found out the complete evolution of the tensor $\mathbf{T}^\circ$, i.e., the values of this tensor since the moment $t=\tau_0(\mathbf{r})$ of the stress occurrence in a given point, so we can find the complete evolution of the stresses tensor $\mathbf{T}$ in this point by using the inverse to $\mathcal{H}_{\tau_0(\mathbf{r})}$ integral transformation $\mathcal{H}^{-1}_{\tau_0(\mathbf{r})}$:

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)} = \mathbf{T}^\circ(\mathbf{r},t) + \int_{\tau_0(\mathbf{r})}^{t}\mathbf{T}^\circ(\mathbf{r},\tau)\,R(t,\tau)\,d\tau. \tag{31}$$

When we use a particular approximation for the creep kernel $K(t,\tau)$ then the expression for the respective relaxation kernel $R(t,\tau)$ may not be known in the closed form or be too bulky. In such situation the procedure of reconstructing the evolution of the tensor $\mathbf{T}$ by numerical treatment of the Volterra integral equation of the 2nd kind

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)} - \int_{\tau_0(\mathbf{r})}^{t}\frac{\mathbf{T}(\mathbf{r},\tau)}{G(\tau)}\,K(t,\tau)\,d\tau = \mathbf{T}^\circ(\mathbf{r},t),$$

for example, by the method of quadratures [52], will be less expensive from a computational point of view and may be even more precise than use of analytic formula (31).

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