Artificial Boundary Method for Anisotropic Problems in Semi-infinite Strips

Yajun Chen, and Qikui Du

Abstract—**In this paper, the artificial boundary method for anisotropic problems in semi-infinite strips is investigated. The exact and approximate boundary conditions on a segment artificial boundary are given. Finite element approximations are applied to the problem in a bounded computational domain and error estimates are obtained. Finally, some numerical examples show the effectiveness of this method.**

Index Terms—**artificial boundary method, anisotropic problem, error estimate.**

I. Introduction

PROBLEMS in semi-infinite strips are encountered in applications involving waveguide or flow around an obstacle in a channel. Artificial boundary method [1]-[2], which is also called coupling method with natural boundary reduction [3]-[5] or DtN method [6]-[7] is a common method to solve such problems numerically. The method may be summarized as follows: (i) Introduce an artificial boundary, which divides the original unbounded domain into two non-overlapping subdomains: a bounded computational domain and an infinite residual domain. (ii) By analyzing the problem in the infinite residual domain, obtain a relation on the artificial boundary involving the unknown function and its derivatives. (iii) Using the relation as a boundary condition, to obtain a well-posed problem in the bounded computational domain. (iv) Solve the problem in the bounded computational domain be the standard finite element methods or some other numerical methods. Other related works can also be found from [8]-[20].

Recently, the authors proposed some new artificial boundary methods and domain decomposition methods based on elliptical arc artificial boundary to solve Poisson problems and anisotropic problems [21]-[24]. In this paper, we investigate the artificial boundary method for anisotropic problems in semi-infinite strips. Let Ω be a strip, and b is the width of the channel Ω. The boundary of domain Ω is decomposed into three disjoint parts: Γ_W, Γ_N, and Γ_S(see Fig. 1). We introduce a Cartesian coordinate system (x, y), such that the ray Γ_S coincides with the x axis.

We consider the following anisotropic problems in two cases:

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) = f, \quad \text{in } \Omega, \\ \mathcal{A}\nabla u\cdot n = 0, \quad \text{on } \Gamma_S \cup \Gamma_N, \\ u = g, \quad \text{on } \Gamma_W, \\ u \text{ is bounded at infinity}, \end{cases} \tag{1}$$

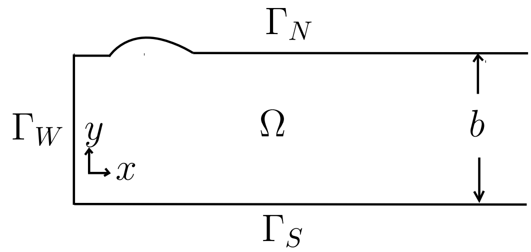

Fig. 1. The Illustration of Domain Ω

and

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) = f, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \Gamma_S \cup \Gamma_N, \\ \mathcal{A}\nabla u\cdot n = h, \quad \text{on } \Gamma_W, \\ u \text{ is vanish at infinity}, \end{cases} \tag{2}$$

where $\mathcal{A} = \begin{pmatrix} k^2 & 0 \\ 0 & 1 \end{pmatrix}$, k is a constant and $0 < k < 1$, u is the unknown function, $f \in L^2(\Omega)$ and $g, h \in L^2(\Gamma_W)$ are given functions, supp(f) is compact.

The rest of the paper is organized as follows. In section 2, we obtain the exact artificial boundary condition. In section 3, we give the equivalent variational problem and it's well-posedness. In section 4, we discuss the finite element approximation and an new error estimate that depends on the finite element mesh, the order of artificial boundary condition and the location of the artificial boundary. Finally, in section 5, we give some numerical examples to show the effectiveness of the method.

II. The exact artificial boundary condition

We introduce a segment artificial boundary $\Gamma_E = \{(x, y)|x = d,\ 0 \le y \le b\}$ to enclose supp(f), which divides Ω into a bounded domain Ω_W and an unbounded domain Ω_E(see Fig. 2).

In the first case, problem (1) confines in Ω_W is

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) = f, \quad \text{in } \Omega_W, \\ \mathcal{A}\nabla u\cdot n = 0, \quad \text{on } \Gamma_{SW} \cup \Gamma_{NW}, \\ u = g, \quad \text{on } \Gamma_W, \end{cases} \tag{3}$$

where $\Gamma_{SW} = \Gamma_S \cap \Omega_W$, $\Gamma_{NW} = \Gamma_N \cap \Omega_W$.

Problem (1) confines in Ω_E is

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) = 0, \quad \text{in } \Omega_E, \\ \mathcal{A}\nabla u\cdot n = 0, \quad \text{on } \Gamma_{SE} \cup \Gamma_{NE}, \\ u \text{ is bounded at infinity}, \end{cases} \tag{4}$$

where $\Gamma_{SE} = \Gamma_S \cap \Omega_E$, $\Gamma_{NE} = \Gamma_N \cap \Omega_E$.

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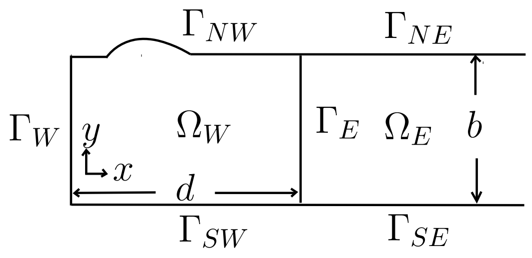

Fig. 2. The Illustration of Domain Ω_W and Ω_E

By the variable transform $x = k\tilde{x}$, $y = \tilde{y}$, the anisotropic equation becomes a Poisson equation in $\widetilde{\Omega}_E$ and the boundary Γ_E becomes another boundary $\widetilde{\Gamma}_E = \{(\tilde{x}, \tilde{y})|\tilde{x} = \frac{d}{k},\ 0 \leq \tilde{y} \leq b\}$.

Problem (3) becomes the following problem

$$\begin{cases} -\Delta u = f, \quad \text{in } \widetilde{\Omega}_W, \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \widetilde{\Gamma}_{SW} \cup \widetilde{\Gamma}_{NW}, \\ u = g, \quad \text{on } \widetilde{\Gamma}_W, \end{cases} \tag{5}$$

where $\widetilde{\Omega}_W = \{(\tilde{x}, \tilde{y})|0 < \tilde{x} < \frac{d}{k}, 0 < \tilde{y} < b\}$, $\widetilde{\Gamma}_{SW} = \{(\tilde{x}, \tilde{y})|0 < \tilde{x} < \frac{d}{k}, \tilde{y} = 0\}$, $\widetilde{\Gamma}_{NW} = \{(\tilde{x}, \tilde{y})|0 < \tilde{x} < \frac{d}{k}, \tilde{y} = b\}$.

Problem (4) becomes the following problem

$$\begin{cases} -\Delta u = 0, \quad \text{in } \widetilde{\Omega}_E, \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \widetilde{\Gamma}_{SE} \cup \widetilde{\Gamma}_{NE}, \\ u \text{ is bounded at infinity}, \end{cases} \tag{6}$$

where $\widetilde{\Omega}_E = \{(\tilde{x}, \tilde{y})|\tilde{x} > \frac{d}{k}, 0 < \tilde{y} < b\}$, $\widetilde{\Gamma}_{SE} = \{(\tilde{x}, \tilde{y})|\tilde{x} > \frac{d}{k}, \tilde{y} = 0\}$, $\widetilde{\Gamma}_{NE} = \{(\tilde{x}, \tilde{y})|\tilde{x} > \frac{d}{k}, \tilde{y} = b\}$.

By separation of variables, we know that the solution of problem (6) has the form

$$u(\tilde{x}, \tilde{y}) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{(\frac{d}{k} - \tilde{x})\frac{n\pi}{b}} \cos\frac{n\pi\tilde{y}}{b}, \tag{7}$$

where

$$a_n = \frac{2}{b}\int_0^b u(\frac{d}{k}, \tilde{y}')\cos\frac{n\pi\tilde{y}'}{b}d\tilde{y}', \quad n = 0, 1, 2, \cdots\cdots. \tag{8}$$

We differentiate (7) with respect to $\tilde{x}$ and set $\tilde{x} = \frac{d}{k}$ to obtain

$$\frac{\partial u}{\partial \tilde{x}}|_{\widetilde{\Gamma}_E} = -\frac{2\pi}{b^2}\sum_{n=1}^{+\infty} n\int_0^b u(\frac{d}{k}, \tilde{y}')\cos\frac{n\pi\tilde{y}}{b}\cos\frac{n\pi\tilde{y}'}{b}d\tilde{y}'. \tag{9}$$

Since

$$\frac{\partial u}{\partial n}|_{\widetilde{\Gamma}_E} = -\frac{\partial u}{\partial \tilde{x}}|_{\widetilde{\Gamma}_E},$$

we obtain the exact artificial boundary condition on $\widetilde{\Gamma}_E$:

$$\begin{aligned} \frac{\partial u}{\partial n}|_{\widetilde{\Gamma}_E} &= \frac{2\pi}{b^2}\sum_{n=1}^{+\infty} n\int_0^b u(\frac{d}{k}, \tilde{y}')\cos\frac{n\pi\tilde{y}}{b}\cos\frac{n\pi\tilde{y}'}{b}d\tilde{y}' \\ &\triangleq \mathcal{K}_1 u(\frac{d}{k}, \tilde{y}). \end{aligned} \tag{10}$$

In practice, we need to truncate the above infinite series by finite terms, let

$$\mathcal{K}_1^N = \frac{2\pi}{b^2}\sum_{n=1}^{N} n\int_0^b u(\frac{d}{k}, \tilde{y}')\cos\frac{n\pi\tilde{y}}{b}\cos\frac{n\pi\tilde{y}'}{b}d\tilde{y}', \tag{11}$$

then we obtain the approximate artificial boundary condition on $\widetilde{\Gamma}_E$:

$$\frac{\partial u}{\partial n}|_{\widetilde{\Gamma}_E} = \mathcal{K}_1^N. \tag{12}$$

For the second case, the exact artificial boundary condition on $\widetilde{\Gamma}_E$ is

$$\begin{aligned} \frac{\partial u}{\partial n}|_{\widetilde{\Gamma}_E} &= \frac{2\pi}{b^2}\sum_{n=1}^{+\infty} n\int_0^b u(\frac{d}{k}, \tilde{y}')\sin\frac{n\pi\tilde{y}}{b}\sin\frac{n\pi\tilde{y}'}{b}d\tilde{y}' \\ &\triangleq \mathcal{K}_2 u(\frac{d}{k}, \tilde{y}), \end{aligned} \tag{13}$$

and the approximate artificial boundary condition on $\widetilde{\Gamma}_E$ is

$$\begin{aligned} \frac{\partial u}{\partial n}|_{\widetilde{\Gamma}_E} &= \frac{2\pi}{b^2}\sum_{n=1}^{N} n\int_0^b u(\frac{d}{k}, \tilde{y}')\sin\frac{n\pi\tilde{y}}{b}\sin\frac{n\pi\tilde{y}'}{b}d\tilde{y}' \\ &\triangleq \mathcal{K}_2^N. \end{aligned} \tag{14}$$

In the following sections, we just consider the equivalent variational problem and finite element approximation of problem (1), we can obtain corresponding result of problem (2) in the same way.

III. The Equivalent Variational Problem

By the exact artificial boundary condition (10), the original problem (1) confines in $\widetilde{\Omega}_W$ is

$$\begin{cases} -\Delta u = f, \quad \text{in } \widetilde{\Omega}_W, \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \widetilde{\Gamma}_{SW} \cup \widetilde{\Gamma}_{NW}, \\ u = g, \quad \text{on } \widetilde{\Gamma}_W, \\ \dfrac{\partial u}{\partial n} = \mathcal{K}_1 u(\dfrac{d}{k}, \tilde{y}), \quad \text{on } \widetilde{\Gamma}_E. \end{cases} \tag{15}$$

By the approximate artificial boundary condition (12), the approximation problem can be described as follows

$$\begin{cases} -\Delta u^N = f, \quad \text{in } \widetilde{\Omega}_W, \\ \dfrac{\partial u^N}{\partial n} = 0, \quad \text{on } \widetilde{\Gamma}_{SW} \cup \widetilde{\Gamma}_{NW}, \\ u^N = g, \quad \text{on } \widetilde{\Gamma}_W, \\ \dfrac{\partial u^N}{\partial n} = \mathcal{K}_1^N, \quad \text{on } \widetilde{\Gamma}_E. \end{cases} \tag{16}$$

Let $V = H^1(\widetilde{\Omega}_W)$, $V_g = \{v \in H^1(\widetilde{\Omega}_W), v|_{\widetilde{\Gamma}_W} = g\}$, then the problem (15) is equivalent to the following variational problem

$$\begin{cases} \text{Find } u \in V_g, \text{ such that} \\ a(u, v) + b(u, v) = f(v), \quad \forall v \in V_0, \end{cases} \tag{17}$$

problem (16) is equivalent to the following variational problem

$$\begin{cases} \text{Find } u^N \in V_g, \text{ such that} \\ a(u^N, v) + b_N(u^N, v) = f(v), \quad \forall v \in V_0, \end{cases} \tag{18}$$

where

$$a(u,v)=\int_{\Omega_W}\mathcal{A}\nabla u\cdot\nabla v dxdy = k\int_{\widetilde{\Omega}_W}\nabla u\cdot\nabla v d\widetilde{x}d\widetilde{y}, \quad (19)$$

$$\begin{aligned} b(u,v)=&k\sum_{n=1}^{+\infty}\frac{2}{n\pi}\int_0^b\int_0^b\frac{\partial u(\frac{d}{k},\tilde{y}')}{\partial\tilde{y}'}\frac{\partial v(\frac{d}{k},\tilde{y})}{\partial\tilde{y}}\\ &\cdot\sin\frac{n\pi\tilde{y}'}{b}\sin\frac{n\pi\tilde{y}}{b}d\tilde{y}'d\tilde{y}, \end{aligned} \quad (20)$$

$$\begin{aligned} b_N(u,v)=&k\sum_{n=1}^{N}\frac{2}{n\pi}\int_0^b\int_0^b\frac{\partial u(\frac{d}{k},\tilde{y}')}{\partial\tilde{y}'}\frac{\partial v(\frac{d}{k},\tilde{y})}{\partial\tilde{y}}\\ &\cdot\sin\frac{n\pi\tilde{y}'}{b}\sin\frac{n\pi\tilde{y}}{b}d\tilde{y}'d\tilde{y}, \end{aligned} \quad (21)$$

$$f(v)=\int_{\Omega_W}fvdxdy = k\int_{\widetilde{\Omega}_W}fvd\tilde{x}d\tilde{y}. \quad (22)$$

For any real number s, we have the equivalent definition of Sobolev spaces $H^s(\widetilde{\Gamma}_E)$ as follows [19]:

$$\forall v\in H^s(\widetilde{\Gamma}_E)\Leftrightarrow v(\frac{d}{k},\tilde{y})=\frac{c_0}{2}+\sum_{n=1}^{+\infty}c_n\cos\frac{n\pi\tilde{y}}{b},$$

$$\text{and } \frac{c_0^2}{2}+\sum_{n=1}^{+\infty}(1+n^2)^s c_n^2<\infty.$$

The norm of $H^s(\widetilde{\Gamma}_E)$ can be defined as follows

$$\|v(\frac{d}{k},\tilde{y})\|_{s,\widetilde{\Gamma}_E}=[\frac{c_0^2}{2}+\sum_{n=1}^{+\infty}(1+n^2)^s c_n^2]^{\frac{1}{2}}.$$

Then we have the following results.

Lemma 1. $b(u,v)$ and $b_N(u,v)$ are both a symmetric, semi-definite and continuous bilinear form on $V\times V$.

Proof. Let

$$u(\frac{d}{k},\tilde{y}')=\frac{a_0}{2}+\sum_{n=1}^{+\infty}a_n\cos\frac{n\pi\tilde{y}'}{b},$$

$$v(\frac{d}{k},\tilde{y})=\frac{c_0}{2}+\sum_{n=1}^{+\infty}c_n\cos\frac{n\pi\tilde{y}}{b},$$

taking the derivative with respect to $\tilde{y}'$ and $\tilde{y}$ we have

$$\frac{\partial u(\frac{d}{k},\tilde{y}')}{\partial\tilde{y}'}=\sum_{n=1}^{+\infty}\frac{n\pi}{b}a_n\sin\frac{n\pi\tilde{y}'}{b},$$

$$\frac{\partial v(\frac{d}{k},\tilde{y})}{\partial\tilde{y}}=\sum_{n=1}^{+\infty}\frac{n\pi}{b}c_n\sin\frac{n\pi\tilde{y}}{b},$$

then we have

$$\begin{aligned} |b(u,v)|&=|\frac{k\pi}{2}\sum_{n=1}^{+\infty}na_nc_n|\\ &\le\frac{k\pi}{2}\|u\|_{\frac{1}{2},\widetilde{\Gamma}_E}\|v\|_{\frac{1}{2},\widetilde{\Gamma}_E}\\ &\le C\|u\|_{1,\widetilde{\Omega}_W}\|v\|_{1,\widetilde{\Omega}_W}. \end{aligned}$$

In the same way, we obtain

$$|b_N(u,v)|=|\frac{k\pi}{2}\sum_{n=1}^{N}na_nc_n|\le C\|u\|_{1,\widetilde{\Omega}_W}\|v\|_{1,\widetilde{\Omega}_W},$$

$$|b(u,u)|=\frac{k\pi}{2}\sum_{n=1}^{+\infty}na_n^2\ge 0,$$

$$|b_N(u,u)|=\frac{k\pi}{2}\sum_{n=1}^{N}na_n^2\ge 0.$$

By using this lemma we have the following theorem.

Theorem 1. The variational problem (17) and (18) have a unique solution on V, respectively.

Proof. It is easy to see that $a(u,v)$ is a symmetric, bounded and coercive bilinear form on $V\times V$. Note that $f(v)$ is a continuous linear function on V and lemma 1, we completed the prove of this theorem by Lax-Milgram theorem.

IV. Finite Element Approximation

Assume that $\mathcal{J}_h$ is a regular and quasi-uniform triangulation of $\widetilde{\Omega}_W$ such that

$$\widetilde{\Omega}_W=\bigcup_{K\in\mathcal{J}_h}K,$$

where K is a triangle and h is the maximal diameter of the triangles. For the sake of simplicity, we assume $g=0$. Let

$$V_h=\{v\in V_0, v|_K \text{ is a linear polynomial, } \forall K\in\mathcal{J}_h\}.$$

The approximation problem of (18) can be described as follows

$$\begin{cases}\text{Find } u_h^N\in V_h, \text{ such that}\\ a(u_h^N,v)+b_N(u_h^N,v)=f(v), \quad \forall v\in V_h.\end{cases} \quad (23)$$

Similar with theorem 1, we can see that the variational problem (23) has a unique solution $u_h^N\in V_h$.

Let $\widetilde{\Gamma}_{d_0}=\{(d_0,\tilde{y})|d_0<\frac{d}{k},0<\tilde{y}<b\}$ be the smallest segment to enclose supp(f), we have

Lemma 2. Suppose u is the solution of the problem (1), $u|_{\widetilde{\Gamma}_{d_0}}\in H^{p-\frac{1}{2}}(\widetilde{\Gamma}_{d_0})$, p is a constant and $p\ge 1$, then for any $v\in V$ we have

$$\begin{aligned} &|b_N(u,v)-b(u,v)|\\ &\le C\frac{e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{(N+1)^{p-1}}\|u\|_{p-\frac{1}{2},\widetilde{\Gamma}_{d_0}}\|v\|_{1,\widetilde{\Omega}_W}, \end{aligned} \quad (24)$$

where C is a constant independent of h, N and d.

Proof. By the formula (7) we have

$$u(d_0,\tilde{y}')=\frac{a_0}{2}+\sum_{n=1}^{+\infty}a_ne^{(d_0-\frac{d}{k})\frac{n\pi}{b}}\cos\frac{n\pi\tilde{y}'}{b}.$$

For any $v\in V$, let

$$v(\frac{d}{k},\tilde{y})=\frac{e_0}{2}+\sum_{n=1}^{+\infty}e_n\cos\frac{n\pi\tilde{y}}{b}.$$

Then we have

$$\begin{aligned}
&|b_N(u,v)-b(u,v)| \\
&=|\sum_{n=N+1}^{+\infty}\frac{2k}{n\pi}\int_0^b\int_0^b\frac{\partial u}{\partial\tilde{y}'}\frac{\partial v}{\partial\tilde{y}}\sin\frac{n\pi\tilde{y}'}{b}\sin\frac{n\pi\tilde{y}}{b}d\tilde{y}'d\tilde{y}| \\
&=|\frac{k\pi}{2}\sum_{n=N+1}^{+\infty}nc_ne^{(d_0-\frac{d}{k})\frac{n\pi}{b}}e_n| \\
&\le\frac{k\pi e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{2(N+1)^{p-1}}|\sum_{n=N+1}^{+\infty}n^pc_ne_n| \\
&\le C\frac{e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{(N+1)^{p-1}}\|u\|_{p-\frac{1}{2},\widetilde{\Gamma}_{d_0}}\|v\|_{1,\widetilde{\Omega}_W}.
\end{aligned}$$

Theorem 2. Suppose $u\in H^2(\widetilde{\Omega}_W)$ is a solution of the problem (1), $u|_{\widetilde{\Gamma}_{d_0}}\in H^{p-\frac{1}{2}}(\widetilde{\Gamma}_{d_0})$, p is a constant and $p\ge 1$, $u_h^N\in V_h$ is the solution of the problem (23), the following error estimate holds

$$\begin{aligned}
&\|u-u_h^N\|_{1,\widetilde{\Omega}_W} \\
&\le C(h\|u\|_{2,\widetilde{\Omega}_W}+\frac{e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{(N+1)^{p-1}}\|u\|_{p-\frac{1}{2},\widetilde{\Gamma}_{d_0}}),
\end{aligned}\tag{25}$$

where C is a constant independent of h, N and d.

Proof. From variational problem (17) and (18) we have

$$\begin{aligned}
&a(u-u_h^N,v)+b_N(u-u_h^N,v) \\
&=b_N(u,v)-b(u,v),\quad\forall v\in V_h.
\end{aligned}$$

For $\forall v\in V_h$, by lemma 2 we have

$$\begin{aligned}
&\|u_h^N-v\|^2_{1,\widetilde{\Omega}_W} \\
&\le C(a(u_h^N-v,u_h^N-v)+b_N(u_h^N-v,u_h^N-v)) \\
&=C(a(u-v,u_h^N-v)+b_N(u-v,u_h^N-v) \\
&\quad+b(u,u_h^N-v)-b_N(u,u_h^N-v)) \\
&\le C(\|u-v\|_{1,\widetilde{\Omega}_W}\|u_h^N-v\|_{1,\widetilde{\Omega}_W} \\
&\quad+|b(u,u_h^N-v)-b_N(u,u_h^N-v)|) \\
&\le C(\|u-v\|_{1,\widetilde{\Omega}_W}\|u_h^N-v\|_{1,\widetilde{\Omega}_W} \\
&\quad+\frac{e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{(N+1)^{p-1}}\|u\|_{p-\frac{1}{2},\widetilde{\Gamma}_{d_0}}\|u_h^N-v\|_{1,\widetilde{\Omega}_W}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|u_h^N-v\|_{1,\widetilde{\Omega}_W}\le C(\|u-v\|_{1,\widetilde{\Omega}_W} \\
&+\frac{e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{(N+1)^{p-1}}\|u\|_{p-\frac{1}{2},\widetilde{\Gamma}_{d_0}}),\quad\forall v\in V_h.
\end{aligned}$$

Notice that

$$\inf_{v\in V_h}\|u-v\|_{1,\widetilde{\Omega}_W}\le Ch\|u\|_{2,\widetilde{\Omega}_W},$$

and by the triangle inequality

$$\|u-u_h^N\|_{1,\widetilde{\Omega}_W}\le\|u-v\|_{1,\widetilde{\Omega}_W}+\|u_h^N-v\|_{1,\widetilde{\Omega}_W},$$

we obtain

$$\begin{aligned}
&\|u-u_h^N\|_{1,\widetilde{\Omega}_W} \\
&\le C(h\|u\|_{2,\widetilde{\Omega}_W}+\frac{e^{(d_0-\frac{d}{k})\frac{(N+1)\pi}{b}}}{(N+1)^{p-1}}\|u\|_{p-\frac{1}{2},\widetilde{\Gamma}_{d_0}}).
\end{aligned}$$

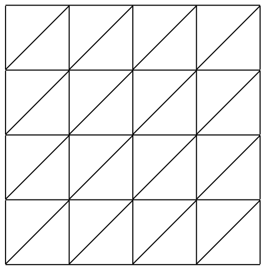

Fig. 3. Mesh h of Subdomain $\widetilde{\Omega}_W$ for Example 1

TABLE I
$L^\infty(\widetilde{\Omega}_W)$ ERRORS WITH DIFFERENT MESH FOR EXAMPLE 1

Mesh	$k=0.2$	$k=0.4$	$k=0.6$	$k=0.8$
h	0.040595	0.042162	0.030543	0.022572
$h/2$	0.037305	0.015811	0.008309	0.005958
$h/4$	0.014159	0.004193	0.002212	0.001512
$h/8$	0.003758	0.001065	0.000557	0.000379

V. NUMERICAL EXAMPLES

We computed some numerical examples to test the effectiveness of the method we developed. The finite element method with linear elements is used in the computation.

Example 1. We consider problem (1), where $\Omega=\{(x,y)|x>0,0<y<b\}$, $\Gamma_W=\{(0,y)|0<y<b\}$, $\Gamma_S=\{(x,0)|x>0\}$, $\Gamma_N=\{(x,b)|x>0\}$ and $b=1$. Let $u(x,y)=e^{-\frac{\pi x}{kb}}\cos\frac{\pi y}{b}$ be the exact solution of original problem and $g=u|_{\Gamma_W}$. Take the artificial boundary $\Gamma_E=\{(x,y)|x=d,0<y<b\}$. By using coordinate transformation $x=k\widetilde{x}$, $y=\widetilde{y}$, we just need to solve the problem as the following

$$\begin{cases}-\Delta u=f,\quad\text{in }\widetilde{\Omega}_W,\\ \dfrac{\partial u}{\partial n}=0,\quad\text{on }\widetilde{\Gamma}_{NW}\cup\widetilde{\Gamma}_{SW},\\ u=g,\quad\text{on }\widetilde{\Gamma}_W,\\ \dfrac{\partial u}{\partial n}=\mathcal{K}u,\quad\text{on }\widetilde{\Gamma}_E,\end{cases}\tag{26}$$

where $\widetilde{\Omega}_W=\{(\tilde{x},\tilde{y})|0<\tilde{x}<\frac{d}{k},0<\tilde{y}<b\}$, $\widetilde{\Gamma}_{SW}=\{(\tilde{x},\tilde{y})|0<\tilde{x}<\frac{d}{k},\tilde{y}=0\}$, $\widetilde{\Gamma}_{NW}=\{(\tilde{x},\tilde{y})|0<\tilde{x}<\frac{d}{k},\tilde{y}=b\}$, $\widetilde{\Gamma}_W=\{(\widetilde{x},\widetilde{y})|\widetilde{x}=0,0<\widetilde{y}<b\}$, and $\widetilde{\Gamma}_E=\{(\widetilde{x},\widetilde{y})|\widetilde{x}=\frac{d}{k},0<\widetilde{y}<b\}$.

Fig. 3 shows the Mesh h of subdomain $\widetilde{\Omega}_W$, Table 1 shows $L^\infty(\widetilde{\Omega}_W)$ errors with different Mesh($N=20$, $d=1$), Fig. 4 shows $L^\infty(\widetilde{\Omega}_W)$ errors with different k($N=20$, $d=1$), Fig. 5 shows $L^\infty(\widetilde{\Gamma}_E)$ errors with different N($k=0.5$, $d=1$), Fig. 6 shows $L^\infty(\widetilde{\Gamma}_E)$ errors with different d($k=0.5$, $N=20$).

The numerical results show that the numerical errors can be affected by the finite element mesh, the order of artificial boundary condition and the location of artificial boundary. Numerical results are in agreement with the error estimates and show the efficiency of our method.

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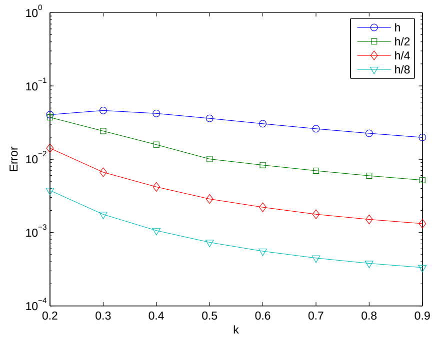

Fig. 4. $L^\infty(\widetilde{\Omega}_W)$ Errors with Different k for Example 1

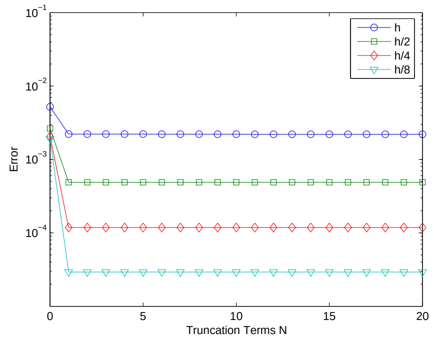

Fig. 5. $L^\infty(\widetilde{\Gamma}_E)$ Errors with Different N for Example 1

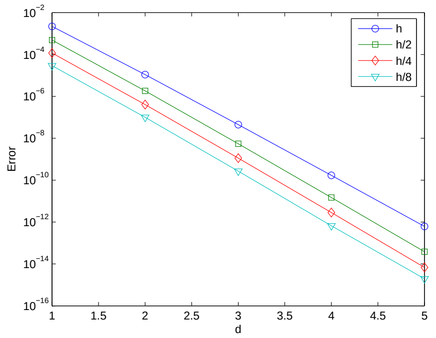

Fig. 6. $L^\infty(\widetilde{\Gamma}_E)$ Errors with Different d for Example 1

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