Finite-Time State Feedback Stabilization for a Class of Uncertain High-Order Nonholonomic Feedforward Systems

Fangzheng Gao, Xiaochun Zhu, Jiacai Huang and Xiulan Wen

***Abstract*—This paper addresses the problem of finite-time stabilization by state feedback for a class of uncertain high-order nonholonomic systens in feedforward-like form. Based on the finite-time stability theorem, and by introducing sign function and necessarily modifying the homogeneous domination approach, a constructive design procedure for state feedback control is given. Together with a novel switching control strategy, the designed controller renders that the states of the closed-loop system are regulated to zero in a finite time. A simulation example is provided to illustrate the effectiveness of the proposed approach.**

***Index Terms*—high-order nonholonomic systems, time-varying delays, state feedback, adding a power integrator nonholonomic systems, global asymptotic stabilization.**

I. Introduction

As an important class of nonlinear systems, nonholonomic systems have attracted a great deal of attention over the past decades because they can be used to model numerous mechanical systems, such as mobile robots, car-like vehicle and under-actuated satellites, see, e.g., [1-4] and the references therein. However, from Brockett necessary condition [5], it is well known that no smooth (or even continuous) time-invariant static state feedback exists for the stabilization of nonholonomic systems. To overcome this difficulty, with the effort of many researchers a number of intelligent approaches have been proposed, which can mainly be classified into discontinuous time-invariant stabilization[6,7], smooth time-varying stabilization[8,9] and hybrid stabilization[10]. Using these valid approaches, the asymptotic issue of nonholonomic systems has been extensively studied [11-20].

Compared to the asymptotic stabilization, the closed-loop system with finite-time convergence usually demonstrates faster convergence rates, higher accuracies and better disturbance rejection properties [21]. Motivated by this, finite-time stabilization of nonholonomic systems has been received intense investigation recently [22-25]. More specifically, based on the finite-time Lyapunov stability theorem, [22] proposed a novel switching finite-time control strategy to nonholonomic systems in chained form with weak drifts. The works [23] and [24] extended the results in [22] to the nonholonomic systems in chained form with uncertain parameters and perturbed terms, respectively. In particular, in [25], the adaptive finite-time stabilization problem was considered for a class of more general nonholonomic systems, which is called high order nonholonomic systems and can be viewed as the extension of the classical nonholonomic systems. However, it should be noted that all above papers are concerned with the systems in feedback-like form (i,e., the x-subsystem of considered systems is a feedback-like system), while nonholonomic systems in feedforward-like form have received little consideration. However, there exist some practical systems which can be transformed to nonholonomic systems in feedforward-like form such as the hopping robot presented in [26]. Therefore, how to design finite-time stabilizing controllers for nonholonomic systems in feedforward-like form is a meaningful work, which has not been fully solved in the existing literature.

Motivated the above discussion, in this paper we focus our attention on solving the problem of finite-time stabilization by state feedback for a class of high order nonholonomic systems in feedforward-like form. The contributions is highlighted as follows. (i) The finite-time stabilization problem of the high order nonholonomic systems, which is neither feedback linearizable nor stabilized by applying the frequently-used backstepping approach or its variants, is studied for the first time. (ii) A sufficient condition on characterizing the nonlinear growth of the nonholonomic feedforward systems for its finite-time stabilization is derived. (iii) Based on the finite-time stability theorem, and by introducing sign function and necessarily modifying the homogeneous domination approach, a systematic state feedback control design procedure is proposed to render the states of closed-loop system to zero in a finite time.

The remainder of this paper is organized as follows. Section II describes the systems to be studied and formulates the control problem. Section III presents the control design procedure and the main results. Section IV gives a simulation example to illustrate the theoretical finding of this paper. Finally, concluding remarks are proposed in Section V.

II. Problem Formulation and Preliminaries

In this paper, we consider the following class of high order nonholonomic systems:

$$\begin{array}{lcl} \dot{x}_0 & = & d_0 u_0^{p_0} + \phi_0(t, x_0) \\ \dot{x}_1 & = & d_1 x_2^{p_1} u_0 + \phi_1(t, x_2, \cdots, x_n, u_0, u_1) \\ \dot{x}_2 & = & d_2 x_3^{p_2} u_0 + \phi_2(t, x_3, \cdots, x_n, u_0, u_1) \\ & \vdots & \\ \dot{x}_{n-1} & = & d_{n-1} x_n^{p_{n-1}} u_0 + \phi_{n-1}(t, x_n, u_0, u_1) \\ \dot{x}_n & = & d_n u_1^{p_n} \end{array} \tag{1}$$

where $(x_0, x)^T = (x_0, x_1, \cdots, x_n)^T \in R^{n+1}$, $u = (u_0, u_1)^T \in R^2$ are the system state and control input,

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respectively. $p_i \in R^{\geq 1}_{odd}$, $i = 1, \cdots, n$ are said to be the high orders of the system. d_i, $i = 1, \cdots, n$ are disturbed virtual control coefficients. ϕ_i, $i = 1, \cdots, n$ are unknown continuous functions, which denote the inputs and states driven uncertainties. Note that the x-subsystem of system (1) has a feedforward-like structure. This implies system (1) is a high order nonholonomic system in feedforward-like form, which is also called as high order nonholonomic feedforward system in this paper.

The objective of this paper is to design a state feedback controller in the form $u_0 = u_0(x_0)$, $u_1 = u_1(x_0, x)$ such that the finite-time regulation of the states are achieved; i.e., $\lim_{t\to T}(|x_0(t)| + |x(t)|) = 0$ and $(x_0(t), x(t)) = (0, 0)$ for any $t \geq T$, where T is a finite time.

To this end, the following assumptions are imposed in this paper.

Assumption 1. For $i = 0, 1, \cdots, n$, there are positive constants c_{i1} and c_{i2} such that

$$c_{i1} \leq d_i \leq c_{i2}$$

Assumption 2. For ϕ_0, there is a positive constant a such that

$$|\phi_0(\cdot)| \leq a|x_0|$$

Assumption 3. For $i = 1, \cdots, n-1$, there are constants $b > 0$ and $\tau \in (-\frac{1}{\sum_{l=1}^{n} p_1 \cdots p_{l-1}}, 0)$ such that

$$|\phi_i(\cdot)| \leq b \sum_{j=i+1}^{n+1} |x_j|^{q_{ij}}$$

where $x_{n+1} = u_1$, $r_1 = 1$, $p_i r_{i+1} = r_i + \tau > 0$, $i = 1, \cdots, n$ and q_{ij} is constant satisfying $q_{ij} > p_i r_{i+1}/r_j$.

Remark 1. Assumptions 1 and 2 are common and similar to those usually imposed on the nonlinear systems [10,13,16]. However, it is worth pointing out that the upper bound of ϕ_i, depends on the state x_{i+1} beside the states $x_{i+2}, \cdots, x_n, u_1$ in Assumption 3, which is less restrictive than that in [27,28] and allows for a much broader class of systems.

In what follows, we review some useful definitions and lemmas which will serve as the basis of the coming control design and performance analysis.

Definition 1[21]**.** Consider a system

$$\dot{x} = f(x) \ with \ f(0) = 0, \quad x \in R^n \tag{2}$$

where $f : U_0 \to R^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \in U_0$ of the origin. By "finite-time convergence," we mean: If, for any initial condition $x(0) \in U$, there is a settling time $T > 0$, such that every solution $x(t)$ with $x(0)$ as its initial condition of (2) is well defined with $x(0) \in U \setminus \{0\}$ for $t \in [0, T)$ and satisfies $\lim_{t\to T} x(t) = 0$ and $x(t) = 0$ for any $t \geq T$. If $U = U_0 = R^n$, the origin is a globally finite-time stable equilibrium.

Lemma 1[21]**.** Consider the nonlinear system described in (2). Suppose there is a C^1 function $V(x)$ defined in a neighborhood $\hat{U} \in R^n$ of the origin, real numbers $c > 0$ and $0 < \alpha < 1$, such that

(i) $V(x)$ is positive definite on $\hat{U}$;

(ii) $\dot{V}(x) + cV^{\alpha}(x) \leq 0, \quad \forall x \in \hat{U}$.

Then, the origin of system (2) is locally finite-time stable with

$$T \leq \frac{V^{1-\alpha}(x(0))}{c(1-\alpha)}$$

for initial condition $x(0)$ in some open neighborhood $U \in \hat{U}$ of the origin. If $U = R^n$ and $V(x)$ is also radially unbounded (i.e., $V(x) \to +\infty$ as $x \to +\infty$), the origin of system (2) is globally finite-time stable.

Definition 2 [27]**.** Weighted Homogeneity: For fixed coordinates $(x_1, \cdots, x_n) \in R^n$ and real numbers $r_i > 0$, $i = 1, \cdots, n$,

- the dilation $\Delta_\varepsilon(x)$ is defined by $\Delta_\varepsilon(x) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n)$ for any $\varepsilon > 0$, where r_i is called the weights of the coordinates. For simplicity, we define dilation weight $\Delta = (r_1, \cdots, r_n)$.
- a function $V \in (R^n, R)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $V(\Delta_\varepsilon(x)) = \varepsilon^{\tau}V(x_1, \cdots, x_n)$ for any $x \in R^n \setminus \{0\}, \varepsilon > 0$.
- a vector field $f \in (R^n, R^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $f_i(\Delta_\varepsilon(x)) = \varepsilon^{\tau + r_i} f_i(x)$, for any $x \in R^n \setminus \{0\}, \varepsilon > 0$, $i = 1, \cdots, n$.
- a homogeneous p-norm is defined as $\|x\|_{\triangle,p} = (\sum_{i=1}^{n} |x_i|^{p/r_i})^{1/p}$ for all $x \in R^n$, for a constant $p \geq 1$. For simplicity, in this paper, we choose $p = 2$ and write $\|x\|_\triangle$ for $\|x\|_{\triangle,2}$.

Lemma 2 [27]. Suppose $V : R^n \to R$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then the following holds:

(i) $\partial V/\partial x_i$ is homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i.

(ii) There is a constant c such that $V(x) \leq c\|x\|^{\tau}_\triangle$. Moreover, if $V(x)$ is positive definite, then $\underline{c}\|x\|^{\tau}_\triangle \leq V(x)$, where $\underline{c}$ is a constant.

Lemma 3 [28]**.** For $x \in R$, $y \in R$, $p \geq 1$ and $c > 0$ are constants, the following inequalities hold: (i) $|x+y|^p \leq 2^{p-1}|x^p+y^p|$, (ii) $(|x|+|y|)^{1/p} \leq |x|^{1/p}+|y|^{1/p} \leq 2^{(p-1)/p}(|x| + |y|)^{1/p}$, (iii) $||x| - |y||^p \leq ||x|^p - |y|^p|$, (iv) $|x|^p + |y|^p \leq (|x| + |y|)^p$, (v) $|[x]^{1/p} - [y]^{1/p}| \leq 2^{1-1/p}|x - y|^{1/p}$, (vi) $|[x]^p - [y]^p| \leq c|x - y|||x - y|^{p-1} + |y|^{p-1}|$.

Lemma 4[28]**.** Let x, y be real variables, then for any positive real numbers a, m and n, one has

$$a|x|^m|y|^n \leq b|x|^{m+n} + \frac{n}{m+n}\Big(\frac{m+n}{m}\Big)^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} |y|^{m+n},$$

where $b > 0$ is any real number.

III. Finite-time controller design

In this section, we give a constructive procedure for the finite-time stabilizer of system (1) by state feedback. The design of finite-time controller is divided into the following two steps:

- We first construct a state feedback controller u_1 to stabilize the x-subsystem in a finite time;
- Then we design a controller u_0 such that the x_0-subsystem is finite-time stable.

A. The controller design of the x-subsystem

For the x_0-subsystem, we choose the control u_0 as

$$u_0 = u_0^* \tag{3}$$

where u_0^* is a positive constant. In this case, the x_0-subsystem becomes

$$\dot{x}_0 = d_0 u_0^{*p_0} + \phi_0(t, x_0) \tag{4}$$

Note that $\phi_0(t, x_0)$ satisfies the linear growth condition. From Assumption 2, it is easy to obtain that the solution of x_0-subsystem is well-defined on $[0, \infty)$. Under the control law (3), the x-subsystem can be written as

$$\begin{array}{rcl} \dot{x}_1 & = & d_1 u_0^* x_2^{p_1} + \phi_1(t, x_2, \cdots, x_n, u_0, u_1) \\ \dot{x}_2 & = & d_2 u_0^* x_3^{p_2} + \phi_2(t, x_3, \cdots, x_n, u_0, u_1) \\ & \vdots & \\ \dot{x}_{n-1} & = & d_{n-1} u_0^* x_n^{p_{n-1}} + \phi_{n-1}(t, x_n, u_0, u_1) \\ \dot{x}_n & = & d_n u_1^{p_n} \end{array} \tag{5}$$

Before designing the controller, we first introduce the following coordinate transformation:

$$z_1 = x_1, \;\; z_i = \frac{x_i}{\varepsilon^{\kappa_i}}, \;\; i = 2, \cdots, n, \;\; v^{p_n} = \frac{u_1^{p_n}}{\varepsilon^{\kappa_n + 1}} \tag{6}$$

where $0 < \varepsilon < 1$ is a constant to be determined later and $\kappa_1 = 0$, $\kappa_{i+1} = \frac{\kappa_i + 1}{p_i}$, $i = 1, \cdots, n-1$.

Then, under this transformation, system (5) is transformed into:

$$\begin{array}{rcl} \dot{z}_1 & = & \varepsilon d_1 u_0^* z_2^{p_1} + f_1(t, z_2, \cdots, z_n, u_1) \\ \dot{z}_2 & = & \varepsilon d_2 u_0^* z_3^{p_2} + f_2(t, z_3, \cdots, z_n, u_1) \\ & \vdots & \\ \dot{z}_{n-1} & = & \varepsilon d_{n-1} u_0^* z_n^{p_{n-1}} + f_{n-1}(t, z_n, u_1) \\ \dot{z}_n & = & \varepsilon d_n v^{p_n} \end{array} \tag{7}$$

where

$$f_i(t, z_{i+1}, \cdots, z_n, u_1) = \frac{\phi_i(t, x_{i+1}, \cdots, x_n, u_1)}{\varepsilon^{\kappa_i}} \tag{8}$$

In the next, we shall construct a continuous state feedback controller by using the homogeneous domination approach.

Step 1. Let $\xi_1 = [z_1]^{1/r_1}$ and choose the Lyapunov function

$$V_1 = W_1 = \int_{z_1^*}^{z_1} \Big[[s]^{1/r_1} - [z_1^*]^{1/r_1} \Big]^{2 - r_1} ds \tag{9}$$

with $z_1^* = 0$. From (7), it follows that

$$\dot{V}_1 \leq -n\varepsilon |\xi_1|^{2+\tau} + \varepsilon d_1 u_0^* [\xi_1]^{2 - r_1} (z_2^{p_1} - z_2^{*p_1}) + \frac{\partial V_1}{\partial z_1} f_1 \tag{10}$$

where the virtual controller is chosen as

$$z_2^* = -\Big(\frac{n}{c_{11} u_0^*} \Big)^{1/p_1} [\xi_1]^{(r_1 + \tau)/p_1} := -\beta_1 [\xi_1]^{r_2} \tag{11}$$

Step k ($k = 2, \cdots, n$)**.** In this step, we can obtain the following property, whose similar proof can be found in [26] and hence is omitted here.

Proposition 1. Suppose at step $k-1$, there is a C^1, proper and positive definite Lyapunov function V_{k-1}, and a set of virtual

$$\begin{array}{rclrcl} z_1^* & = & 0, & \xi_1 & = & [z_1]^{1/r_1} - [z_1^*]^{1/r_1} \\ z_2^* & = & -\beta_1 [\xi_1]^{r_2}, & \xi_2 & = & [z_2]^{1/r_2} - [z_2^*]^{1/r_2} \\ & \vdots & & & \vdots & \\ z_k^* & = & -\beta_{k-1} [\xi_{k-1}]^{r_k}, & \xi_k & = & [z_k]^{1/r_k} - [z_k^*]^{1/r_k} \end{array} \tag{12}$$

with $\beta_i > 0$, $i = 1, \cdots, k$, being constants, such that

$$\begin{array}{l} \dot{V}_{k-1} \leq -(n - k + 2)\varepsilon \sum_{i=1}^{k-1} |\xi_i|^{2+\tau} \\ \quad + \varepsilon d_{k-1} u_0^* [\xi_{k-1}]^{2 - r_{k-1}} (z_k^{p_{k-1}} - z_k^{*p_{k-1}}) \\ \quad + \sum_{i=1}^{k-1} \frac{\partial V_{k-1}}{\partial z_i} f_i \end{array} \tag{13}$$

Then the kth Lyapunov function

$$V_k = V_{k-1} + W_k = V_{k-1} + \int_{z_k^*}^{z_k} \Big[[s]^{1/r_k} - [z_k^*]^{1/r_k} \Big]^{2 - r_k} ds \tag{14}$$

is C^1, positive definite and proper, and there exists a C^0 virtual controller $z_{k+1}^* = -\beta_k [\xi_k]^{r_{k+1}}$ such that

$$\begin{array}{l} \dot{V}_k \leq -(n - k + 1)\varepsilon \sum_{i=1}^{k} |\xi_i|^{2+\tau} \\ \quad + \varepsilon d_k u_0^* [\xi_k]^{2 - r_k} (z_{k+1}^{p_k} - z_{k+1}^{*p_k}) \\ \quad + \sum_{i=1}^{k} \frac{\partial V_k}{\partial z_i} f_i \end{array} \tag{15}$$

Hence at step n, choosing

$$V_n = \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} \int_{z_k^*}^{z_k} \Big[[s]^{1/r_k} - [z_k^*]^{1/r_k} \Big]^{2 - r_k} ds \tag{16}$$

and

$$v = z_{n+1}^* = -\beta_n [\xi_n]^{r_{n+1}} \tag{17}$$

From Proposition 1, we arrive at

$$\dot{V}_n \leq -\varepsilon \sum_{i=1}^{n} |\xi_i|^{2+\tau} + \sum_{i=1}^{n} \frac{\partial V_n}{\partial z_i} f_i \tag{18}$$

Hence, the following result is obtained.

Lemma 5. For the nonlinear system (7) under Assumptions 1 and 3, the state feedback controller (17) renders the origin of the closed-loop system is semi-globally finite-time stable.

Proof. Since V_n is positive definite and proper with respect to $z = (z_1, \cdots, z_n)^T$, by introducing the dilation weight $\Delta = (r_1, \cdots, r_n)$, from Definition 1, it can be shown that V_n is homogeneous of degree 2 with respect to Δ. By Lemma 2, there is a constant $\bar{c}_1$, such that

$$V_n \leq \bar{c}_1 \|z\|_\Delta^2 \tag{19}$$

where $\bar{c}_1 > 0$ and $\|z\|_\Delta = \sqrt{(\sum_{i=1}^n |z_i|^{2/r_i})}$. Similarly, since the $\sum_{i=1}^n |\xi_i|^{2+\tau}$ is homogeneous of degree $2+\tau$, by Lemma 2 there is a constant $\bar{c}_2$ such that

$$\dot{V}_n \leq -\varepsilon \bar{c}_2 \|z\|_\Delta^{2+\tau} + \sum_{i=1}^{n} \frac{\partial V_n}{\partial z_i} f_i \tag{20}$$

According to (6), (8) and Assumption 3, we have

$$|f_i(\cdot)| = \Big| \frac{\phi_i(\cdot)}{\varepsilon^{\kappa_i}} \Big| \leq b \sum_{j=i+1}^{n+1} \frac{\varepsilon^{q_{ij}\kappa_j}}{\varepsilon^{\kappa_i}} |z_j|^{q_{ij}} \tag{21}$$

Note that by Assumption 3, for $i+1 \le j \le n$,

$$\begin{array}{rl} & q_{ij}\kappa_j - \kappa_i \\ > & \dfrac{\kappa_j(r_i+\tau)}{r_j} - \kappa_i \\ = & \dfrac{\kappa_j(\tau\kappa_i + 1/(p_1\cdots p_{i-1})+\tau)}{\tau\kappa_j + 1/(p_1\cdots p_{j-1})} - \kappa_i \\ = & \dfrac{\tau\kappa_j + \kappa_j/(p_1\cdots p_{i-1}) - \kappa_i/(p_1\cdots p_{j-1})}{\tau\kappa_j + 1/(p_1\cdots p_{j-1})} \\ = & \dfrac{\tau\kappa_j + (1+p_i+p_i\cdots p_{j-2})/(p_1\cdots p_{j-1})}{\tau\kappa_j + 1/(p_1\cdots p_{j-1})} \\ \ge & 1 \end{array} \quad (22)$$

which in turn implies that there is a constant $\alpha_i > 0$ such that for $0 < \varepsilon < 1$,

$$\begin{array}{rcl} |f_i(\cdot)| & \le & b\varepsilon^{1+\alpha_i} \displaystyle\sum_{j=i+1}^{n+1} |z_j|^{q_{ij}} \\ & = & b\varepsilon^{1+\alpha_i} \displaystyle\sum_{j=i+1}^{n+1} |z_j|^{q_{ij}-(r_i+\tau)/r_j} |z_j|^{(r_i+\tau)/r_j} \end{array} \quad (23)$$

Next, we decide the gain ε to guarantee the attractivity for a given region. To this end, for an arbitrarily large number $K > 1$, define compact sets $N = \{\vartheta | \|\vartheta\| \le K, \vartheta \in R^n\}$ and $\Omega = \{z | V_n(z) \le M, M = max_{z\in N} V_n(z)\}$. Clearly, Ω is a non-empty compact set and in this set z is bounded.

From (23), we know that in the compact set Ω there is a constant δ_i such that

$$|f_i(\cdot)| \le \delta_i \varepsilon^{1+\alpha_i} \sum_{j=1}^{i} |z_j|^{(r_i+\tau)/r_j} \quad (24)$$

Noting that for $i = 1, \cdots, n$, $\partial V_n/\partial z_i$ is homogeneous of degree $2 - r_i$, we know that

$$\frac{\partial V_n}{\partial z_i}(|z_1|^{(r_i+\tau)/r_1} + \cdots + |z_n|^{(r_i+\tau)/r_n}) \quad (25)$$

is homogeneous of degree $2+\tau$.

With (24) and (25) in mind, we can find a positive constant θ_i such that

$$\left|\frac{\partial V_n}{\partial z_i}\right| |f_i| \le \theta_i \varepsilon^{1+\alpha_i} \|z\|_{\triangle}^{2+\tau} \quad (26)$$

where $\alpha = min_{1\le i\le n}\{\alpha_i\} > 0$.

Substituting (26) into (20) yields

$$\dot{V}_n \le -\varepsilon(\bar{c}_2 - \sum_{i=1}^{n} \theta_i \varepsilon^{\alpha}) \|z\|_{\triangle}^{2+\tau} \quad (27)$$

Apparently, by choosing a small enough ε, the right-hand side of (27) is negative definite. Therefore, the trajectory of the closed-loop system starting from Ω will stay in the compact set Ω forever. Furthermore, it can be deduced from (19) and (27) that there is a constant $\bar{c}_3$ such that

$$\dot{V}_n \le -\bar{c}_3 V_n^{(2+\tau)/2} \quad (28)$$

By Lemma 1 ($V_n = V$, $c = \bar{c}_3$ and $(2+\tau)/2 < 1$), (28) leads to the conclusion that the closed-loop system (7) and (17) is finite-time stable with its settling time T_1 satisfying

$$T_1 \le \frac{-2V_n^{(-\tau)/2}(0)}{\bar{c}_3\tau} \quad (29)$$

Note that by definition of of compact sets N and Ω, we have the following relation for any arbitrarily large set N

$$(\xi_1, \cdots, \xi_n)^T \in N \Rightarrow \|Z\| \le K \Rightarrow Z \in \Omega \quad (30)$$

Consequently, we have the conclusion that starting from any points in N, the trajectory will stay in the compact set Ω and tend to the origin in a finite time, i.e., the closed-loop system (7)+(17) is semi-globally finite-time stable. Thus, the proof is completed.

Since (6) is an equivalent transformation, the closed-loop system consisting of (5), $u_1^{p_n} = \varepsilon^{\kappa_n+1}\upsilon^{p_n}$ in (6) and (17), has the same properties as the system (7) and (17), that is, system (5) is semi-globally finite-time regulated at origin with the settling time T_1.

B. The controller design of the x_0-subsystem

From the discussion of the above subsection, we know that $x(t) \equiv 0$ when $t \ge T_1$. Therefore, we just need to stabilize the x_0-subsystem in a finite time. When $t \ge T_1$, for the x_0-subsystem, we can take the following control law

$$u_0^{p_0} = g_0[x_0]^{\alpha_0}, \quad 0 < \alpha_0 < 1 \quad (31)$$

$$g_0 = -\frac{1}{c_{01}}\big(k_0 + \psi_0(x_0)\big) \quad (32)$$

where k_0, α_0 are positive constants, and $\psi_0(x_0) \ge a|x_0|^{1-\alpha_0} \ge 0$ is a smooth function. For instance, we can simply choose $\psi_0(x_0) = a(1+x_0^2)$.

Taking the Lyapunov function $V_0 = x_0^2/2$, a simple computation gives

$$\dot{V}_0 \le -k_0 x_0^{1+\alpha_0} \le -k_0 V_0^{(1+\alpha_0)/2} \quad (33)$$

Thus by Lemma 1, x_0 tends to 0 within a settling time denoted by T_2 and

$$T_2 \le \frac{2V_0^{(1-\alpha_0)/2}(0)}{k_0(1-\alpha_0)} \quad (34)$$

Up to now, we have finished the finite-time state feedback stabilizing controller design of the system (1). Consequently, the following theorem can be obtained to summarize the main results of the paper.

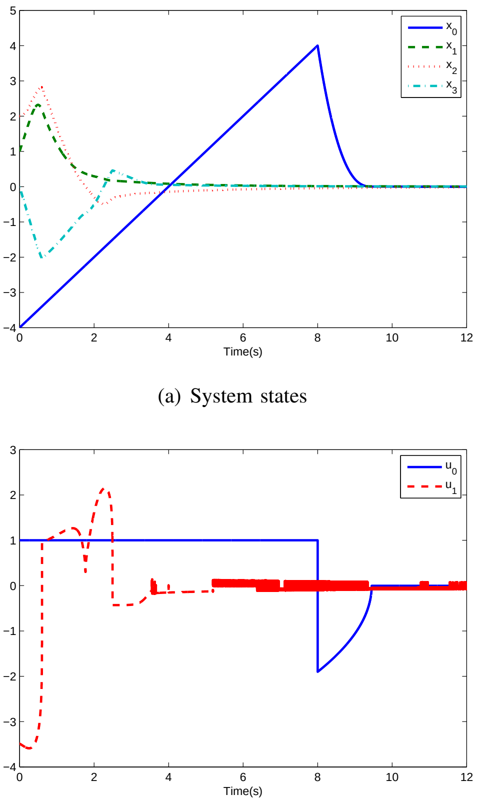
Theorem 1. Under Assumptions 1-3, if the proposed control design procedure together with the above switching control strategy is applied to high order nonholonomic feedforward system (1), then the states of closed-loop system are semi-globally regulated to zero in a finite time.

IV. Simulation Example

To verify the proposed controller, we consider the following high order nonholonomic feedforward system:

$$\begin{array}{rcl} \dot{x}_0 & = & u_0^3 \\ \dot{x}_1 & = & x_2^{5/3}u_0 + x_3^2 \\ \dot{x}_2 & = & x_3 + u_1 sin^2 x_3 \\ \dot{x}_3 & = & u_1 \end{array} \quad (35)$$

where $x_i, i = 0, 1, 2, 3$, are system states, u_0 and u_1 are control inputs. It is worth pointing out that system (35) cannot be finite-time stabilized by using the existing control methods because of the presence of feedforward-like terms x_3^4 and $u_1 sin^2 x_3$. Choose $\tau = -1/6$, then $r_1 = 1, r_2 = 1/2$,



(a) System states

(b) Control inputs

Fig. 1. The responses of the closed-loop system (35)–(37).

$r_3 = 1/3$ and $r_4 = 1/6$. By Lemma 4, it can be verified that $|\phi_1| \le |x_3|^4$ and $|\phi_2| \le \frac{2}{3}|sinx_3|^3 + \frac{1}{3}|u_1|^3 \le (|x_3|^3 + |u_1|^3)$, that is, Assumption 3 is satisfied with $q_{13} = 2$, $q_{23} = q_{24} = 3$ and $b = 1$. By choosing $u_0^* = 1$ and following the design procedure given in Section 3, we can design a state feedback controller

$$u_1 = -\varepsilon^{13/5}\beta_3\Big[[x_3/\varepsilon^{8/5})]^3 + \beta_2^3[x_2/\varepsilon^{3/5}]^2 + \beta_2^3\beta_1^2[x_1]\Big]^{1/6} \tag{36}$$

to render the x-subsystem of (35) semi-globally finite-time stable with a settling time T_1. Then, when $t \ge T_1$, for the x_0-subsystem, we switch the control input u_0 to

$$u_0^5 = -k_0[x_0]^{1/3} \tag{37}$$

where β_1, β_2, β_3, ε and k_0 are appropriate positive constants. The simulation is carried out with the following choices: $\varepsilon = 0.8$, $\beta_1 = 1$, $\beta_2 = 1.5$, $\beta_3 = 4$, $k_0 = 3$ and the initial value $(x_0(0), x_1(0), x_2(0), x_3(0))= (-4, 2, 1, 0)$. The responses of the closed-loop system (35)-(37) are shown in Figure 3, from which the validness of the controller is demonstrated.

V. Conclusion

This paper has solved the problem of finite-time stabilization by state feedback for a class of uncertain high order nonholonomic feedforward systems. With the help of the homogeneous domination approach, a constructive design procedure for state feedback control is given. Together with a novel switching control strategy, the designed controller can guarantee that the closed-loop system states are finite-time regulated to zero. It should be noted that the proposed controller can only work well when the whole state vector is measurable. Therefore, a natural and more interesting problem is how to design an output feedback stabilizing controller for the systems studied in the paper when only partial state vector is measurable, which is now under our further investigation.

References

[1] Y. Takita, H. Date, S. Ohkawa, “High-speed driving of lateral guided robotic vehicle with a rear wheel steer mechanism controlled by SSM,” *Engineering Letters*, vol.20, no.1, pp.20-27, 2012.

[2] R.M. Mahamood, “Improving the performance of adaptive PDPID control of two-link flexible robotic manipulator with ILC, ” *Engineering Letters*, vol.20, no.3, pp.259–270, 2012.

[3] Y. L. Shang, D.H. Hou, F. Z. Gao, “Global output feedback stabilization of nonholonomic chained form systems with communication delay, ” *IAENG International Journal of Applied Mathematics*, vol.46, no.3, pp. 367–371, 2016.

[4] Y. L. Shang, D.H. Hou, F. Z. Gao, “Finite-time output feedback stabilization for a class of uncertain high order nonholonomic systems, ” *Engineering Letters*, vol.25, no.1, pp. 39–45, 2017.

[5] R. W. Brockett, “Asymptotic stability and feedback stabilization, ” in: R.W. Brockett, R.S. Millman, H.J. Sussmann (Eds.), *Differentical Geometry Control Theory*, pp. 2961–2963, 1983.

[6] A. Astolfi, “Discontinuous control of nonholonomic systems, ”*Systems and Control Letters*, vol.27, no.1, pp. 37–45, 1996.

[7] W. L. Xu, W. Huo, “Variable structure exponential stabilization of chained systems based on the extended nonholonomic integrator, ”*Systems and Control Letters*, vol.41, no.4, pp.225–235, 2000.

[8] Z. P. Jiang, “Iterative design of time-varying stabilizers for multi-input systems in chained form, ”*Systems and Control Letters*, vol.28, no.5, pp. 255–262, 1996

[9] H. L. Yuan, Z. H. Qu, “Smooth time-varying pure feedback control for chained nonholonomic systems with exponential convergent rate, ”*IET Control Theory and Applications*, vol.4, no.7, pp.1235–1244, 2010.

[10] C. de Wit Canudas, B. Siciliano, G. Bastin (Eds.), *Theory of Robot Control*, Springer, London, 1996.

[11] Z. P. Jiang, “Robust exponential regulation of nonholonomic systems with uncertainties, *Automatica*, vol.36, no.2, pp.189–209, 2000.

[12] S. S. Ge, Z. P. Wang, T. H. Lee, “ Adaptive stabilization of uncertain nonholonomic systems by state and output feedback, *Automatica*, vol.39, no.8, pp.1451–1460, 2003.

[13] X. Y. Zheng, Y. Q. Wu, “Adaptive output feedback stabilization for nonholonomic systems with strong nonlinear drifts, ” *Nonlinear Analysis: Theory, Methods and Applications*, vol.70, no.2, pp. 904–920, 2009.

[14] Y. Q. Wu, G. L. Ju, X. Y. Zheng, “Adaptive output feedback control for nonholonomic systems with uncertain chained form, ” *International Journal of Systems Science*, vol.41, no.12, pp. 1537–1547, 2010.

[15] F. Z. Gao, F. S. Yuan, H. J. Yao, “Robust adaptive control for nonholonomic systems with nonlinear parameterization, ” *Nonlinear Analysis: Real World Applications*, vol.11, no.4, pp. 3242–3250, 2010.

[16] F. Z. Gao, F. S. Yuan, Y. Q. Wu, “State-feedback stabilisation for stochastic non-holonomic systems with time-varying delays, ” *IET Control Theory and Applications*, vol.6, no.17, pp. 2593–2600, 2012.

[17] X. W. Mu, J. M. Yu, G. F. Cheng, “Adaptive regulation of high order nonholonomic systems, ” *Applied Mathematics and Mechanics*, vol. 27, pp. 501–507, 2006.

[18] X. Y. Zheng, Y. Q. Wu, “Controller design of high order nonholonomic system with nonlinear drifts, ” *International Journal of Automation and Computing*, vol.6, no.3, pp. 240–244, 2009.

[19] F. Z. Gao, F. S. Yuan, H. J. Yao, X. W. Mu, “Adaptive stabilization of high order nonholonomic systems with strong nonlinear drifts, ” *Applied Mathematical Modelling*, vol.35, no.9, pp.4222–4233, 2011.

[20] J. Zhang, Y. Liu, “Adaptive stabilization of a class of high-order uncertain nonholonomic systems with unknown control coefficients, ” *International Journal of Adaptive Control and Signal Processing*, vol.27, no.5, pp.368–385, 2013.

[21] S. P. Bhat, D. S. Bernstein, “Finite-time stability of continuous autonomous systems, ” *SIAM Journal on Control and Optimization*, vol.38, no.3, pp. 751–766, 2000.

[22] Y. Hong, J. Wang, Z. Xi, “Stabilization of uncertain chained form systems within finite settling time, ” *IEEE Transactions on Automatic Control*, vol.50, no.9, pp.1379–1384, 2005.

[23] F. Z. Gao, Y. L. Shang, F.S. Yuan, “Robust adaptive finite-time stabilization of nonlinearly parameterized nonholonomic systems, ” *Acta applicandae mathematicae*, vol.123, no.1, pp.157–173, 2013.

[24] Y. Q. Wu, F. Z. Gao, Z. G. Liu, “Finite-time state feedback stabilization of nonholonomic systems with low-order nonlinearities, ” *IET Control Theory and Applications*, vol.9, no.10, pp.1553-1560, 2015.
[25] F. Z. Gao, F. S. Yuan, “Adaptive finite-time stabilization for a class of uncertain high order nonholonomic systems, ” *ISA Transactions*, vol.54, pp.75–82, 2015.
[26] F. Z. Gao, Y. Q. Wu, Z. C. Zhang, “Finite-time stabilization of uncertain nonholonomic systems in feedforward-like form by output feedback, ” *ISA Transactions*, vol.59, pp.125–132, 2015.
[27] L. Rosier, “Homogeneous lyapunov functions for homogeneous continuous vector fields, ” *Systems and Control Letters*, vol.19, no.6, pp. 467–473, 1992.
[28] C. J. Qian, W. Lin, “A continuous feedback approach to global strong stabilization of nonlinear systems, ”*IEEE Transactions on Automatic Control*, vol.46, no.7, pp.1061–1079, 2001.