The Distribution of the Absolute Maximum of the Discontinuous Stationary Random Process with Raileigh and Gaussian Components

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Abstract—**The purpose of this research is to find the asymptotically exact expressions for the distribution function and for the probability that the absolute maximum of the sum of statistically independent homogeneous Gaussian and Rayleigh random processes with nondifferentiable covariance function will exceed the specified threshold. In this study, the applicability boundaries of the introduced theoretical formulas are also determined by means of statistical simulation. The recommendations are presented concerning the application of the obtained expressions depending on the observation interval length and the interrelation of Gaussian and Rayleigh components of the analyzed random process.**

Index Terms—**Rayleigh random process, Gaussian random process, absolute maximum, probability distribution, level crossing probability**

I. Introduction

A. Relevance of the Study

Finding the random processes absolute maxima probabilistic distributions is an important problem of applied value [1]-[4]. The analysis of the extreme values of random processes is a common task in physics and technology, as well as in biology, medicine, economy, etc. In particular, the random processes absolute maxima probabilistic distributions have to be determined while evaluating the complex engineering systems reliability and examining the surfaces roughness, or the maximum deviations and stability of mechanical structures, etc. In the statistical radio physics, finding the distributions of absolute maxima of random processes is the task arising during the analysis of fluctuating phenomena of the radio wave propagation in various media, and it can also be faced when examining the noise influence on threshold radio-electronic devices and tracking measurers and when testing the efficiency of procedures for detection and processing of the radio physical signals observed against interferences, etc.

The theory of extreme values of Gaussian processes is the most developed one to date [5]-[13], because such processes are widely applied in the simulation of many physical phenomena and are also characterized by relative simplicity of their mathematical description. In many cases, Gaussian model of the random process confirms that the real physical phenomena is due to the central limit theorem [14]. But in statistical radio physics and radio engineering the issue regards non-Gaussian random processes also [15]-[19]. One type of such processes is Rayleigh random process, or the two-degrees-of-freedom χ-process [20], [21]; the one describes the envelope of a narrow-band Gaussian random process [20], [21], and it is observed, for example, in radio receivers at the envelope demodulator output when Gaussian fluctuation noise is present at the receiver input [21].

In practice, Rayleigh random process can be observed together with Gaussian random process. The presence of the Gaussian noise component is often caused by the thermal noise obeying Gaussian probability distribution and arising in electronics nodes and elements. The appearance of the Gaussian noise component at the output of the receiver with the envelope demodulator is often the result of the thermal noise presence in the receiver stages located behind the demodulator. Random processes with Rayleigh and Gaussian components are commonly revealed in radio engineering systems when processing the stochastic signals with deterministic and random components. Such random process particularly describes the decision statistics for the optimal detector of the two-component radio signal [22] with unknown time of arrival observed against Gaussian additive noise while the signal is absent at the receiver input. In this case, in order to calculate the type I error probability (false alarm probability) it is required to find the probability of exceeding the set threshold h by the absolute maximum of the stationary random process with independent Rayleigh and Gaussian components.

B. The Results of the Previous Studies

Despite the considerable attention that researchers pay to

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the problem of the extreme values of random processes, the exact expression for the probability that the threshold is to be exceeded by the absolute maximum of the random process is still undetermined, even for the stationary Gaussian random processes [5]-[12]. Exact results are obtained in some special cases only, for example, for the triangular correlation function of Gaussian process when the process determination interval length does not exceed its correlation time [23], [24]. Generally, it is possible to find the asymptotically exact (with h increasing) expressions only for the probability of the absolute maximum of Gaussian random process exceeding the level h [5], [7]-[10], [12]. Otherwise some approximations can only be found, including either upper or lower bounds of this probability.

If the correlation function of the random process is continuously differentiable two times at least, then it is possible to use a common method to find the asymptotically exact (with h increasing) expressions for the probability that the level h will be exceeded by the random process. This method is based on the calculation of the average number of the process excursions for the arbitrary level h, as it is presented in [5], [12], [25]. Then, it is necessary to take into account that, for Gaussian or Rayleigh processes, the distribution of the number of the process excursions for the level h converges to the Poisson law with h increasing [5], [7], [11], [26]-[28]. Then, the expression can be written for the average number of excursions, and then – the asymptotically exact (with h increasing) expression for the probability that the level h will be exceeded by the absolute maximum of the stationary Gaussian or Rayleigh random processes, as seen in, for example, [12], [17], [18], [29]. This method can also be used for the sum of the statistically independent Gaussian or Rayleigh random processes, as it is implemented in, for example, [30] for the sum of Gaussian and non-Gaussian processes.

However in a number of applications the simulations of the random processes are used, whose correlation functions are continuous, but nondifferentiable at the maximum point. The derivatives of such correlation functions have the discontinuities of the first kind in the specified point and, therefore, the random processes corresponding to them are called the discontinuous (irregular) ones [31]. It must be emphasized that the realizations of the discontinuous processes are continuous in mean square [20], [21], due to the continuity of their correlation functions. A widely known example of the discontinuous process is the random process with triangular correlation function. The discontinuous random processes represent an issue in many cases of processing signals against interferences [29], [31], [32]. As the correlation functions of discontinuous processes are nondifferentiable at the maximum point, the technique [5], [25] proposed to calculate the average number of excursions is inapplicable for these random processes.

In a number of papers, the method is considered that enables the calculation of asymptotic distributions of an absolute maximum of Gaussian random process without the conditionality of continuous differentiability of its correlation function. For the first time, this method subsequently termed as the double sum method [7] has been applied in paper [33]. It has been demonstrated that the flow of A-outputs [7] of the stationary discontinuous Gaussian process for h level is the asymptotically Poisson one under $h \to \infty$. Here, the asymptotic expression for the average number of A-outputs has also been obtained. The paper [33] contained a number of errors which were later corrected in [34], [35]. Based on these results, it is possible to find the asymptotically exact (with h increasing) expression for the probability that the threshold h will be exceeded by the absolute maximum of the discontinuous stationary Gaussian random process [7], [12], [29]. The similar expressions for discontinuous stationary Rayleigh, generalized Rayleigh and Hoyt random processes are found in [17], [18], [29]. The specified expressions are obtained accounting for the local Markovian properties of these processes [29] and by applying the results from [36] valid for Markov process.

C. The Goal of the Paper

However, the problem of finding the asymptotic distributions of the absolute maximum of the sum of Rayleigh and Gaussian processes remains still unresolved. In this study, the asymptotically exact (with h increasing) expressions are obtained for the distribution function $F_m(h)$ and the probability $\alpha(h) = 1 - F_m(h)$ that the threshold h will be exceeded by the absolute maximum of the two-component random process that is the sum of the statistically independent discontinuous stationary Rayleigh and Gaussian random processes.

II. Problem Statement

A. The Model of the Random Process

The two-component processes

$$L(l) = L_0(l) + L_1(l), \qquad l \in [\Lambda_1, \Lambda_2], \tag{1}$$

has been considered, representing the sum of the statistically independent stationary Gaussian $L_0(l)$ and Rayleigh $L_1(l)$ random processes within the interval $[\Lambda_1, \Lambda_2]$. It can be presupposed that the stationary Gaussian random process $L_0(l)$ has the mathematical expectation (ME) $M_0 = \langle L_0(l) \rangle$ and the dispersion $\sigma_0^2 = \langle [L_0(l) - M_0]^2 \rangle$ and for the normalized covariance function (CF) $R_0(l_2 - l_1) = \langle [L_0(l_1) - M_0][L_0(l_2) - M_0] \rangle / \sigma_0^2$ of this process the following relation is fulfilled when $|\Delta l| = |l_2 - l_1| \to 0$:

$$R_0(\Delta l) = 1 - a_0 |\Delta l| + \mathrm{o}(|\Delta l|), \tag{2}$$

while for $|\Delta l| \to \infty$, it is $R_0(\Delta l) = \mathrm{o}(\ln^{-1} |\Delta l|)$. Here a_0 is the constant determining the correlation and the spectral properties of the process $L_0(l)$ [17], [18], [21]; $\mathrm{o}(x)$ denotes the higher-order infinitesimal terms compared with x and $\langle \cdot \rangle$ designate the averaging over all the realizations of the random process. Then Gaussian random process can be presented in the form of

$$L_0(l) = M_0 + \sigma_0 \xi_0(l), \tag{3}$$

where $\xi_0(l) = [L_0(l) - M_0]/\sigma_0$ is the stationary centered Gaussian random process with unit dispersion and CF $\langle \xi_0(l_1)\xi_0(l_2)\rangle = R_0(l_2 - l_1)$ allowing the representation (2).

The Rayleigh random process $L_1(l)$ is introduced in the form of [20], [21]

$$L_1(l) = \sqrt{L_{11}^2(l) + L_{12}^2(l)}, \tag{4}$$

where $L_{11}(l)$ and $L_{12}(l)$ are statistically independent centered stationary Gaussian random processes with the same dispersions $\sigma_1^2 = \langle L_{1k}^2(l)\rangle$ and the normalized CFs $R_1(l_2 - l_1) = \langle L_k(l_1)L_k(l_2)\rangle/\sigma_1^2$, $k = 1,2$, assuming the following asymptotic relation for $|\Delta l| \to 0$:

$$R_1(\Delta l) = 1 - a_1|\Delta l| + o(|\Delta l|), \tag{5}$$

while $R_1(\Delta l) = o(\ln^{-1}|\Delta l|)$ for $|\Delta l| \to \infty$. Here a_1 is the constant determining the correlation and the spectral properties of the processes $L_{1k}(l)$ [17], [18], [21].

The process (4) can be written as

$$L_1(l) = \sigma_1\sqrt{\xi_{11}^2(l) + \xi_{12}^2(l)}, \tag{6}$$

where $\xi_{11}(l) = L_{11}(l)/\sigma_1$, $\xi_{12}(l) = L_{12}(l)/\sigma_1$ are statistically independent centered Gaussian random processes with the unit dispersions and the same CFs $\langle \xi_{1k}(l_1)\xi_{1k}(l_2)\rangle = R_1(l_2 - l_1)$, $k = 1,2$, allowing the representation (5). It should be noted that the random processes $\xi_{11}(l)$, $\xi_{12}(l)$ and $\xi_0(l)$ are also the statistically independent ones.

Thus, the studied random process (1) is presented as follows

$$L(l) = M_0 + \sigma_0\xi_0(l) + \sigma_1\sqrt{\xi_{11}^2(l) + \xi_{12}^2(l)}, \quad l \in [\Lambda_1, \Lambda_2], \tag{7}$$

Here the parameters σ_0 and σ_1 characterize the corresponding contribution of Gaussian $L_0(l)$ and Rayleigh $L_1(l)$ components to the resultant random process $L(l)$ (1).

According to (2), (5), the normalized CFs $R_0(\Delta l)$ and $R_1(\Delta l)$ of Gaussian and Rayleigh components and, therefore, CF of the random process (1) are continuous nondifferentiable at the point $\Delta l = 0$, as the first-order derivatives of the functions $R_0(\Delta l)$ and $R_1(\Delta l)$ have the discontinuity of the first kind under $\Delta l = 0$. Thus, the random processes $L_0(l)$ and $L_1(l)$, as well as the random process $L(l)$, are discontinuous (irregular) ones [29], [31], [32]. At the same time, the realizations of the random processes $L_0(l)$, $L_1(l)$, $L(l)$ are continuous in mean square [20], [21], as the normalized CFs $R_0(\Delta l)$ and $R_1(\Delta l)$ are continuous at $\Delta l = 0$.

B. The Representation of the Maximum of the Two-Component Random Process

The absolute (greatest) maximum L_m of the analyzed random process (1) within the definitional interval $l \in [\Lambda_1, \Lambda_2]$ is then considered, i.e.

$$L_m = \sup_{l\in[\Lambda_1,\Lambda_2]} L(l) = \sup_{l\in[\Lambda_1,\Lambda_2]} [L_0(l) + L_1(l)], \tag{8}$$

Rayleigh random process (4) can be presented in the form of

$$L_1(l) = \sup_{\varphi\in[-\pi,\pi]} L_{1e}(l,\varphi), \tag{9}$$

as a result of the maximization of the equivalent random field

$$L_{1e}(l) = L_{11}(l)\cos(\varphi) + L_{12}(l)\sin\varphi \tag{10}$$

by a variable $\varphi \in [-\pi, \pi]$ for each fixed $l \in [\Lambda_1, \Lambda_2]$. The equivalent field (10) is the homogeneous centered Gaussian random field with the dispersion σ_1^2. Similarly to (6), it is convenient to write the random field (10) as

$$L_{1e}(l) = \sigma_1[\xi_{11}(l)\cos\varphi + \xi_{12}(l)\sin\varphi]. \tag{11}$$

Then, taking into account (9), (11) the value L_m (8) can be presented in the form of

$$L_m = \sup_{l,\varphi\in\Theta} L_\varphi(l,\varphi), \tag{12}$$

i.e. as the absolute maximum value of the random field

$$L_\varphi(l,\varphi) = M_0 + \sigma_0\xi_0(l) + \sigma_1[\xi_{11}(l)\cos\varphi + \xi_{12}(l)\sin\varphi] \tag{13}$$

within the domain Θ of the values l, φ is set by the conditions $l \in [\Lambda_1, \Lambda_2]$, $\varphi \in [-\pi, \pi]$. The random field (13) is homogeneous Gaussian random field with ME M_0 and CF

$$K(\Delta l, \Delta\varphi) = \sigma_0^2 R_0(\Delta l) + \sigma_1^2 R_1(\Delta l)\cos(\Delta\varphi). \tag{14}$$

And the dispersion of the random field (13) is equal to $\sigma^2 = \sigma_0^2 + \sigma_1^2$.

According to (2), (5), the following asymptotic relation is fulfilled for CF (14) of the random field (13) for $\Delta l \to 0$, $\Delta\varphi \to 0$:

$$K(\Delta l, \Delta\varphi) = \sigma^2\left[1 - \eta|\Delta l| - \vartheta\Delta\varphi^2/2\right] + o(|\Delta l|) + o(\Delta\varphi^2), \quad (15)$$

where

$$\eta = \left(a_0\sigma_0^2 + a_1\sigma_1^2\right)/\left(\sigma_0^2 + \sigma_1^2\right), \quad \vartheta = \sigma_0^2/\left(\sigma_0^2 + \sigma_1^2\right) \le 1. \quad (16)$$

Here the parameter $1/\eta$ characterizes the correlation interval of the random process (1) [17], [18], and the value $0 \le \vartheta \le 1$ sets the relative contribution of Rayleigh components into the random term of the studied process (1). If $a_0 = a_1$, then $\eta = 1$.

Thus, the absolute maximum L_m of the two-component random process $L(l)$ (1) within the interval $l \in [\Lambda_1, \Lambda_2]$ can be presented as the absolute maximum (12) of Gaussian random field $L_\varphi(l, \varphi)$ (13) within the domain Θ of the values *l*, φ set by the conditions $l \in [\Lambda_1, \Lambda_2]$, $\varphi \in [-\pi, \pi]$.

III. The Distribution of the Absolute Maximum

A. Common Provisions

It is useful to find the distribution function $F_m(h) = P[L_m < h]$ of the absolute maximum L_m of the random process $L(l)$ (1) within the interval $l \in [\Lambda_1, \Lambda_2]$, as well as the associated probability $\alpha(h) = P[L_m > h] = 1 - F_m(h)$ of level *h* being exceeded by the value L_m. Hereinafter, $P[A]$ means the probability of event *A*. At that, the representation (12) of the value L_m is applied as the absolute maximum of homogeneous Gaussian random field $L_\varphi(l, \varphi)$ (13) within the definitional domain Θ set by the conditions $l \in [\Lambda_1, \Lambda_2]$, $\varphi \in [-\pi, \pi]$.

The exact expression for the distribution function of the absolute maximum of homogeneous Gaussian random field for arbitrary level *h* and size of the definitional domain Θ is unknown. Following [5], [7], [10], [26], [29], the asymptotically exact (with *h* increasing) expressions for the distribution function $F_m(h)$ are obtained using two methods.

The first method is based on the calculation of the average number of A-outputs of homogeneous Gaussian random field for level *h*, taking into account that, for $h \to \infty$, the A-outputs flow of this field is asymptotically Poisson [7]. The second method is based on the application of the local additive approximation (LAA) method [37], [38].

B. Application of the Average Number of A-outputs of the Gaussian Random Field

It can be assumed that the A-outputs flow of the homogeneous Gaussian random field $L_\varphi(l, \varphi)$ over level *h* is asymptotically Poisson, if $h \to \infty$ [7]. According to [7], A-output of the random field $L_\varphi(l, \varphi)$ over level *h* is such crossing of level *h* by the field realization in some point (l_0, φ_0) that presupposes that, under $\rho > 0$, there exists the range of values of *l* and φ adjacent to the point (l_0, φ_0) and set by the conditions $l \in [l_0 - \rho/2, l_0 + \rho/2]$, $\varphi \in [\varphi_0 - \rho, \varphi_0)$ and $l \in [l_0 - \rho/2, l_0)$, $\varphi = \varphi_0$, within which the realization of the field $L_\varphi(l, \varphi)$ does not exceed *h*. The Poisson character of A-outputs flow of the field $L_\varphi(l, \varphi)$ over level *h* means [7] that the distribution of the number of the field A-outputs over this level is described by the Poisson law [14],]14]. Then, similarly to [12], [17], [29], under $h \to \infty$, the distribution function $F_m(h)$ can be presented as

$$F_m(h) = \exp[-\Pi(h)]. \quad (17)$$

Here $\Pi(h)$ is the average number of A-outputs of the random field $L_\varphi(l, \varphi)$ over level *h* within the domain Θ where the absolute maximum L_m is searched.

In [7], the asymptotically exact (with level *h* increasing) expression is introduced for the average number of A-outputs of the homogeneous Gaussian random field over level *h*. After applying this expression to the random field $L_\varphi(l, \varphi)$ (13) with CF (15) and taking into account that the field definitional domain Θ is set by the conditions $l \in [\Lambda_1, \Lambda_2]$, $\varphi \in [-\pi, \pi]$, it can be obtained

$$\Pi(h) = H_a V\eta\sqrt{\vartheta}u^2 \exp\left(-u^2/2\right)/2\sqrt{\pi}. \quad (18)$$

Here H_a is the Pickands constant [7], the values η, ϑ are determined from (16),

$$u = (h - M_0)/\sigma \quad (19)$$

is the normalized level and $V = 2\pi(\Lambda_2 - \Lambda_1) = 2\pi\lambda$ is the area of definitional domain Θ of the random field $L_\varphi(l, \varphi)$ within which A-outputs are considered, while $\lambda = \Lambda_2 - \Lambda_1$ and 2π are the sizes of definitional domain Θ by variables *l* and φ, correspondingly. According to [7], the Pickands constant H_a, in case of CF (15), is equal to

$$H_a = \lim_{v \to \infty}\left[H_a^*(v)/v^2\right],$$

where $H_a^*(v) = 1 + \int_0^\infty P[\sup_{l,\varphi \in \Theta} \chi(l, \varphi) > y]\exp(y)\,dy$ and $\chi(l, \varphi)$ is Gaussian random field with ME $M_\chi(l, \varphi) = -|l| - \varphi^2$ and CF $K_\chi(l_1, l_2, \varphi_1, \varphi_2) = |l_1| + |l_2| + \varphi_1^2 + \varphi_2^2 - |l_1 - l_2| - (\varphi_1 - \varphi_2)^2$. Such constant H_a is calculated, for example, in [39] and it is equal to $H_a = 1/\sqrt{\pi}$.

It must be emphasized that the method [7] to calculate the average number of A-outputs of the random field presupposes that the volume of the reduced domain Θ is equal to $\Omega = V\eta\sqrt{\vartheta}$ [29] and should not be too small (at least, it cannot be equal to 0) [7], [29]. In other words, the reduced sizes [29]

$$m_l = \lambda\eta\,, \qquad m_\varphi = 2\pi\sqrt{\vartheta} \tag{20}$$

of the domain Θ should not be too small by variables l and φ, respectively. If this condition is not satisfied, then the formula (18) has poor accuracy, if the values of h are not great enough. Under $m_l = 0$ or $m_\varphi = 0$, the formula (18) is inapplicable and it provides the value $\Pi(h) \equiv 0$ so that, according to (17), the distribution function $F_m(h)$ is for all h. It should be noted that the value $m_\varphi = 0$ is reached, if $\vartheta = 0$, while Rayleigh component $L_1(l)$ is absent. Therefore, if $\vartheta = 0$, when the random process (1) has Gaussian component only, the formula (18) is inapplicable. The greater are the sizes m_l and m_φ (20), the more exact is the asymptotic approximation (18) for the finite values of h [29].

Following [29], the asymptotically exact expressions (17), (18) is applied for finite values of level h, while the conditions

$$m_l >> 1\,, \qquad \vartheta > 0 \tag{21}$$

are fulfilled. Similarly to [29], the function (18) is considered as nonmonotonic by the variable u. Indeed, the boundary value $u_0 = \sqrt{2}$ exists, and thus the increasing monotonic function (18) becomes the decreasing one with the level u decreasing under $u < u_0$. Then the distribution function $F_m(h)$ (17) increases with level u decreasing under $u < u_0$. But this contradicts the meaning of the average number $\Pi(h)$ of A-outputs and the distribution function $F_m(h)$ that cannot decrease and increase respectively with level u decreasing. On the other hand, under general considerations, it is clear that $F_m(h) \to 0$, while $h \to -\infty$. Therefore, the step approximation [29] can be used:

$$F_m(h) = \begin{cases} \exp[-\Pi(h)], & u \ge u_0\,, \\ 0\,, & u < u_0\,, \end{cases} \tag{22}$$

where $\Pi(h)$ is determined from (18). By substituting (18) into (22), it can be finally obtained:

$$F_m(h) = \begin{cases} \exp\left[-m_l\sqrt{\vartheta}u^2\exp\left(-u^2/2\right)\right], & u \ge \sqrt{2}\,, \\ 0\,, & u < \sqrt{2}\,, \end{cases} \tag{23}$$

where m_l is the reduced length (20) of the interval $[\Lambda_1, \Lambda_2]$ of the possible values of l and the values η and ϑ are determined from (16). The accuracy of the expression (23) increases with the normalized values of level u (19) and the values m_l and m_φ (20) that are equivalent to λ and ϑ (16) increase.

Considering the limiting cases, it can be presupposed that Gaussian component is absent, i.e. $M_0 = 0$, $\sigma_0 = 0$. Then in (23) it should be set $\vartheta = 1$ and the known asymptotically exact (with increasing u) expression [17], [18], [29] for the distribution function of the absolute maximum of the discontinuous stationary Rayleigh random process is obtained:

$$F_m(h) = \begin{cases} \exp\left[-m_l u^2\exp\left(-u^2/2\right)\right], & u \ge \sqrt{2}\,, \\ 0\,, & u < \sqrt{2}\,, \end{cases} \tag{24}$$

where, according to (16), (19), (20), $m_l = \lambda a_1$ and $u = h/\sigma_1$.

If Rayleigh component is absent, i.e. $\sigma_1 = 0$, then in (23) it is necessary to set $\vartheta = 0$. However, under $\vartheta = 0$ the formula (23) is not applicable. Therefore, in this case, instead of (23), the known asymptotically exact (with u increasing) expression [12], [29] for the distribution function of the absolute maximum of the discontinuous stationary Gaussian random process should be used:

$$F_m(h) = \begin{cases} \exp\left[-m_l u\exp\left(-u^2/2\right)/\sqrt{2\pi}\right], & u \ge 1\,, \\ 0\,, & u < 1\,, \end{cases} \tag{25}$$

where, according to (16), (19), (20), $m_l = \lambda a_0$ and $u = h/\sigma_{01}$.

The analysis of the distribution function $F_m(h)$ shows that it increases when ϑ decreases, i.e. with the decrease of the Rayleigh component contribution. Therefore, the distribution function (25) of the absolute maximum of Gaussian process can be considered as the upper bound of the distribution $F_m(h)$ for all the possible ϑ. Then the dependence (23) can be specified for small values of ϑ majorizing it by the dependence (25). The function (23) already reaches the boundary value (25) at the levels $u < 1/\sqrt{2\pi\vartheta}$.

C. Application of the Local Additive Approximation Method

It should be reminded that the formula (23) obtained on the basis of the results from [7] is inapplicable in the limiting case of Gaussian random process, when $\vartheta = 0$. Under small values of ϑ, when the Rayleigh component contribution is small, the formula (23) provides considerably overstated values for the probability $F_m(h)$. In order to eliminate this disadvantage, the LAA method can be applied for the calculation of the distribution function $F_m(h)$.

It is known [10], [29] that probability of the realization of the homogeneous random field $L(x_1, x_2, \ldots, x_N)$ exceeding the high level h are defined by the local properties of probabilistic field characteristics in the small neighborhood of an arbitrary point from the field Θ definitional domain. On the other hand, for the full probabilistic description of Gaussian random field, it is enough to specify its ME and CF [14], [20], [21]. Then, under high values of h the distribution function $F_m(h) = P[L_m < h]$ of the absolute maximum L_m of homogeneous Gaussian field

$L(x_1, x_2, \dots, x_N)$ is determined by ME M of this field, and also by the behavior of field CF $K(\Delta x_1, \Delta x_2, \dots, \Delta x_N)$ in the small neighborhood of the values $\Delta x_1 = \Delta x_2 = \dots = \Delta x_N = 0$. Therefore, while calculating the asymptotically exact (with h increasing) expression for the function $F_m(h)$, the examined field $L(x_1, x_2, \dots, x_N)$ can be changed by the equivalent homogeneous Gaussian field $L_e(x_1, x_2, \dots, x_N)$ with the same ME M and the paticular CF $K_e(\Delta x_1, \Delta x_2, \dots, \Delta x_N)$ allowing the following asymptotic representation when $\varepsilon = \max(|\Delta x_1|, |\Delta x_2|, \dots, |\Delta x_N|) \to 0$:

$$K_e(\Delta x_1, \Delta x_2, \dots, \Delta x_N) = K(\Delta x_1, \Delta x_2, \dots, \Delta x_N) + o(\varepsilon). \tag{26}$$

The condition (26) means that CFs of initial and the equivalent random fields coincide asymptotically in the small neighborhood of an arbitrary point from the field definitional domain.

According to LAA method, the equivalent field should be chosen as the sum

$$L_e(x_1, x_2, \dots, x_N) = M + \sum_{j=1}^{N} L_{ej}(x_j), \tag{27}$$

where M is the constant ME of the examined field; N is the field definitional domain dimension; and $L_{ej}(x_j)$, $j = 1, 2, \dots, N$ are statistically independent centered Gaussian random processes. According to (27), MEs of initial and equivalent random fields are equal. CFs $K_{ej}(\Delta x_j)$ of the random processes $L_{ej}(x_j)$ shall be chosen so that their sum $\sum_{j=1}^{N} K_{ej}(\Delta x_j)$ is equal to CF $K_e(\Delta x_1, \Delta x_2, \dots, \Delta x_N)$ of the equivalent random field satisfies the condition (26).

It should be noted that the equality of MEs and CFs of the examined and the equivalent random fields in the small neighborhood of an arbitrary point from the field definitional domain provides the convergence of the examined field to the equivalent field by distribution, with the size of the specified neighborhood decreasing.

The representation of the initial field $L(x_1, x_2, \dots, x_N)$ as the sum (27) of the constant field ME and statistically independent centered Gaussian random processes $L_{ej}(x_j)$ allows expressing the distribution function $F_m(h)$ of the absolute field maximum as the convolution of the distributions of the absolute maxima of equivalent random processes $L_{ej}(x_j)$, $j = 1, 2, \dots, N$. Thus, the equivalent processes should be chosen so that the distributions of their absolute maxima are known or can be easy calculated.

LAA method is applied to find the asymptotically exact (with h increasing) expression for the distribution function $F_m(h)$ of the absolute maximum of the homogeneous Gaussian field $L_\varphi(l, \varphi)$ (13). For this, under $|\Delta l| \to 0$, $|\Delta\varphi| \to 0$, CF (14) of the field (13) is presented as the sum

$$K(\Delta l, \Delta\varphi) = K_0(\Delta l) + K_1(\Delta\varphi) + o(|\Delta l|) + o(\Delta\varphi^2), \tag{28}$$

where, according to (15), the functions permit the following asymptotic representations

$$K_0(\Delta l) = \sigma^2 (2 - \vartheta)\left[1 - 2\eta|\Delta l|/(2 - \vartheta) + o(|\Delta l|)\right]/2, \text{ if } |\Delta l| \to 0, \tag{29}$$

$$K_1(\Delta\varphi) = \sigma^2 \vartheta\left[1 - \Delta\varphi^2 + o(\Delta\varphi^2)\right]/2, \text{ if } |\Delta\varphi| \to 0,$$

and η and ϑ are determined from (16). Such functions can be interpreted as CFs of some random processes.

The statistically independent stationary centered Gaussian random processes $L_{\varphi 0}(l)$ and $L_{\varphi 1}(\varphi)$ are introduced with CFs

$$K_0(\Delta l) = \sigma_{\varphi 0}^2 \begin{cases} 1 - \eta_0|\Delta l|, & |\Delta l| \geq 1/\eta_0, \\ 0, & |\Delta l| < 1/\eta_0, \end{cases} \tag{30}$$

$$K_1(\Delta\varphi) = \sigma_{\varphi 1}^2 \begin{cases} 1 - \Delta\varphi^2, & |\Delta\varphi| \to 0, \\ 0, & |\Delta\varphi| \to \infty, \end{cases} \tag{31}$$

respectively, where

$$\sigma_{\varphi 0}^2 = \sigma^2 (2 - \vartheta)/2, \qquad \sigma_{\varphi 1}^2 = \sigma^2 \vartheta/2 \tag{32}$$

are the dispersions of the random processes $L_{\varphi 0}(l)$ and $L_{\varphi 1}(\varphi)$, while $\sigma_{\varphi 0}^2 + \sigma_{\varphi 1}^2 = \sigma^2$ and

$$\eta_0 = 2\eta/(2 - \vartheta). \tag{33}$$

When $|\Delta l| \to 0$, $|\Delta\varphi| \to 0$, CFs (30), (31) of the random processes $L_{\varphi 0}(l)$ and $L_{\varphi 1}(\varphi)$ permit the asymptotic representations (29), correspondingly. According to (28), CFs of the Gaussian random field $L_\varphi(l, \varphi)$ and the sum of random processes $L_{\varphi 0}(l) + L_{\varphi 1}(\varphi)$ coincide asymptotically, while $|\Delta l| \to 0$, $|\Delta\varphi| \to 0$. Then, according to LAA method, while calculating the asymptotically exact (with h increasing) expression for the distribution function $F_m(h)$, the random field $L_\varphi(l, \varphi)$ (13) can be changed by the sum $M_0 + L_{\varphi 0}(l) + L_{\varphi 1}(\varphi)$, where M_0 is ME of the random field $L_\varphi(l, \varphi)$. As a result, the value L_m of the absolute maximum of the random field $L_\varphi(l, \varphi)$ can be presented in the form of

$$L_m = \sup_{l, \varphi \in \Theta} L_\varphi(l, \varphi) = M_0 + L_{m0} + L_{m1}, \tag{34}$$

where L_{m0} and L_{m1} are statistically independent random variables

$$L_{m0} = \sup_{l \in [\Lambda_1, \Lambda_2]} L_{\varphi 0}(l), \qquad L_{m1} = \sup_{\varphi \in [-\pi, \pi]} L_{\varphi 1}(\varphi) \tag{35}$$

equal to the values of the absolute maxima of the random processes $L_{\varphi 0}(l)$ and $L_{\varphi 1}(\varphi)$ within the intervals $\lambda \in [\Lambda_1, \Lambda_2]$ and $\varphi \in [-\pi, \pi]$, accordingly.

$F_0(x) = P[L_{m0} < x]$ and $F_1(x) = P[L_{m1} < x]$ are designated as the distribution functions of the absolute maxima L_{m0} and L_{m1} (35) of the random processes $L_{\varphi 0}(l)$ and $L_{\varphi 1}(\varphi)$ within the intervals $\lambda \in [\Lambda_1, \Lambda_2]$ and $\varphi \in [-\pi, \pi]$, accordingly, and $w_1(x) = F_1(x)/dx$ – as the probability density of the random variable L_{m1}. Then, taking into account that the values L_{m0} and L_{m1} (35) are statistical independent, the distribution function $F_m(h) = P[L_m < h]$ of the sum (34) can be presented as the convolution of the distributions of these values [20], [21]

$$F_m(h) = \int_{-\infty}^{\infty} F_0(h - x - M_0) w_1(x) dx . \tag{36}$$

The expressions for the functions $F_0(x)$ and $w_1(x)$ introduced in (36) are specified.

The Case of the Big Definitional Domain

The exact expressions for the distribution function $F_0(x)$ and the probability density $w_1(x)$ of the absolute maxima of stationary Gaussian random processes $L_{\varphi 0}(l)$ and $L_{\varphi 1}(\varphi)$ are unknown for arbitrary definitional intervals and levels *x*. Similarly to [37], it can be shown that, in order to find the asymptotically exact (with *h* increasing) expression for the distribution function $F_m(h)$, it would be sufficient to use the asymptotically exact (with *h* increasing) approximations of the functions $F_0(x)$ and $w_1(x)$ in (36). Such approximations are obtained, for example, in [29]. If the reduced length m_φ of the definitional interval $[\Lambda_1, \Lambda_2]$ of the random process $L_{\varphi 0}(l)$ is big enough, i.e. [29]

$$m_\varphi = (\Lambda_2 - \Lambda_1)\eta_0 = 2m_l/(2 - \vartheta) >> 1 , \tag{37}$$

where η_0 is determined according to (33), then, from [29], it can be obtained

$$F_0(x) = \begin{cases} \exp\left[-m_\varphi x \exp\left(-x^2/2\sigma_{\varphi 0}^2\right)/\sigma_{\varphi 0}\sqrt{2\pi}\right], & x/\sigma_{\varphi 0} \geq 1, \\ 0, & x/\sigma_{\varphi 0} < 1. \end{cases} \tag{38}$$

As $0 \leq \vartheta \leq 1$ by definition, therefore, it is $m_l \leq m_\varphi \leq 2m_l$ and the condition $m_\varphi >> 1$ (37) is equivalent to the condition $m_l >> 1$ (21). For the random process $L_{\varphi 1}(\varphi)$ specified within the interval $\varphi \in [-\pi, \pi]$, by applying the results [29], it is:

$$F_1(x) = \begin{cases} \exp\left[-\sqrt{2}\exp\left(-x^2/2\sigma_{\varphi 1}^2\right)\right], & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{39}$$

From (39), the probability density $w_1(x) = F_1(x)/dx$ is obtained in the form of

$$w_1(x) = \begin{cases} \exp\left(-\sqrt{2}\right)\delta(x) + \left(\sqrt{2}x/\sigma_{\varphi 1}^2\right)\times \\ \times\exp\left[-x^2/2\sigma_{\varphi 1}^2 - \sqrt{2}\exp\left(-x^2/2\sigma_{\varphi 1}^2\right)\right], & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{40}$$

where $\delta(x)$ is the delta function [20]. By substituting (38), (40) into (36), it is:

$$\begin{aligned} F_m(h) = \sqrt{2} \int_0^{\left(u - \sqrt{(2-\vartheta)/2}\right)\sqrt{2/\vartheta}} & y \exp\Bigg\{ -\frac{2m_l}{\sqrt{\pi(2-\vartheta)^3}} \times \\ & \times \left(u - y\sqrt{\frac{\vartheta}{2}}\right) \exp\left[-\frac{1}{2-\vartheta}\left(u - y\sqrt{\frac{\vartheta}{2}}\right)\right] - \\ & - \frac{y^2}{2} - \sqrt{2}\exp\left(-\frac{y^2}{2}\right)\Bigg\} dy + \\ & + \exp\left[-\sqrt{2} - \frac{2m_l}{\sqrt{\pi(2-\vartheta)^3}} \exp\left(-\frac{u^2}{2-\vartheta}\right)\right] + \\ & + \exp\left[-\sqrt{2} - \frac{2m_l}{\sqrt{\pi(2-\vartheta)^3}} \exp\left(-\frac{u^2}{2-\vartheta}\right)\right], \end{aligned} \tag{41}$$

if $u \geq \sqrt{(2-\vartheta)/2}$, and $F_m(h) = 0$, if $u \geq \sqrt{(2-\vartheta)/2}$. Here *u* is the normalized level (19). Under $\vartheta = 0$, when Rayleigh component of the random process (1) is absent, the formula (41) is simplified and, with the condition of standardization for the probability density $w_1(x)$ taking into account, transforms into (25). The accuracy of the formula (41) increases with *u* (19) and m_l (21).

While calculating the integral (36), instead of (38)-(40), simpler but less exact expressions can be used

$$F_0(x) \approx 1 - m_\varphi x \exp\left(-x^2/2\sigma_{\varphi 0}^2\right)/\sigma_{\varphi 0}\sqrt{2\pi} ,$$
$$F_1(x) \approx 1 - \sqrt{2}\exp\left(-x^2/2\sigma_{\varphi 1}^2\right), \; w_1(x) \approx \sqrt{2}x\exp\left(-x^2/2\sigma_{\varphi 1}^2\right)/\sigma_{\varphi 1}^2 \tag{42}$$

that are valid under very large values of *x*. The formulas (42) can be obtained by applying the expansion $\exp(-y) \approx 1 - y$ for $y << 1$ in (38)-(40). In this case, it is convenient to overwrite the expression (36) in the following way: $F_m(h) = 1 - \int_{-\infty}^{\infty}[1 - F_0(h - x - M_0) w_1(x)]dx$. Substituting (42) in here, integration and account are carried out only for the terms of the higher-order by *u*, thus getting simpler but less exact formula for the function $F_m(h)$ that is valid under very large values of u:

$$F_m(h) \approx 1 - m_\varphi \sqrt{\vartheta} u^2 \exp\left(-u^2/2\right). \quad (43)$$

The accuracy of the formula (43) increases with u (19) and m_l (21).

It should be noted that the expression (43) can be also obtained from (23), if it is assumed that the value of u is very large and that the asymptotic expansion $\exp(-x) \approx 1 - x$ is used if $x << 1$. Therefore, the formulas (23) and (41) coincide asymptotically in case of the large values of u.

The expressions (23), (41) and (43) obtained above are compared. As an example, in Fig. 1 the dependences of the probability $\alpha = 1 - F_m(h)$ from the level u under $m_l = 10$ and various ϑ are shown. Curves 1 correspond to $\vartheta = 0.7$, curves 2 – to $\vartheta = 0.1$, curves 3 – to $\vartheta = 0.002$. Solid lines are obtained from (41), dashed lines are obtained from (23), and dash-dotted lines are obtained from (43). By dotted lines the bounds of the probability α are plotted. The lower bound G corresponds to the presence of Gaussian component only ($\vartheta = 0$) and is calculated by the formula (25). The upper bound R corresponds to the presence of Rayleigh component only ($\vartheta = 1$) and is calculated by applying (24).

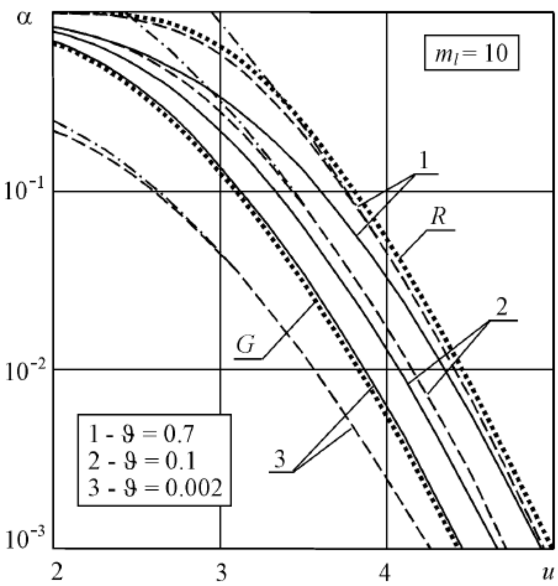

Fig. 1. The comparison of approximations of the probability of threshold exceeding.

The analysis of the obtained results shows that, under small $\vartheta \le 0.01$, the formulas (23), (43) provide considerably understated values of α which go beyond the lower bound G under $u < 1/\sqrt{2\pi\vartheta}$. The lower the ratio ϑ is, the greater values of u are required in order to provide the satisfactory accuracy for (23), (43). If $\vartheta = 0$, the formulas (23), (43) are inapplicable for the arbitrary finite levels u. Therefore, if ϑ is low, the formula (41) is preferable. On the other hand, while $\vartheta \ge 0.03$, the computational results provided with the help of the formulas (23), (43) and (41) coincide satisfactorily, if the parameter m_l satisfies the relation $m_l > 1$ and the level u is not too small (so that values $\alpha < 0.2 - 0.3$ are provided).

The formula (43) satisfactorily approximates the dependence (23), if the values of u are not too small and if $\alpha < 0.2 - 0.3$. However, under big m_l and small u the formula (43) produces the values $F_m(h) < 0$ and that results in probability values greater than 1: $\alpha > 1$. But this is against the meaning of probability, as its values should always lie within the range from 0 to 1. Besides, under small ϑ, the formula (43), together with (23), provides considerably understated values of probability α. At the same time, the formula (43) is very simple, and that is the reason why it can be useful in the analytical calculations, accounting for the restrictions specified above.

The Case of the Small Definitional Domain

The expressions (23), (41) for the distribution function of $F_m(h)$ are found when either the condition $m_l >> 1$ (21) or the equivalent condition $m_\varphi >> 1$ (37) are satisfied. Therefore, under $m_l \lesssim 1$ or $m_\varphi \lesssim 1$ (of the order of unity and less), the expressions (23), (41) can be of low accuracy, as, in our calculations, the asymptotically exact formulas (18), (38) have been used when $m_l >> 1$.

Application of the results from [23] makes it possible to write the exact expression for the distribution function $F_r(x,m)$ of the absolute maximum of the stationary centered Gaussian random process $r(l)$ with unit dispersion and triangular CF $K(\Delta l) = \max(0, 1 - |\Delta l|)$. If the process $r(l)$ is specified within the interval $l \in [0, m]$ by the length $0 \le m \le 1$, then we get [24], [37]:

$$F_r(x,m) = P[\sup_{l\in[0,m]} r(l) < x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \Phi\left[\frac{x - y(1-m)}{\sqrt{m(2-m)}}\right] \times \\ \times \exp\left(-\frac{y^2}{2}\right) dy - \frac{mx}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Phi\left(x\sqrt{\frac{m}{2-m}}\right) - \\ - \frac{\sqrt{m(2-m)}}{2\pi} \exp\left(-\frac{x^2}{2-m}\right). \quad (44)$$

Here $\Phi(x) = \int_{-\infty}^{x} \exp\left(-t^2/2\right) dt / \sqrt{2\pi}$ is the probability integral [20]. Then the probability density $w_r(x,m) = dF_r(x,m)/dx$ of the absolute maximum of the process $r(l)$ is equal to

$$w_r(x,m) = \frac{2 + m\left(x^2 - 1\right)}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Phi\left(x\sqrt{\frac{m}{2-m}}\right) - \\ - \frac{\sqrt{m(2-m)}}{2\pi} x \exp\left(-\frac{x^2}{2-m}\right). \quad (45)$$

It will be taken into account that the Gaussian random process $L_{\varphi 0}(l)$ has the triangular CF (30). Thus, if the condition $m_\varphi = 2m_l/(2-\vartheta) \le 1$ holds, or if

$$m_l \le 1 - \vartheta/2, \tag{46}$$

which is equivalent, then (44), (45) can be applied, and there can be written the exact expressions for the distribution function $F_0(x) = P[L_{m0} < x]$ and the corresponding probability density $w_0(x) = dF_0(x)/dx$ of the absolute maximum L_{m0} of the process $L_{\varphi 0}(l)$:

$$F_0(x) = F_r(x/\sigma_{\varphi 0}, m_\varphi), \quad w_0(x) = w_r(x/\sigma_{\varphi 0}, m_\varphi)/\sigma_{\varphi 0}. \tag{47}$$

In order to find the distribution function $F_m(h)$ (36), while the condition (46) is fulfilled, the exact expressions (47) can be used instead of the approximate expression (38). Moreover, it is convenient to overwrite the function $F_m(h)$ in the form of

$$F_m(h) = \int_{-\infty}^{\infty} F_1(h - x - M_0) w_0(x) dx. \tag{48}$$

By substituting (39), (47) into (48) and taking into account (32), (37), it is:

$$\begin{aligned} F_m(h) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{u\sqrt{2/(2-\vartheta)}} & \left\{ \left[1 + \frac{m_l(y^2-1)}{2-\vartheta} \right] \exp\left(-\frac{y^2}{2} \right) \times \right. \\ & \times \Phi\left(y\sqrt{\frac{m_l}{2-\vartheta-m_l}} \right) + \frac{\sqrt{m_l(2-\vartheta-m_l)}}{\sqrt{2\pi}(2-\vartheta)} \times \\ & \left. \times y \exp\left(-\frac{y^2}{2} \frac{2-\vartheta}{2-\vartheta-m_l} \right) \right\} \times \\ & \times \exp\left\{ -\sqrt{2} \exp\left[-\left(\frac{u}{\sqrt{\vartheta}} - y\sqrt{\frac{2-\vartheta}{2\vartheta}} \right)^2 \right] \right\} dy, \end{aligned} \tag{49}$$

when the condition (46) is satisfied. Here *u* is determined from (19). The accuracy of the expression (49) increases with *u*.

The limiting cases are going to be considered. Under $\vartheta = 0$, when Rayleigh component is absent, the formula (49) is simplified and transforms into the exact expression for the distribution function of the absolute maximum of the stationary Gaussian random process

$$F_m(h) = F_r(u, m_l), \tag{50}$$

where the function $F_r(u, m_l)$ is determined from (44). If, in addition, $m_l = 0$, then equation (49) transforms into the already known exact expression $F_m(h) = \Phi(u)$ for the distribution function of Gaussian random variable [14], [20], [21].

On the other hand, under $m_l = 0$ and $\vartheta > 0$, the formula (49) provides only the asymptotically exact (with *u* increasing) expression for the distribution function $F_m(h)$. However, if $m_l = 0$, then the exact expression of this function can be written too. For this purpose, it is taken into account that under $m_l = 0$ the analyzed random process $L(l)$ (1) degenerates into the sum of statistically independent Gaussian L_0 and Rayleigh L_1 random variables. Therefore, the distribution function $F_m(h)$ can be presented as the convolution [20], [21]

$$F_m(h) = \int_0^{\infty} F_{00}(h - y) w_{10}(y) dy,$$

where $w_{10}(x) = x \exp(-x^2/2\sigma_1^2)/\sigma_1^2$, if $x \ge 0$, and $w_{10}(x) = 0$, if $x < 0$, is the probability density of Rayleigh random variable and $F_{00}(x) = \Phi[(x - M_0)/\sigma_0]$ is the distribution function of Gaussian variable L_0 [14], [20], [21]. Calculating the last integral leads us to

$$F_m(h) = \Phi\left(\frac{u}{\sqrt{1-\vartheta}} \right) - \sqrt{\vartheta} \exp\left(-\frac{u^2}{2} \right) \Phi\left(u\sqrt{\frac{\vartheta}{1-\vartheta}} \right). \tag{51}$$

The formulas (23) and (49) are further compared, while the condition (46) holds. As an example, in Fig. 2 the dependences of the probability $\alpha = 1 - F_m(h)$ are plotted from the level *u* (19) under $m_l = 0.25$ and two values of ϑ. Curves 1 correspond to $\vartheta = 0.7$, and curves 2 – to $\vartheta = 0.002$. Solid lines are obtained by means of (49), and dashed lines – by (23) derived in asymptotics $m_l >> 1$ (21). By dotted line, the lower bound for the probability α is shown reached in limiting case of Gaussian random process ($\vartheta = 0$) and calculated by (50). By dash-dotted lines, the lower bounds are designated for the probability α reached under $m_l = 0$ and calculated by (51).

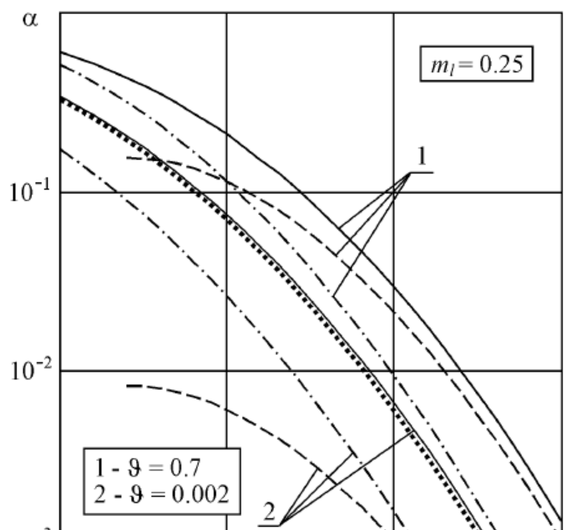

Fig. 2. The comparison of approximations of the probability of threshold

From Fig. 2 it can be seen that the formula (23) provides the understated values of the probability α which might considerably go beyond the exact lower bounds of this probability under small ϑ or not too big u. The lower the values ϑ and m_l are, the higher the level u required to ensure the satisfactory accuracy of the formula (23) is. On the other hand, the more complex formula (49) does not have these disadvantages.

IV. The Results of Statistical Simulation

In order to test the accuracy of the found asymptotic formulas for the distribution function $F_m(h)$ and in order to determine the limits of applicability of these formulas under finite values of u, there have been conducted the statistical computer simulation of the absolute maximum L_m of the random process $L(l)$ (1) within the definitional intervals $[0,\lambda]$ of various length λ.

During the simulation, the realizations of Gaussian $L_0(l)$ (3) and Rayleigh $L_1(l)$ (6) components of the random process $L(l)$ have been formed under zero ME $M_0 = 0$ and unit dispersion $\sigma^2 = \sigma_0^2 + \sigma_1^2 = 1$. Here the value of ϑ (16) characterizing the relative contribution of Rayleigh component in the analyzed random process underwent changed in the limits from 0 to 1. Identical and triangular CFs $R_0(l)$ and $R_1(l)$ of Gaussian component $L_0(l)$ and quadratures $L_{11}(l)$, $L_{12}(l)$ of Rayleigh component $L_1(l)$ have been selected, i.e. they were equal to

$$R_i(l) = R(l) = \max(0, 1 - |\Delta l|), \qquad i = 0,1 . \tag{52}$$

Such CFs satisfy the conditions (2), (5) and they correspond to the case of $\eta = 1$, $m_l = \lambda$.

In the simulation with the specified step Δl, the samples $L_j = L(j\Delta l)$, $j = 1,2,\ldots,\{m_l/\Delta l\}$ have been generated of the realizations of the random process $L(l)$ (1) within the interval $l \in [0, m_l]$ with the set length $m_l = \lambda$ and at the discrete moments of time $l_j = j\Delta l$. Here {} designate the integer part. In order to form the samples L_j, the following formula has been used, according to (7):

$$\begin{aligned} L_j &= \sigma_0 \xi_{0j} + \sigma_1 \sqrt{\xi_{1j}^2 + \xi_{2j}^2} = \\ &= \sqrt{1-\vartheta}\,\xi_{0j} + \sqrt{\vartheta\left(\xi_{1j}^2 + \xi_{2j}^2\right)}, \end{aligned} \tag{53}$$

where $\xi_{0j} = \xi_0(j\Delta l)$, $\xi_{1j} = \xi_{11}(j\Delta l)$, $\xi_{2j} = \xi_{12}(j\Delta l)$ are the samples of centered stationary Gaussian processes with CFs (52) and ϑ is determined from (16). The discretization step Δl selected has not been greater than 0.01, and that ensured that the root-mean-square error ε of the step approximation of the continuous realizations of random processes $\xi_0(\Delta l)$, $\xi_{11}(\Delta l)$, $\xi_{12}(\Delta l)$ based on the samples (53) did not exceed $\varepsilon = \sqrt{2[1 - R(\Delta l/2)]} = \sqrt{\Delta l} = 0.01$ (i.e. 10 %). The samples ξ_{0j}, ξ_{1j}, ξ_{2j} have been calculated by the moving summation method [40] as

$$\xi_{ij} = \frac{1}{\sqrt{\mu}} \sum_{k=j}^{j+\mu-1} \gamma_{ik} , \qquad i = 0,1,2 , \tag{54}$$

where $\mu = \{1/\Delta l\} \geq 100$ is the number of summands in the sum (54) and γ_{ik}, $i = 0,1,2$, $k = 1,2,\ldots,\{m_l/\Delta l\} + \mu - 1$ are statistically independent Gaussian random variables with zero MEs and unit dispersions. Each random variable γ_{ik} has been formed based on the central limit theorem [40], [41] by the summation method of K independent random numbers ζ_{ikn}, $n = 1,2,\ldots,N$, uniformly distributed within the interval [0,1] by means of the nonlinear Cornish-Fisher correction [38], [41]:

$$\gamma_{ik} = \beta_{ik} + \left(\beta_{ik}^3 - 3\beta_{ik}\right)/20N , \; \beta_{ik} = \sqrt{\frac{12}{N}} \sum_{n=1}^{N} (\zeta_{ikn} - 0.5) . \tag{55}$$

The number of summands N in the sum (55), following [38], has been selected to be equal to 5 providing the high calculation rate and the required quality of the obtained random variables. To form the sequence of independent random numbers ζ_{ikn} the program transmitter URAND [42] was used.

Based on the samples L_j generated according to (53)-(55), it has been determined that the value L_m that is the absolute maximum of the realization $L(l)$ within the interval $l \in [0, m_l]$ as the value of the greatest sample L_j for all $j = 1,2,\ldots,\{m_l/\Delta l\}$.

During the simulation, no independent realizations $L(l)$ have been formed being higher than $Q_r = 10^5$ for each chosen pair of the values of m_l and ϑ. Moreover, the dependent testing method [40] has been used, when the samples L_j have been calculated for various ϑ by the formulas (53)-(55) on the basis of the same samples ξ_{ij} realizations. Thus, on the basis of the obtained array of Q_r values of L_m for each pair of parameters m_l, ϑ, the experimental probability $\alpha(u) = 1 - F_m(u)$ of threshold u exceeding has been calculated for various values of u. This probability has been determined as the relative frequency of the variable L_m exceeding threshold u.

The error of the obtained experimental values of the probability of level u has been evaluated being exceeded by the value L_m of the absolute maximum of the simulated random process $L(l)$. An experimental estimate of the probability $\alpha(u)$ has been designated as $\tilde{\alpha}$ and its true value as α_0. Then, for a great number of tests $Q_r >> 1$, the confidence probability P_d for the relative confidence interval δ is equal to [14]

$$P_d = P\left[\left|\tilde{\alpha} - \alpha_0\right| / \tilde{\alpha} < \delta\right] \approx 2\Phi\left[\delta\sqrt{\tilde{\alpha} Q_r / (1 - \tilde{\alpha})}\right] - 1 .$$

The value P_d means the probability that the relative deviation of the estimate $\tilde{\alpha}$ from the true value α_0 does not exceed δ. By applying this formula for the quantity of tests $Q_r = 10^5$, it has been discovered that, with confidential probability of 0.9, confidence intervals boundaries deviate from the experimental values of the probability α no more than by $\delta = 0.016$ (1.6 %.) under $\alpha > 0.1$; no more than by $\delta = 0.052$ (5.2 %.) under $\alpha > 0.01$; and no more than by $\delta = 0.17$ (17 %.) under $\alpha > 0.001$.

Some experimental values of α obtained by simulation are plotted by circles, squares, triangles and rhombuses in Figs. 3-6. The corresponding theoretical dependences $\alpha(u)$ are also shown there. Solid lines have been calculated by (41) under $m_l \geq 1$ or by (49) under $m_l < 1$ found by LAA method. Dashed lines are drawn with the help of the formula (23) obtained by the analysis of the average number of the random filed A-outputs.

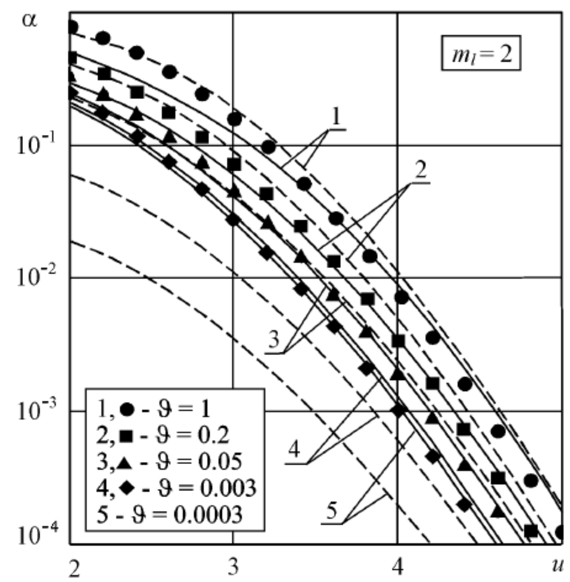

Fig. 3. The probability of threshold exceeded under various contribution of Rayleigh process.

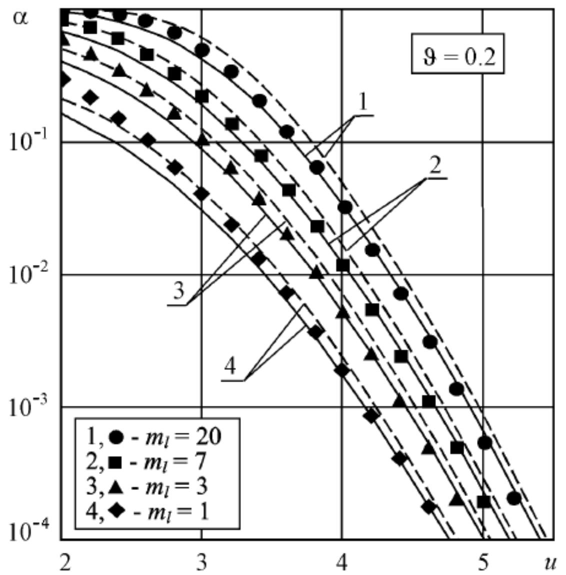

Fig. 4. The probability of threshold exceeded under various definitional interval length of the random process.

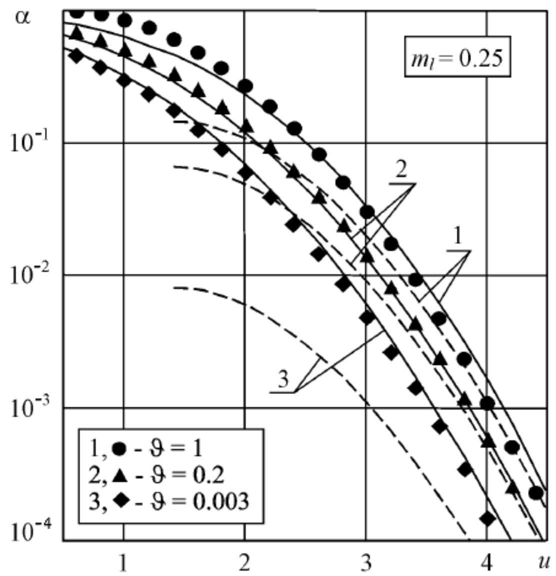

Fig. 5. The probability of threshold exceeded under various contribution of Rayleigh process.

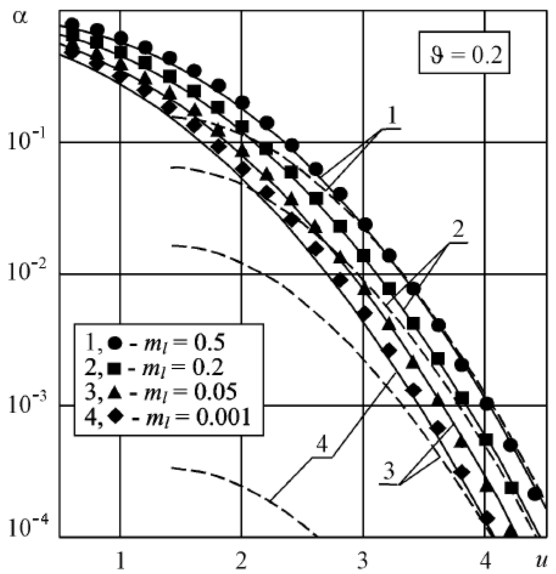

Fig. 6. The probability of threshold exceeded under various definitional interval length of the random process.

In Figs. 3, 4 the case of $m_l \geq 1$ is presented and in Figs. 5, 6 – the case of $m_l < 1$.

Results in Fig. 3 were obtained for $m_l = 2$ and various values of ϑ. Curves 1 and circles correspond to $\vartheta = 1$, curves 2 and squares – to $\vartheta = 0.2$, curves 3 and triangles – to $\vartheta = 0.05$, curves 4 and rhombuses – $\vartheta = 0.003$, and curves 5 – to $\vartheta = 0.0003$. Experimental values for $\vartheta = 0.0003$ are not shown as they practically coincide with the rhombuses corresponding to $\vartheta = 0.003$.

Results in Fig. 4 have been obtained for $\vartheta = 0.2$ and various values of m_l. Curves 1 and circles correspond to $m_l = 20$, curves 2 and squares – to $m_l = 7$, curves 3 and triangles – to $m_l = 3$, curves 4 and rhombuses – to $m_l = 1$. From Figs. 3, 4, it can be seen that the formula (23) provides the significantly understated values of the probability α under small $\vartheta \leq 0.01$, while the formula (41) has the satisfactory accuracy for all the possible values of $0 \leq \vartheta \leq 1$. As the formula (43) satisfactorily approximate the dependence (23) under not too small u (see Fig. 1), then it has the same disadvantage as the formula (23).

Results in Fig. 5 have been obtained for $m_l = 0.25$ and various values of ϑ. Curves 1 and circles correspond to $\vartheta = 1$, curves 2 and triangles – to $\vartheta = 0.2$, curves 3 and rhombuses – to $\vartheta = 0.003$. Results in Fig. 6 have been obtained for $\vartheta = 0.2$ and various values of m_l. Curves 1 and circles correspond to $m_l = 0.5$, curves 2 and squares – to $m_l = 0.2$, curves 3 and triangles – to $m_l = 0.05$, curves 4 and rhombuses – $m_l = 0.001$.

From Figs. 5, 6 it can be seen that the formula (23) provides the significantly understated values of the probability α under small values of m_l and ϑ. In addition, the accuracy of this formula is rapidly deteriorating with m_l and ϑ decreasing. At the same time, the formula (49) obtained by LAA method has a good accuracy under arbitrary values of ϑ, u, if $m_l \le 1 - \vartheta/2$.

V. CONCLUSION

In order to calculate the probability of level crossing of the absolute maximum of the sum of statistically independent homogeneous Gaussian and Rayleigh random processes with nondifferentiable covariance function under $m_l > 1 - \vartheta/2$, the formula (41) obtained by LAA method is more preferable as it has a good accuracy for all the possible values ϑ. The simpler formula (23) obtained on the basis of the analysis of the average number of A-outputs of the random field can be used, if $\vartheta \ge 0.03$ at least. It should be also noted that, instead of (23), it is possible to apply the simplified asymptotic formula (43), if level u is not too small thus providing the values of $\alpha < 0.2 - 0.3$. When $m_l \le 1 - \vartheta/2$, the formula (49) obtained by LAA method can recommended for use.

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