Rough Set Models induced by Serial Fuzzy Relations Approach in Semigroups

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Abstract—**This work presents rough sets in approximation spaces based on overlaps of successor classes with respect to level in closed unit intervals under serial fuzzy relations between two universes. Some related properties are verified. In semigroup structures, the concepts of rough semigroups, rough ideals and rough completely prime ideals in approximation spaces under transitive and compatible fuzzy relations are introduced. Next, sufficient conditions of them are provided. Finally, relationships between rough semigroups (resp. rough ideals and rough completely prime ideals) and their homomorphic images are investigated. These relationships are presented in purport of necessary and sufficient conditions.**

Index Terms—**rough set, semigroup, rough semigroup, rough ideal, rough completely prime ideal, serial fuzzy relation, transitive fuzzy relation, compatible fuzzy relation**

I. Introduction

To solve problems in uncertain data under information sciences with computational technologies in terms of crisp sets, Pawlak's rough set theory offers an alternative classical tool for such the problem-solving. This theory was proposed by Pawlak [1] in 1982 which is an approximation processing model based on the foundation of an approximation space induced by an equivalence relation on a universal set. Given an equivalence relation on a universal set and a non-empty subset of the universal set, the Pawlak's rough set of the given set is defined by a pair of two sets, called the Pawlak's upper and Pawlak's lower approximations where the difference between the Pawlak's upper approximation and the Pawlak's lower approximation (also called the Pawlak's boundary region) is a non-empty set. The Pawlak's upper approximation is the union of all the equivalence classes which have a non-empty intersection with the given set. The Pawlak's lower approximation is the union of all the equivalence classes which are subset of the given set. As studied above, the Pawlak's rough set model has been being used in algebraic systems [2]–[15], expert systems with applications [16], knowledge-based systems [17], computers and electrical engineerings [18], measurements [19], approximate reasonings [20] etc. Because of new trend, Pawlaks rough set theory has been becoming an information management tool in the area of artificial intelligence.

Based on the Pawlak's rough set induced by an equivalence relation, extended notions have been studied with different arbitrary binary relations (briefly, binary relations). Especially, Yao [21] introduced roughness models using successor neighborhoods induced by binary relations [$SN_\theta(u) := \{u' \in U : (u, u') \in \theta\}$ denotes a successor neighborhood of u induced by a binary relation θ on a universal set U where u is an element in U]. Lately, Mareay [22] introduced rough sets via cores of successor neighborhoods induced by binary relations [$CSN_\theta(u) := \{u' \in U : SN_\theta(u) = SN_\theta(u')\}$ denotes a core of a successor neighborhood of u induced by a binary relation θ on a universal set U where u is an element in U]. If a binary relation on a universal set has a property that an equivalence relation, then the Yao's rough set and the Mareay's rough set are generalizations of the Pawlak's rough set.

In 1965, Zadeh [23] introduced a classical notion of fuzzy set theory. Based on this point, Zadeh (see [24], [25]) introduced the concept of fuzzy relations in 1971. This classical fuzzy set theory has possible uses in various fields, such as information sciences [26], algebraic systems [27], computers and engineerings [28] etc.

The semigroup structure (see [29]) is an algebraic system with respect to extensive applications, such as the semigroup provide an algebraic framework for modeling and investigating the key factors in dynamical systems under algebraic engineerings [30] etc. For combinations of semigroup theory and Pawlak's rough set theory, Kuroki [4] proposed the notions of upper and lower approximation semigroups (resp. ideals) in semigroups induced by congruence relations, and provided sufficient conditions of upper and lower approximation semigroups (resp. ideals) in 1997. In 2006, Xiao and Zhang [7] proposed the notions of upper and lower approximation completely prime ideals in semigroups induced by congruence relations, and provided sufficient conditions of upper and lower approximation completely prime ideals. Also, they verified the relationship between upper and lower approximation completely prime ideals (resp. ideals) and the homomorphic image of upper and lower approximation completely prime ideals (resp. ideals) under homomorphism problems. Under combinations of semigroup theory, Pawlak's rough set theory and fuzzy set theory, Wang and Zhan [13] introduced the concepts of upper and lower approximation semigroups (resp. ideals and completely prime ideals) induced by special congruence relations induced by fuzzy ideals, and also provided sufficient conditions of upper and lower approximation semigroups (resp. ideals and completely prime ideals) in 2016.

The main purpose of this paper is developments of the rough set theory induced by fuzzy relations on universal sets and semigroups. After providing some preliminary concepts of fundamental fuzzy relations and semigroups in Section II, we introduce a rough set in an approximation space based on overlaps of successor classes with respect to level in a closed

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unit interval under a fuzzy relation between two universes, and verify some interesting properties in Section III. In Section IV, we give concepts of rough semigroups, rough ideals and rough completely prime ideals in approximation spaces under transitive and compatible fuzzy relations on semigroups. Next, we provide sufficient conditions of them. In Section V, we investigate relationships between rough semigroups (resp. rough ideals and rough completely prime ideals) and their homomorphic images. In the end, we give a conclusion of the research in Section VI.

II. Preliminaries

In this section we review some important definitions which will be refered in the subsequent sections.

Throughout this work, we suppose that U and V denote two non-empty universal sets.

Definition 1. [23] A *fuzzy set* of U is defined as a function from U to the closed unit interval $[0,1]$.

Definition 2. [26] Let $\mathcal{F}(U\times V)$ be a family of all fuzzy sets of $U\times V$. An element in $\mathcal{F}(U\times V)$ is referred to as a *fuzzy relation from U to V*. An element in $\mathcal{F}(U\times V)$ is called a *fuzzy relation on U* if $U=V$. For a fuzzy relation $\Phi\in\mathcal{F}(U\times V)$ and elements $u\in U$, $v\in V$, the value of $\Phi(u,v)$ in $[0,1]$ representing the *membership grade of the relation between u and v under Φ*. If $\Phi\in\mathcal{F}(U\times V)$ where $U:=\{u_1,u_2,u_3,...,u_m\}$ and $V:=\{v_1,v_2,v_3,...,v_n\}$, then the fuzzy relation Φ is represented by the matrix as

$$\begin{pmatrix}\Phi(u_1,v_1) & \Phi(u_1,v_2) & \Phi(u_1,v_3) & \cdots & \Phi(u_1,v_n)\\ \Phi(u_2,v_1) & \Phi(u_2,v_2) & \Phi(u_2,v_3) & \cdots & \Phi(u_2,v_n)\\ \Phi(u_3,v_1) & \Phi(u_3,v_2) & \Phi(u_3,v_3) & \cdots & \Phi(u_3,v_n)\\ \vdots & \vdots & \vdots & \ldots & \vdots\\ \Phi(u_m,v_1) & \Phi(u_m,v_2) & \Phi(u_m,v_3) & \cdots & \Phi(u_m,v_n)\end{pmatrix}.$$

Definition 3. [26] Let Φ be a fuzzy relation from U to V. Φ is called *serial* if for all $u\in U$, there exists $v\in V$ such that $\Phi(u,v)=1$.

Definition 4. [26] Let Φ be a fuzzy relation on U.

(1) Φ is called *reflexive* if for all $u\in U$, $\Phi(u,u)=1$.

(2) Φ is called *symmetric* if for all $u_1,u_2\in U$,
$$\Phi(u_1,u_2)=\Phi(u_2,u_1).$$

(3) Φ is called *transitive* if for all $u_1,u_2\in U$,
$$\Phi(u_1,u_2)\geq\vee_{u_3\in U}(\Phi(u_1,u_3)\wedge\Phi(u_3,u_2)).$$

(4) Φ is called a *similarity fuzzy relation* if it is reflexive, symmetric and transitive.

A *semigroup* [29] $(S,\circledast)$ is defined as an algebraic system where S is a non-empty set and $\circledast$ is an associative binary operation on S. Throughout this paper, S denotes a semigroup. A non-empty subset X of S is called a *subsemigroup* [31] of S if $X^2\subseteq X$. A non-empty subset X of S is called a *left (right) ideal* [31] of S if $SX\subseteq X$ ($XS\subseteq X$), and if it is both a left ideal and a right ideal of S, then it is called an *ideal* [31]. An ideal X of S is called a *completely prime ideal* [31] of S if for all $s_1,s_2\in S$, $s_1s_2\in X$ implies $s_1\in X$ or $s_2\in X$.

Definition 5. [31] Let Φ be a fuzzy relation on S. Φ is called *compatible* if for all $s_1,s_2,s_3\in S$,
$$\Phi(s_1s_3,s_2s_3)\geq\Phi(s_1,s_2)\text{ and }\Phi(s_3s_1,s_3s_2)\geq\Phi(s_1,s_2).$$

III. Rough set models induced by serial fuzzy relations

In this section we introduce a novel rough set induced by a serial fuzzy relation between two universes. Then we give a real-world example and verify to some interesting properties.

In the initial point, we construct new classes in Definitions 6 and 7, and examine to some related properties as the following.

Definition 6. Let $\iota\in[0,1]$ and let Φ be a fuzzy relation from U to V. For an element $u\in U$,
$$S_\Phi(u;\iota):=\{v\in V:\Phi(u,v)\geq\iota\}$$
is called a *successor class of u with respect to ι-level under Φ*.

Remark 1. Let $\iota\in[0,1]$. If Φ is a serial fuzzy relation from U to V, then $S_\Phi(u;\iota)\neq\emptyset$ for all $u\in U$.

Definition 7. Let $\iota\in[0,1]$ and let Φ be a serial fuzzy relation from U to V. For an element $u_1\in U$,
$$OS_\Phi(u_1;\iota):=\{u_2\in U:S_\Phi(u_1;\iota)\cap S_\Phi(u_2;\iota)\neq\emptyset\}$$
is called an *overlap of the successor class of u_1 with respect to ι-level under Φ*.

We denote by $\mathcal{OS}_\Phi(U;\iota)$ the collection of $OS_\Phi(u;\iota)$ for all $u\in U$.

From Definition 7, the following proposition can be easily obtained.

Proposition 1. *Let $\iota\in[0,1]$ and let Φ be a serial fuzzy relation from U to V. Then, $u\in OS_\Phi(u;\iota)$ for all $u\in U$.*

Proposition 2. *Let $\iota\in[0,1]$ and let Φ be a serial fuzzy relation on U. Then we have the following statements.*

(1) If Φ is reflexive, then we have $S_\Phi(u;\iota)\subseteq OS_\Phi(u;\iota)$ for all $u\in U$.

(2) If Φ is a similarity fuzzy relation, then $S_\Phi(u;\iota)$ and $OS_\Phi(u;\iota)$ are identical classes for all $u\in U$.

Proof: The proof is straightforward, so we omit it. ■

In the following, we give a new rough set induced by a serial fuzzy relation.

Definition 8. Let $\iota\in[0,1]$ and let Φ be a serial fuzzy relation from U to V. The triple $(U,V,\mathcal{OS}_\Phi(U;\iota))$ is referred to as an *approximation space based on $\mathcal{OS}_\Phi(U;\iota)$* (briefly, *$\mathcal{OS}_\Phi(U;\iota)$-approximation space*). If $U=V$, then the triple $(U,V,\mathcal{OS}_\Phi(U;\iota))$ is substituted by a pair $(U,\mathcal{OS}_\Phi(U;\iota))$.

Definition 9. Let $(U,V,\mathcal{OS}_\Phi(U;\iota))$ be an $\mathcal{OS}_\Phi(U;\iota)$-approximation space. For a non-empty subset X of U, we define three sets as follows:

$\overline{\Phi}(X;\iota):=\{u\in U:OS_\Phi(u;\iota)\cap X$ is a non-empty set$\}$,
$\underline{\Phi}(X;\iota):=\{u\in U:OS_\Phi(u;\iota)\subseteq X\}$ and
$\Phi_{bnd}(X;\iota):=\overline{\Phi}(X;\iota)-\underline{\Phi}(X;\iota)$.

Then

(1) $\overline{\Phi}(X;\iota)$ is referred to as an *upper approximation of X in $(U,V,\mathcal{OS}_\Phi(U;\iota))$* (briefly, *$\mathcal{OS}_\Phi(U;\iota)$-upper approximation of X*).

(2) $\underline{\Phi}(X;\iota)$ is referred to as a *lower approximation of* X *in* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ (briefly, $\mathcal{OS}_\Phi(U;\iota)$*-lower approximation of* X).
(3) $\Phi_{bnd}(X;\iota)$ is referred to as a *boundary region of* X *in* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ (briefly, $\mathcal{OS}_\Phi(U;\iota)$*-boundary region of* X).
(4) If $\Phi_{bnd}(X;\iota)\neq\emptyset$, then $\Phi(X;\iota):=(\overline{\Phi}(X;\iota),\underline{\Phi}(X;\iota))$ is referred to as a *rough set of* X *in* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ (briefly, $\mathcal{OS}_\Phi(U;\iota)$*-rough set of* X).
(5) If $\Phi_{bnd}(X;\iota)=\emptyset$, then X is referred to as a *definable set in* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ (briefly, $\mathcal{OS}_\Phi(U;\iota)$*-definable set*).

Here we present an example as the following.

Example 1. *Let* $U:=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ *be a set of electrical discharge machines (EDM) in an aerospace industry of a leading company, and let* $V:=\{v_1,v_2,v_3,v_4,v_5\}$ *be a set of components of each elements in* U*. For a fuzzy relation* $\Phi\in\mathcal{F}(U\times V)$ *and elements* $u\in U$, $v\in V$*, the number* $\Phi(u,v)$ *in the closed unit interval* $[0,1]$ *is defined as a damage value of* u *with respect to* v *under* Φ*. The damage values of all electrical discharge machines in* U *with respect to components in* V *under* Φ *are given as the following matrix.*

$$\begin{pmatrix} 0.9 & 0.8 & 0.9 & 0.9 & 0.8\\ 0.8 & 0.8 & 0.9 & 0.8 & 0.8\\ 0.9 & 0.8 & 0.8 & 0.8 & 0.8\\ 0.7 & 0.9 & 0.8 & 0.8 & 0.8\\ 0.8 & 0.9 & 0.8 & 0.7 & 0.8\\ 0.7 & 0.8 & 0.8 & 0.8 & 0.9 \end{pmatrix}$$

Let $\iota=0.9$ *be a maximum damage value of the usable level. Suppose that a measurement expert committee assign* $X:=\{u_1,u_3,u_5\}$ *which is a non-empty set of electrical discharge machines for the discharge under the global evaluation. Then the assessment of* X *in an approximation space* $(U,V,\mathcal{OS}_\Phi(U;0.9))$ *is derived by the process as the following. According to Definition 6, it follows that*
$S_\Phi(u_1;0.9):=\{v_1,v_3,v_4\}$,
$S_\Phi(u_2;0.9):=\{v_3\}$,
$S_\Phi(u_3;0.9):=\{v_1\}$,
$S_\Phi(u_4;0.9):=\{v_2\}$,
$S_\Phi(u_5;0.9):=\{v_2\}$ *and*
$S_\Phi(u_6;0.9):=\{v_5\}$.
According to Definition 7, it follows that
$OS_\Phi(u_1;0.9):=\{u_1,u_2,u_3\}$,
$OS_\Phi(u_2;0.9):=\{u_1,u_2\}$,
$OS_\Phi(u_3;0.9):=\{u_1,u_3\}$,
$OS_\Phi(u_4;0.9):=\{u_4,u_5\}$,
$OS_\Phi(u_5;0.9):=\{u_4,u_5\}$ *and*
$OS_\Phi(u_6;0.9):=\{u_6\}$.
According to Definition 9, it follows that
$\overline{\Phi}(X;0.9):=\{u_1,u_2,u_3,u_4,u_5\}$,
$\underline{\Phi}(X;0.9):=\{u_3\}$ *and*
$\Phi_{bnd}(X;0.9):=\{u_1,u_2,u_4,u_5\}$.
Thus we get $\Phi(X;0.9):=(\{u_1,u_2,u_3,u_4,u_5\},\{u_3\})$ *is a* $\mathcal{OS}_\Phi(U;0.9)$*-rough set of* X*. As a consequence,*
(1) u_1,u_2,u_3,u_4 *and* u_5 *are possibly electrical discharge machines for the discharge,*
(2) u_3 *is certainly electrical discharge machine for the discharge,*
(3) u_1,u_2,u_4 *and* u_5 *cannot be determined whether four students are electrical discharge machines for the discharge or not.*

In what follows, Example 1 leads to Definition 10 as the following.

Definition 10. Let $(U,V,\mathcal{OS}_\Phi(U;\iota))$ be an $\mathcal{OS}_\Phi(U;\iota)$-approximation space and let X be a non-empty subset of U. $\overline{\Phi}(X;\iota)$ is called a *non-empty* $\mathcal{OS}_\Phi(U;\iota)$*-upper approximation of* X *in* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ if $\overline{\Phi}(X;\iota)$ is a non-empty subset of U. Similarly, we can define non-empty $\mathcal{OS}_\Phi(U;\iota)$-lower approximations. $\Phi(X;\iota)$ is referred to as a *non-empty* $\mathcal{OS}_\Phi(U;\iota)$*-rough set* if $\overline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(U;\iota)$-upper approximation and $\underline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(U;\iota)$-lower approximation.

Proposition 3. *Let* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ *be an* $\mathcal{OS}_\Phi(U;\iota)$*-approximation space. If* X *and* Y *are non-empty subsets of* U*, then we have the following statements.*
(1) $\overline{\Phi}(U;\iota)=U$ *and* $\underline{\Phi}(U;\iota)=U$.
(2) $\overline{\Phi}(\emptyset;\iota)=\emptyset$ *and* $\underline{\Phi}(\emptyset;\iota)=\emptyset$.
(3) $X\subseteq\overline{\Phi}(X;\iota)$ *and* $\underline{\Phi}(X;\iota)\subseteq X$.
(4) $\overline{\Phi}(X\cup Y;\iota)=\overline{\Phi}(X;\iota)\cup\overline{\Phi}(Y;\iota)$ *and* $\underline{\Phi}(X\cap Y;\iota)=\underline{\Phi}(X;\iota)\cap\underline{\Phi}(Y;\iota)$.
(5) $\overline{\Phi}(X\cap Y;\iota)\subseteq\overline{\Phi}(X;\iota)\cap\overline{\Phi}(Y;\iota)$ *and* $\underline{\Phi}(X\cup Y;\iota)\supseteq\underline{\Phi}(X;\iota)\cup\underline{\Phi}(Y;\iota)$.
(6) If $X\subseteq Y$*, then* $\overline{\Phi}(X;\iota)\subseteq\overline{\Phi}(Y;\iota)$ *and* $\underline{\Phi}(X;\iota)\subseteq\underline{\Phi}(Y;\iota)$.

Proof: The proof is straightforward, so we omit it. ■

Definition 11. Let $(U,V,\mathcal{OS}_\Phi(U;\iota))$ be an $\mathcal{OS}_\Phi(U;\iota)$-approximation space and let X be a non-empty subset of U. If $\underline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(U;\iota)$-lower approximation of X in $(U,V,\mathcal{OS}_\Phi(U;\iota))$ and $\underline{\Phi}(X;\iota)$ is a proper subset of X, then X is called a *set over non-empty interior set*.

Proposition 4. *Let* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ *be an* $\mathcal{OS}_\Phi(U;\iota)$*-approximation space and let* X *be a non-empty subset of* U*. If* X *is a set over non-empty interior set, then* $\Phi(X;\iota)$ *is a non-empty* $\mathcal{OS}_\Phi(U;\iota)$*-rough set of* X *in* $(U,V,\mathcal{OS}_\Phi(U;\iota))$.

Proof: Suppose that X is a set over non-empty interior set. Then we have that $\underline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(U;\iota)$-lower approximation and $\underline{\Phi}(X;\iota)\subset X$. By Proposition 3 (3), we obtain that $\emptyset\neq X\subseteq\overline{\Phi}(X;\iota)$. Thus we get $\overline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(U;\iota)$-upper approximation. We shall verify that $\Phi_{bnd}(X;\iota)\neq\emptyset$. Suppose that $\Phi_{bnd}(X;\iota)=\emptyset$. Then, $\overline{\Phi}(X;\iota)=\underline{\Phi}(X;\iota)$. From Proposition 3 (3), once again, it follows that $\underline{\Phi}(X;\iota)=X$, a contradiction. Therefore, $\Phi_{bnd}(X;\iota)\neq\emptyset$. Consequently, $\Phi(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(U;\iota)$-rough set of X. ■

Proposition 5. *Let* $(U,V,\mathcal{OS}_\Phi(U;\iota))$ *be an* $\mathcal{OS}_\Phi(U;\iota)$*-approximation space and let* $(U,V,\mathcal{OS}_\Omega(U;\kappa))$ *be an* $\mathcal{OS}_\Omega(U;\kappa)$*-approximation space. If* $\iota\geq\kappa$ *and* $\Phi\subseteq\Omega$*, then* $\overline{\Phi}(X;\iota)\subseteq\overline{\Omega}(X;\kappa)$ *for every non-empty subset* X *of* U.

Proof: Let X be a non-empty subset of U and let u_1 be an element in $\overline{\Phi}(X;\iota)$. Then, $OS_\Phi(u_1;\iota)\cap X\neq\emptyset$. Thus there exists $u_2\in OS_\Phi(u_1;\iota)\cap X$. Hence we get

$S_{\Phi}(u_1;\iota) \cap S_{\Phi}(u_2;\iota) \neq \emptyset$. Thus there exists $u_3 \in U$ such that $u_3 \in S_{\Phi}(u_1;\iota)$ and $u_3 \in S_{\Phi}(u_2;\iota)$. Then, $\Phi(u_1,u_3) \geq \iota$ and $\Phi(u_2,u_3) \geq \iota$. By the assumption, we obtain that

$$\Omega(u_1,u_3) \geq \Phi(u_1,u_3) \geq \iota \geq \kappa$$

and

$$\Omega(u_2,u_3) \geq \Phi(u_2,u_3) \geq \iota \geq \kappa.$$

Hence we have $u_3 \in S_{\Omega}(u_1;\kappa) \cap S_{\Omega}(u_2;\kappa)$, and so $S_{\Omega}(u_1;\kappa) \cap S_{\Omega}(u_2;\kappa) \neq \emptyset$. Thus, $u_2 \in OS_{\Omega}(u_1;\kappa) \cap X$. Whence $OS_{\Omega}(u_1;\kappa) \cap X \neq \emptyset$. Hence $u_1 \in \overline{\Omega}(X;\kappa)$. Therefore, $\overline{\Phi}(X;\iota) \subseteq \overline{\Omega}(X;\kappa)$. ∎

Proposition 6. *Let $(U,V,\mathcal{OS}_{\Phi}(U;\iota))$ be an $\mathcal{OS}_{\Phi}(U;\iota)$-approximation space and let $(U,V,\mathcal{OS}_{\Omega}(U;\kappa))$ be an $\mathcal{OS}_{\Omega}(U;\kappa)$-approximation space. If $\iota \geq \kappa$ and $\Phi \subseteq \Omega$, then $\underline{\Omega}(X;\kappa) \subseteq \underline{\Phi}(X;\iota)$ for every non-empty subset X of U.*

Proof: Let X be a non-empty subset of U. Then we prove that $\underline{\Omega}(X;\kappa) \subseteq \underline{\Phi}(X;\iota)$. Indeed, let u_1 be an element in $\underline{\Omega}(X;\kappa)$. Then, $OS_{\Omega}(u_1;\iota) \subseteq X$. We shall show that $OS_{\Phi}(u_1;\iota) \subseteq OS_{\Omega}(u_1;\kappa)$. Let $u_2 \in OS_{\Phi}(u_1;\iota)$. Then we have $S_{\Phi}(u_1;\iota) \cap S_{\Phi}(u_2;\iota) \neq \emptyset$. Thus there exists $u_3 \in U$ such that $u_3 \in S_{\Phi}(u_1;\iota) \cap S_{\Phi}(u_2;\iota)$. Hence $\Phi(u_1,u_3) \geq \iota$ and $\Phi(u_2,u_3) \geq \iota$. By the assumption, we obtain that

$$\Omega(u_1,u_3) \geq \Phi(u_1,u_3) \geq \iota \geq \kappa$$

and

$$\Omega(u_2,u_3) \geq \Phi(u_2,u_3) \geq \iota \geq \kappa.$$

Thus we get that $u_3 \in S_{\Omega}(u_1;\kappa) \cap S_{\Omega}(u_2;\kappa)$, and so $S_{\Omega}(u_1;\kappa) \cap S_{\Omega}(u_2;\kappa) \neq \emptyset$. Thus, $u_2 \in OS_{\Omega}(u_1;\kappa)$, which yields $OS_{\Phi}(u_1;\iota) \subseteq OS_{\Omega}(u_1;\kappa) \subseteq X$. Therefore, $u_1 \in \underline{\Phi}(X;\iota)$. This means that $\underline{\Omega}(X;\kappa) \subseteq \underline{\Phi}(X;\iota)$. ∎

IV. Roughness in Semigroups

In this section we propose rough set models in semigroups induced by transitive and compatible fuzzy relations, which mainly include rough semigroups, rough ideals and rough completely prime ideals. Then we provide to sufficient conditions of them and give some intriguing properties and examples.

Definition 12. Let $(S,\mathcal{OS}_{\Phi}(S;\iota))$ be an $\mathcal{OS}_{\Phi}(S;\iota)$-approximation space. $(S,\mathcal{OS}_{\Phi}(S;\iota))$ is called an $\mathcal{OS}_{\Phi}(S;\iota)$-*approximation space type TCF* if Φ is a transitive and compatible fuzzy relation.

Proposition 7. *If $(S,\mathcal{OS}_{\Phi}(S;\iota))$ is an $\mathcal{OS}_{\Phi}(S;\iota)$-approximation space type TCF, then*

$$(OS_{\Phi}(s_1;\iota))(OS_{\Phi}(s_2;\iota)) \subseteq OS_{\Phi}(s_1s_2;\iota)$$

for all $s_1,s_2 \in S$.

Proof: Let s_1 and s_2 be two elements in S. Suppose that $s_3 \in (OS_{\Phi}(s_1;\iota))(OS_{\Phi}(s_2;\iota))$. Then there exists $s_4 \in OS_{\Phi}(s_1;\iota)$ and exists $s_5 \in OS_{\Phi}(s_2;\iota)$ such that $s_3 = s_4s_5$. Thus $S_{\Phi}(s_1;\iota)\cap S_{\Phi}(s_4;\iota) \neq \emptyset$ and $S_{\Phi}(s_2;\iota)\cap S_{\Phi}(s_5;\iota) \neq \emptyset$. Let $s_6 \in S_{\Phi}(s_1;\iota)\cap S_{\Phi}(s_4;\iota)$ and $s_7 \in S_{\Phi}(s_2;\iota)\cap S_{\Phi}(s_5;\iota)$. Then we have $\Phi(s_1,s_6) \geq \iota$, $\Phi(s_4,s_6) \geq \iota$, $\Phi(s_2,s_7) \geq \iota$ and $\Phi(s_5,s_7) \geq \iota$. Since Φ is transitive and compatible, we have

$$\begin{aligned}\Phi(s_1s_2,s_6s_7) &\geq \vee_{s_8\in S}\left(\Phi(s_1s_2,s_8)\wedge\Phi(s_8,s_6s_7)\right)\\ &\geq \Phi(s_1s_2,s_1s_7)\wedge\Phi(s_1s_7,s_6s_7)\\ &\geq \Phi(s_2,s_7)\wedge\Phi(s_1,s_6)\\ &\geq \iota\wedge\iota\\ &= \iota.\end{aligned}$$

Hence $\Phi(s_1s_2,s_6s_7) \geq \iota$, and so $s_6s_7 \in S_{\Phi}(s_1s_2;\iota)$. Since Φ is transitive and compatible, once again, we have

$$\begin{aligned}\Phi(s_4s_5,s_6s_7) &\geq \vee_{s_9\in S}\left(\Phi(s_4s_5,s_9)\wedge\Phi(s_9,s_6s_7)\right)\\ &\geq \Phi(s_4s_5,s_6s_5)\wedge\Phi(s_6s_5,s_6s_7)\\ &\geq \Phi(s_4,s_6)\wedge\Phi(s_5,s_7)\\ &\geq \iota\wedge\iota\\ &= \iota.\end{aligned}$$

Hence $\Phi(s_4s_5,s_6s_7) \geq \iota$, and so $s_6s_7 \in S_{\Phi}(s_4s_5;\iota)$. Thus $S_{\Phi}(s_1s_2;\iota)\cap S_{\Phi}(s_4s_5;\iota) \neq \emptyset$. Therefore, $s_3 \in OS_{\Phi}(s_1s_2;\iota)$. Consequently, $(OS_{\Phi}(s_1;\iota))(OS_{\Phi}(s_2;\iota)) \subseteq OS_{\Phi}(s_1s_2;\iota)$. ∎

In what follows, we give an example to illustrate that the property in Proposition 7 is essential.

Example 2. *Let $S := \{s_1,s_2,s_3,s_4,s_5,s_6\}$ be the semigroup with multiplication rules defined by the TABLE I.*

TABLE I
THE MULTIPLICATION TABLE ON S

$\cdot$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_2	s_3	s_3	s_5	s_6
s_2	s_2	s_2	s_2	s_2	s_2	s_6
s_3	s_3	s_2	s_3	s_3	s_5	s_6
s_4	s_3	s_2	s_3	s_3	s_5	s_6
s_5	s_5	s_2	s_5	s_5	s_5	s_6
s_6	s_6	s_6	s_6	s_6	s_6	s_6

Define the membership grades of relationship between any two elements in S under the fuzzy relation Φ on S as the following:

$$\begin{pmatrix}1&0&0&0&0&0\\0&0&1&0&1&0\\0&0&1&0&1&0\\0&0&0&1&0&0\\0&0&1&0&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

Then it is easy to check that Φ is transitive and compatible. For $\iota = 0.9$, the successor classes of each elements in S with respect to 0.9-level under Φ are as follows:
$S_{\Phi}(s_1;0.9) := \{s_1\}$,
$S_{\Phi}(s_2;0.9) := \{s_3,s_5\}$,
$S_{\Phi}(s_3;0.9) := \{s_3,s_5\}$,
$S_{\Phi}(s_4;0.9) := \{s_4\}$,
$S_{\Phi}(s_5;0.9) := \{s_3,s_5\}$ *and*
$S_{\Phi}(s_6;0.9) := \{s_6\}$.
Hence the overlaps of successor classes of each elements in S with respect to 0.9-level under Φ are as follows:
$OS_{\Phi}(s_1;0.9) := \{s_1\}$,
$OS_{\Phi}(s_2;0.9) := \{s_2,s_3,s_5\}$,
$OS_{\Phi}(s_3;0.9) := \{s_2,s_3,s_5\}$,
$OS_{\Phi}(s_4;0.9) := \{s_4\}$,

$OS_{\Phi}(s_5; 0.9) := \{s_2, s_3, s_5\}$ *and*
$OS_{\Phi}(s_6; 0.9) := \{s_6\}$.
Here it is straightforward to check that for all $s, s' \in S$,

$$(OS_{\Phi}(s; 0.9))(OS_{\Phi}(s'; 0.9)) \subseteq OS_{\Phi}(ss'; 0.9).$$

The next example show that

$$(OS_{\Phi}(s; \iota))(OS_{\Phi}(s'; \iota)) = OS_{\Phi}(ss'; \iota)$$

for $\iota \in [0, 1]$.

Example 3. *Let* $S := \{s_1, s_2, s_3, s_4, s_5, s_6\}$ *be the semigroup with multiplication rules defined by the TABLE II.*

TABLE II
THE MULTIPLICATION TABLE ON S

$\cdot$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_1	s_1	s_1	s_1	s_6
s_2	s_1	s_2	s_2	s_2	s_5	s_6
s_3	s_1	s_2	s_3	s_2	s_5	s_6
s_4	s_1	s_2	s_2	s_4	s_5	s_6
s_5	s_1	s_5	s_5	s_5	s_5	s_6
s_6	s_6	s_6	s_6	s_6	s_6	s_6

Define the membership grades of relationship between any two elements in S *under the fuzzy relation* Φ *on* S *as the following.*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then it is easy to verify that Φ *is transitive and compatible. For* $\iota = 0.9$*, the successor classes of each elements in* S *with respect to* 0.9*-level under* Φ *are as follows:*
$S_{\Phi}(s_1; 0.9) := \{s_5\}$,
$S_{\Phi}(s_2; 0.9) := \{s_2, s_3, s_4\}$,
$S_{\Phi}(s_3; 0.9) := \{s_2, s_3, s_4\}$,
$S_{\Phi}(s_4; 0.9) := \{s_2, s_3, s_4\}$,
$S_{\Phi}(s_5; 0.9) := \{s_5\}$ *and*
$S_{\Phi}(s_6; 0.9) := \{s_6\}$.
Hence overlaps of successor classes of each elements in S *with respect to* 0.9*-level under* Φ *are as follows:*
$OS_{\Phi}(s_1; 0.9) := \{s_1, s_5\}$,
$OS_{\Phi}(s_2; 0.9) := \{s_2, s_3, s_4\}$,
$OS_{\Phi}(s_3; 0.9) := \{s_2, s_3, s_4\}$,
$OS_{\Phi}(s_4; 0.9) := \{s_2, s_3, s_4\}$,
$OS_{\Phi}(s_5; 0.9) := \{s_1, s_5\}$ *and*
$OS_{\Phi}(s_6; 0.9) := \{s_6\}$.
Here it is straightforward to check that for all $s, s' \in S$,

$$(OS_{\Phi}(s; 0.9))(OS_{\Phi}(s'; 0.9)) = OS_{\Phi}(ss'; 0.9).$$

Based on this point, the property can be considered as a special case of Proposition 7. This example leads to the following definition.

Definition 13. Let $(S, \mathcal{OS}_{\Phi}(S; \iota))$ be an $\mathcal{OS}_{\Phi}(S; \iota)$-approximation space type TCF. The collection $\mathcal{OS}_{\Phi}(S; \iota)$ is called *complete induced by* Φ (briefly, Φ*-complete*) if for all $s_1, s_2 \in S$,

$$(OS_{\Phi}(s_1; \iota))(OS_{\Phi}(s_2; \iota)) = OS_{\Phi}(s_1 s_2; \iota).$$

Definition 14. Let $(S, \mathcal{OS}_{\Phi}(S; \iota))$ be an $\mathcal{OS}_{\Phi}(S; \iota)$-approximation space type TCF. If $\mathcal{OS}_{\Phi}(S; \iota)$ is complete induced by Φ, then Φ is called a *complete fuzzy relation.* $(S, \mathcal{OS}_{\Phi}(S; \iota))$ is called an $\mathcal{OS}_{\Phi}(S; \iota)$*-approximation space type CPF* if Φ is complete.

Proposition 8. *If* $(S, \mathcal{OS}_{\Phi}(S; \iota))$ *is an* $\mathcal{OS}_{\Phi}(S; \iota)$*-approximation space type TCF, then*

$$(\overline{\Phi}(X; \iota))(\overline{\Phi}(Y; \iota)) \subseteq \overline{\Phi}(XY; \iota),$$

for every non-empty subsets X, Y *of* S.

Proof: Let X and Y be two non-empty subsets of S. Suppose that $s_1 \in (\overline{\Phi}(X; \iota))(\overline{\Phi}(Y; \iota))$. Then there exists $s_2 \in \overline{\Phi}(X; \iota)$ and exists $s_3 \in \overline{\Phi}(Y; \iota)$ such that $s_1 = s_2 s_3$. Thus we have $OS_{\Phi}(s_2; \iota) \cap X \neq \emptyset$ and $OS_{\Phi}(s_3; \iota) \cap Y \neq \emptyset$. Then there exist $s_4, s_5 \in S$ such that $s_4 \in OS_{\Phi}(s_2; \iota) \cap X$ and $s_5 \in OS_{\Phi}(s_3; \iota) \cap Y$. From Proposition 7, it follows that $s_4 s_5 \in (OS_{\Phi}(s_2; \iota))(OS_{\Phi}(s_3; \iota)) \subseteq OS_{\Phi}(s_2 s_3; \iota)$ and $s_4 s_5 \in XY$. Thus $OS_{\Phi}(s_2 s_3; \iota) \cap XY \neq \emptyset$, which yields $s_1 = s_2 s_3 \in \overline{\Phi}(XY; \iota)$. Therefore, $(\overline{\Phi}(X; \iota))(\overline{\Phi}(Y; \iota)) \subseteq \overline{\Phi}(XY; \iota)$. ■

Proposition 9. *If* $(S, \mathcal{OS}_{\Phi}(S; \iota))$ *is an* $\mathcal{OS}_{\Phi}(S; \iota)$*-approximation space type CPF, then*

$$(\underline{\Phi}(X; \iota))(\underline{\Phi}(Y; \iota)) \subseteq \underline{\Phi}(XY; \iota),$$

for every non-empty subsets X, Y *of* S.

Proof: Let X and Y be two non-empty subsets of S. Suppose that $s_1 \in (\underline{\Phi}(X; \iota))(\underline{\Phi}(Y; \iota))$. Then there exist $s_2 \in \underline{\Phi}(X; \iota)$ and $s_3 \in \underline{\Phi}(Y; \iota)$ such that $s_1 = s_2 s_3$, and so $OS_{\Phi}(s_2; \iota) \subseteq X$ and $OS_{\Phi}(s_3; \iota) \subseteq Y$. Since Φ is complete, we get $OS_{\Phi}(s_2 s_3; \iota) = OS_{\Phi}(s_2; \iota) OS_{\Phi}(s_3; \iota) \subseteq XY$. Thus $OS_{\Phi}(s_2 s_3; \iota) \subseteq XY$. Hence $s_1 = s_2 s_3 \in \underline{\Phi}(XY; \iota)$. Therefore, $(\underline{\Phi}(X; \iota))(\underline{\Phi}(Y; \iota)) \subseteq \underline{\Phi}(XY; \iota)$. ■

In the following, a rough set approach to semigroups will be introduced. Now we consider the following example.

Example 4. *According to Example 3, we suppose that* $X := \{s_1, s_3, s_5, s_6\}$ *is a subset of* S*. Then we have that* $\overline{\Phi}(X; \iota) = S$ *and* $\underline{\Phi}(X; \iota) := \{s_1, s_5, s_6\}$*. Here it is easy to verify that* $\overline{\Phi}(X; \iota)$ *and* $\underline{\Phi}(X; \iota)$ *are subsemigroups, ideals and completely prime ideals of* S*. Moreover, we also have* $\Phi_{bnd}(X; \iota)$ *is a non-empty set. Existences of subsemigroups, ideals and completely prime ideals of* S *under transitive and compatible fuzzy relations in this example lead to Definition 15 as the following .*

Definition 15. Let $(S, \mathcal{OS}_{\Phi}(S; \iota))$ be an $\mathcal{OS}_{\Phi}(S; \iota)$-approximation space type TCF and let X be a non-empty subset of S. The non-empty $\mathcal{OS}_{\Phi}(S; \iota)$-upper approximation $\overline{\Phi}(X; \iota)$ of X in $(S, \mathcal{OS}_{\Phi}(S; \iota))$ is called an $\mathcal{OS}_{\Phi}(S; \iota)$*-upper approximation semigroup* if it is a subsemigroup of S. The non-empty $\mathcal{OS}_{\Phi}(S; \iota)$-lower approximation $\underline{\Phi}(X; \iota)$ of X in $(S, \mathcal{OS}_{\Phi}(S; \iota))$ is called a $\mathcal{OS}_{\Phi}(S; \iota)$*-lower approximation semigroup* if it is a subsemigroup of S. The non-empty $\mathcal{OS}_{\Phi}(S; \iota)$-rough set $\Phi(X; \iota)$ of X in $(S, \mathcal{OS}_{\Phi}(S; \iota))$ is called a $\mathcal{OS}_{\Phi}(S; \iota)$*-rough semigroup* if $\overline{\Phi}(X; \iota)$ is an $\mathcal{OS}_{\Phi}(S; \iota)$-upper approximation semigroup and $\underline{\Phi}(X; \iota)$ is a $\mathcal{OS}_{\Phi}(S; \iota)$-lower approximation semigroup. Similarly, we can define $\mathcal{OS}_{\Phi}(S; \iota)$-rough (completely prime) ideals.

Theorem 1. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type TCF. If X is a subsemigroup of S, then $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation semigroup.*

Proof: Suppose that X is a subsemigroup of S. Then, $XX \subseteq X$. By Proposition 3 (3), we obtain that

$$\emptyset \neq X \subseteq \overline{\Phi}(X;\iota).$$

Hence we get that $\overline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(S;\iota)$-upper approximation. From Proposition 3 (6), it follows that $\overline{\Phi}(XX;\iota) \subseteq \overline{\Phi}(X;\iota)$. By Proposition 8, we obtain that

$$(\overline{\Phi}(X;\iota))(\overline{\Phi}(X;\iota)) \subseteq \overline{\Phi}(XX;\iota) \subseteq \overline{\Phi}(X;\iota).$$

Thus $\overline{\Phi}(X;\iota)$ is a subsemigroup of S. Therefore, $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation semigroup. ■

Theorem 2. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is a subsemigroup of S with $\underline{\Phi}(X;\iota) \neq \emptyset$, then $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation semigroup.*

Proof: Suppose that X is a subsemigroup of S. Then, $XX \subseteq X$. Obviously, $\underline{\Phi}(X;\iota)$ is a non-empty $\mathcal{OS}_\Phi(S;\iota)$-lower approximation. From Proposition 3 (6), it follows that $\underline{\Phi}(XX;\iota) \subseteq \underline{\Phi}(X;\iota)$. By Proposition 9, we get that

$$(\underline{\Phi}(X;\iota))(\underline{\Phi}(X;\iota)) \subseteq \underline{\Phi}(XX;\iota) \subseteq \underline{\Phi}(X;\iota).$$

Thus $\underline{\Phi}(X;\iota)$ is a subsemigroup of S. Therefore, $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation semigroup. ■

The following corollary is immediate consequences of Proposition 4, Theorem 1 and Theorem 2.

Corollary 1. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is a subsemigroup of S over non-empty interior set, then $\Phi(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-rough semigroup.*

Observe that, in Corollary 1, the converse is not true in general. We present an example as the following.

Example 5. *According to Example 3, suppose that $X := \{s_1, s_3, s_4, s_5\}$ is a subset of S, then we have $\overline{\Phi}(X;0.9) := \{s_1, s_2, s_3, s_4, s_5\}$ and $\underline{\Phi}(X;0.9) := \{s_1, s_5\}$. Thus we see that $\Phi_{bnd}(X;0.9) \neq \emptyset$. Hence it is straightforward to check that $\overline{\Phi}(X;0.9)$ is an $\mathcal{OS}_\Phi(S;0.9)$-upper approximation semigroup and $\underline{\Phi}(X;0.9)$ is a $\mathcal{OS}_\Phi(S;0.9)$-lower approximation semigroup. However, X is not a subsemigroup of S. Consequently, $\Phi(X;0.9)$ is a $\mathcal{OS}_\Phi(S;0.9)$-rough semigroup, but X is not a subsemigroup of S.*

Theorem 3. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type TCF. If X is an ideal of S, then $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation ideal.*

Proof: Suppose that X is an ideal of S. Then we have $SX \subseteq X$. From Proposition 3 (6), it follows that $\overline{\Phi}(SX;\iota) \subseteq \overline{\Phi}(X;\iota)$. By Proposition 3 (1), we obtain that $\overline{\Phi}(S;\iota) = S$. From Proposition 8, it follows that

$$S(\overline{\Phi}(X;\iota)) = (\overline{\Phi}(S;\iota))(\overline{\Phi}(X;\iota)) \subseteq \overline{\Phi}(SX;\iota) \subseteq \overline{\Phi}(X;\iota).$$

Hence $\overline{\Phi}(X;\iota)$ is a left ideal of S.

Similarly, we can prove that $\overline{\Phi}(X;\iota)$ is a right ideal of S. Therefore we have $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation ideal. ■

Theorem 4. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is an ideal of S with $\underline{\Phi}(X;\iota) \neq \emptyset$, then $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation ideal.*

Proof: Suppose that X is an ideal of S. Then we have $SX \subseteq X$. From Proposition 3 (6), it follows that $\underline{\Phi}(SX;\iota) \subseteq \underline{\Phi}(X;\iota)$. By Proposition 3 (1), we obtain that $\underline{\Phi}(S;\iota) = S$. From Proposition 9, it follows that

$$S(\underline{\Phi}(X;\iota)) = (\underline{\Phi}(S;\iota))(\underline{\Phi}(X;\iota)) \subseteq \underline{\Phi}(SX;\iota) \subseteq \underline{\Phi}(X;\iota).$$

Thus $\underline{\Phi}(X;\iota)$ is a left ideal of S.

Similarly, we can prove that $\underline{\Phi}(X;\iota)$ is a right ideal of S. Thus $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation ideal. ■

The following corollary is immediate consequences of Proposition 4, Theorem 3 and Theorem 4.

Corollary 2. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is an ideal of S over non-empty interior set, then $\Phi(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-rough ideal.*

Observe that, in Corollary 2, the converse is not true in general. We present an example as the following.

Example 6. *According to Example 3, we suppose that $X := \{s_1, s_4, s_5, s_6\}$ is a subset of S, then we have $\overline{\Phi}(X;0.9) = S$ and $\underline{\Phi}(X;0.9) := \{s_1, s_5, s_6\}$. Thus we see that $\Phi_{bnd}(X;0.9) \neq \emptyset$. Obviously, $\overline{\Phi}(X;0.9)$ is an $\mathcal{OS}_\Phi(S;0.9)$-upper approximation ideal, and it is straightforward to check that $\underline{\Phi}(X;0.9)$ is a $\mathcal{OS}_\Phi(S;0.9)$-lower approximation ideal. However, X is not an ideal of S. Consequently, $\Phi(X;0.9)$ is a $\mathcal{OS}_\Phi(S;0.9)$-rough ideal, but X is not an ideal of S.*

Theorem 5. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is a completely prime ideal of S, then $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation completely prime ideal.*

Proof: We prove that $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation completely prime ideal. In fact, since X is an ideal of S, by Theorem 3, we have that $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation ideal. Let $s_1, s_2 \in S$ such that $s_1s_2 \in \overline{\Phi}(X;\iota)$. Then, by the Φ-complete property of $\mathcal{OS}_\Phi(S;\iota)$, we get that

$$(OS_\Phi(s_1;\iota))(OS_\Phi(s_2;\iota)) \cap X = OS_\Phi(s_1s_2;\iota) \cap X \neq \emptyset.$$

Thus there exist $s_3 \in OS_\Phi(s_1;\iota)$ and $s_4 \in OS_\Phi(s_2;\iota)$ such that $s_3s_4 \in X$. Since X is a completely prime ideal, we have $s_3 \in X$ or $s_4 \in X$. Hence we have $OS_\Phi(s_1;\iota) \cap X \neq \emptyset$ or $OS_\Phi(s_2;\iota) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Phi}(X;\iota)$ or $s_2 \in \overline{\Phi}(X;\iota)$. Therefore, $\overline{\Phi}(X;\iota)$ is a completely prime ideal of S. As a consequence, $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation completely prime ideal. ■

Theorem 6. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is a completely prime ideal of S with $\underline{\Phi}(X;\iota) \neq \emptyset$, then $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation completely prime ideal.*

Proof: Since X is an ideal of S, by Theorem 4, $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation ideal. Let $s_1, s_2 \in S$

such that $s_1 s_2 \in \underline{\Phi}(X;\iota)$. Since Φ is complete, we have

$$(OS_\Phi(s_1;\iota))(OS_\Phi(s_2;\iota)) = OS_\Phi(s_1 s_2;\iota) \subseteq X.$$

Suppose that $s_1 \notin \underline{\Phi}(X;\iota)$. Then, $OS_\Phi(s_1;\iota)$ is not a subset of X. Thus there exists $s_3 \in S$ such that $s_3 \in OS_\Phi(s_1;\iota)$ but $s_3 \notin X$. For each $s_4 \in OS_\Phi(s_2;\iota)$,

$$s_3 s_4 \in (OS_\Phi(s_1;\iota))(OS_\Phi(s_2;\iota)) \subseteq X.$$

Whence $s_3 s_4 \in X$. Since X is a completely prime ideal and $s_3 \notin X$, we have $s_4 \in X$. Thus $OS_\Phi(s_2;\iota) \subseteq X$, which yields $s_2 \in \underline{\Phi}(X;\iota)$. Hence we get $\underline{\Phi}(X;\iota)$ is a completely prime ideal of S. Therefore, $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation completely prime ideal. ■

The following corollary is immediate consequences of Proposition 4, Theorem 5 and Theorem 6.

Corollary 3. *Let $(S, \mathcal{OS}_\Phi(S;\iota))$ be an $\mathcal{OS}_\Phi(S;\iota)$-approximation space type CPF. If X is a completely prime ideal of S over non-empty interior set, then $\Phi(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-rough completely prime.*

Observe that, in Corollary 3, the converse is not true in general. We present an example as the following.

Example 7. *According to Example 3, we suppose that $X := \{s_1, s_2, s_5, s_6\}$ is a subset of S, then we have $\overline{\Phi}(X;0.9) = S$ and $\underline{\Phi}(X;0.9) := \{s_1, s_5, s_6\}$. Thus we see that $\Phi_{bnd}(X;0.9) \neq \emptyset$. Obviously, $\overline{\Phi}(X;0.9)$ is an $\mathcal{OS}_\Phi(S;0.9)$-upper approximation completely prime ideal, and it is straightforward to check that $\underline{\Phi}(X;0.9)$ is a $\mathcal{OS}_\Phi(S;0.9)$-lower approximation completely prime ideal. Here we can verify that X is an ideal of S, but it is not a completely prime ideal of S since $s_3 s_4 = s_2 \in X$ but $s_3 \notin X$ and $s_4 \notin X$. As a consequence, $\Phi(X;0.9)$ is a $\mathcal{OS}_\Phi(S;0.9)$-rough completely prime ideal, but X is not a completely prime ideal of S.*

V. Homomorphic Images of Roughness in Semigroups

In this section we investigate relationships between rough semigroups (resp. rough ideals, rough completely prime ideals) and their homomorphic images. Throughout this section, T denotes a semigroup.

Proposition 10. *Let f be an epimorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. Then the following statements hold.*

(1) For all $s_1, s_2 \in S$, $s_1 \in OS_\Phi(s_2;\iota)$ if and only if $f(s_1) \in OS_\Omega(f(s_2);\iota)$.
(2) $f(\overline{\Phi}(X;\iota)) = \overline{\Omega}(f(X);\iota)$ for every non-empty subset X of S.
(3) $f(\underline{\Phi}(X;\iota)) \subseteq \underline{\Omega}(f(X);\iota)$ for every non-empty subset X of S.
(4) If f is injective, then $f(\underline{\Phi}(X;\iota)) = \underline{\Omega}(f(X);\iota)$ for every non-empty subset X of S.
(5) If Ω is transitive and compatible, then Φ is transitive and compatible.

Proof: (1) Let $s_1, s_2 \in S$ be two elements in S. Suppose that $s_1 \in OS_\Phi(s_2;\iota)$. Then we have $f(s_1), f(s_2) \in T$ and $S_\Phi(s_1;\iota) \cap S_\Phi(s_2;\iota) \neq \emptyset$. Thus there exists $s_3 \in S$ such that $s_3 \in S_\Phi(s_1;\iota) \cap S_\Phi(s_2;\iota)$. Hence $\Phi(s_1, s_3) \geq \iota$ and $\Phi(s_2, s_3) \geq \iota$. By the assumption, we obtain that

$$\Omega(f(s_1), f(s_3)) = \Phi(s_1, s_3) \geq \iota$$

and

$$\Omega(f(s_2), f(s_3)) = \Phi(s_2, s_3) \geq \iota.$$

Thus we get $f(s_3) \in S_\Omega(f(s_1);\iota) \cap S_\Omega(f(s_2);\iota)$. Hence we have $S_\Omega(f(s_1);\iota) \cap S_\Omega(f(s_2);\iota) \neq \emptyset$. Therefore we get $f(s_1) \in OS_\Omega(f(s_2);\iota)$.

Conversely, it is easy to verify that $s_1 \in OS_\Phi(s_2;\iota)$ whenever $f(s_1) \in OS_\Omega(f(s_2);\iota)$ for all $s_1, s_2 \in S$.

(2) Let X be a non-empty subset of S. We verify firstly that $f(\overline{\Phi}(X;\iota)) = \overline{\Omega}(f(X);\iota)$. Suppose that $t_1 \in f(\overline{\Phi}(X;\iota))$. Then there exists $s_1 \in \overline{\Phi}(X;\iota)$ such that $f(s_1) = t_1$. Therefore we have $OS_\Phi(s_1;\iota) \cap X \neq \emptyset$. Thus there exists $s_2 \in S$ such that $s_2 \in OS_\Phi(s_1;\iota)$ and $s_2 \in X$. By the argument (1), we obtain that $f(s_2) \in OS_\Omega(f(s_1);\iota)$ and $f(s_2) \in f(X)$. Then, $OS_\Omega(f(s_1);\iota) \cap f(X) \neq \emptyset$, and so $t_1 = f(s_1) \in \overline{\Omega}(f(X);\iota)$. Thus $f(\overline{\Phi}(X;\iota)) \subseteq \overline{\Omega}(f(X);\iota)$.

On the other hand, let $t_2 \in \overline{\Omega}(f(X);\iota)$. Then there exists $s_3 \in S$ such that $f(s_3) = t_2$, and so $OS_\Omega(f(s_3);\iota) \cap f(X) \neq \emptyset$. Thus there exists $s_4 \in X$ such that $f(s_4) \in f(X)$ and $f(s_4) \in OS_\Omega(f(s_3);\iota)$. By the argument (1), we get that $s_4 \in OS_\Phi(s_3;\iota)$, and so we have $OS_\Phi(s_3;\iota) \cap X \neq \emptyset$. Hence $s_3 \in \overline{\Phi}(X;\iota)$, and so $t_2 = f(s_3) \in f(\overline{\Phi}(X;\iota))$. Thus we get $\overline{\Omega}(f(X);\iota) \subseteq f(\overline{\Phi}(X;\iota))$. This implies that $f(\overline{\Phi}(X;\iota)) = \overline{\Omega}(f(X);\iota)$.

(3) Let X be a non-empty subset of S. Let $t_1 \in f(\underline{\Phi}(X;\iota))$. Then there exists $s_1 \in \underline{\Phi}(X;\iota)$ such that $f(s_1) = t_1$. Thus $OS_\Phi(s_1;\iota) \subseteq X$. We shall prove that $OS_\Omega(t_1;\iota) \subseteq f(X)$. Let $t_2 \in OS_\Omega(t_1;\iota)$. Then there exists $s_2 \in S$ such that $f(s_2) = t_2$. Thus we have $f(s_2) \in OS_\Omega(f(s_1);\iota)$. By the argument (1), we obtain that $s_2 \in OS_\Phi(s_1;\iota)$, and so $s_2 \in X$. Hence we have $t_2 = f(s_2) \in f(X)$, and thus, $OS_\Omega(t_1;\iota) \subseteq f(X)$. Therefore we have $t_1 \in \underline{\Omega}(f(X);\iota)$. As a consequence, $f(\underline{\Phi}(X;\iota)) \subseteq \underline{\Omega}(f(X);\iota)$.

(4) Let X be a non-empty subset of S. We only need to prove that $\underline{\Omega}(f(X);\iota) \subseteq f(\underline{\Phi}(X;\iota))$. Let $t_1 \in \underline{\Omega}(f(X);\iota)$. Then there exists $s_1 \in S$ such that $f(s_1) = t_1$. Thus we have $OS_\Omega(f(s_1);\iota) \subseteq f(X)$. We shall show that $OS_\Phi(s_1;\iota) \subseteq X$. Let $s_2 \in OS_\Phi(s_1;\iota)$. Then, by the argument (1), we have $f(s_2) \in OS_\Omega(f(s_1);\iota)$. Hence $f(s_2) \in f(X)$. Thus there exists $s_3 \in X$ such that $f(s_3) = f(s_2)$. By the assumption, we have $s_2 \in X$, and so $OS_\Phi(s_1;\iota) \subseteq X$. Hence $s_1 \in \underline{\Phi}(X;\iota)$, and so $t_1 = f(s_1) \in f(\underline{\Phi}(X;\iota))$. Thus $\underline{\Omega}(f(X);\iota) \subseteq f(\underline{\Phi}(X;\iota))$.

By the argument (3), we get $f(\underline{\Phi}(X;\iota)) \subseteq \underline{\Omega}(f(X);\iota)$. Consequently, $f(\underline{\Phi}(X;\iota)) = \underline{\Omega}(f(X);\iota)$.

(5) The proof is straightforward, so we omit it. ■

Proposition 11. *Let f be an epimorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If Ω is complete, then Φ is complete.*

Proof: Let s_1, s_2 be two elements in S. Suppose that $s_3 \in OS_\Phi(s_1 s_2;\iota)$. Then, by Proposition 10 (1), we get that $f(s_3) \in OS_\Omega(f(s_1 s_2);\iota)$. Since f is a homomorphism and

Ω is complete, we have

$$\begin{aligned} f(s_3) \in & OS_\Omega(f(s_1s_2);\iota) \\ = & OS_\Omega(f(s_1)f(s_2);\iota) \\ = & (OS_\Omega(f(s_1);\iota))(OS_\Omega(f(s_2);\iota)). \end{aligned}$$

Thus there exists $t_1 \in OS_\Omega(f(s_1);\iota)$ and exists $t_2 \in OS_\Omega(f(s_2);\iota)$ such that $f(s_3) = t_1t_2$. Since f is surjective, there exist $s_4, s_5 \in S$ such that $f(s_4) = t_1$ and $f(s_5) = t_2$. From

$$f(s_4)f(s_5) = f(s_3) \in (OS_\Omega(f(s_1);\iota))(OS_\Omega(f(s_2);\iota)),$$

it follows that $f(s_4) \in OS_\Omega(f(s_1);\iota)$ and $f(s_5) \in OS_\Omega(f(s_2);\iota)$. By Proposition 10 (1), we obtain that $s_4 \in OS_\Phi(s_1;\iota)$ and $s_5 \in OS_\Phi(s_2;\iota)$. Since f is a homomorphism, we have $f(s_3) = f(s_4)f(s_5) = f(s_4s_5)$. Since f is injective, we get $s_3 = s_4s_5$. Thus we get $s_3 \in OS_\Phi(s_1;\iota)OS_\Phi(s_2;\iota)$. Therefore we get that $OS_\Phi(s_1s_2;\iota) \subseteq OS_\Phi(s_1;\iota)OS_\Phi(s_2;\iota)$.

On the other hand, by Proposition 7 and Proposition 10 (5), we obtain that $OS_\Phi(s_1;\iota)OS_\Phi(s_2;\iota) \subseteq OS_\Phi(s_1s_2;\iota)$. Thus $OS_\Phi(s_1;\iota)OS_\Phi(s_2;\iota) = OS_\Phi(s_1s_2;\iota)$. Hence $\mathcal{OS}_\Phi(S;\iota)$ is Φ-complete. Therefore, Φ is complete. ■

Theorem 7. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$ type TCF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation semigroup if and only if $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_\Omega(T;\iota)$-upper approximation semigroup.*

Proof: Suppose that $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation semigroup. Then, by Proposition 10 (2), we obtain that

$$\begin{aligned} (\overline{\Omega}(f(X);\iota))(\overline{\Omega}(f(X);\iota)) = & (f(\overline{\Phi}(X;\iota)))(f(\overline{\Phi}(X;\iota))) \\ = & f((\overline{\Phi}(X;\iota))(\overline{\Phi}(X;\iota))) \\ \subseteq & f(\overline{\Phi}(X;\iota)) \\ = & \overline{\Omega}(f(X);\iota). \end{aligned}$$

Hence we get $\overline{\Omega}(f(X);\iota)$ is a subsemigroup of T. Thus $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_\Omega(T;\iota)$-upper approximation semigroup.

Conversely, we suppose that $s_1 \in (\overline{\Phi}(X;\iota))(\overline{\Phi}(X;\iota))$. From Proposition 10 (2), it follows that

$$\begin{aligned} f(s_1) \in & f((\overline{\Phi}(X;\iota))(\overline{\Phi}(X;\iota))) \\ = & (f(\overline{\Phi}(X;\iota)))(f(\overline{\Phi}(X;\iota))) \\ = & (\overline{\Omega}(f(X);\iota))(\overline{\Omega}(f(X);\iota)) \\ \subseteq & \overline{\Omega}(f(X);\iota) \\ = & f(\overline{\Phi}(X;\iota)). \end{aligned}$$

Thus there exists $s_2 \in \overline{\Phi}(X;\iota)$ such that $f(s_1) = f(s_2)$. Hence we have $OS_\Phi(s_2;\iota) \cap X \neq \emptyset$. Since f is injective, we have $s_1 = s_2$. Thus we get $OS_\Phi(s_1;\iota) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Phi}(X;\iota)$. Hence $(\overline{\Phi}(X;\iota))(\overline{\Phi}(X;\iota)) \subseteq \overline{\Phi}(X;\iota)$. Thus $\overline{\Phi}(X;\iota)$ is a subsemigroup of S. Therefore, $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation semigroup. ■

Theorem 8. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$ type TCF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation semigroup if and only if $\underline{\Omega}(f(X);\iota)$ is a $\mathcal{OS}_\Omega(T;\iota)$-lower approximation semigroup.*

Proof: By Proposition 10 (4) and using the similar method in the proof of Theorem 7, we can prove that the statement holds. ■

The following corollary is immediate consequences of Theorems 7 and 8.

Corollary 4. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$ type TCF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\Phi(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-rough semigroup if and only if $\Omega(f(X);\iota)$ is a $\mathcal{OS}_\Omega(T;\iota)$-rough semigroup.*

Theorem 9. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$ type TCF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation ideal if and only if $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_\Omega(T;\iota)$-upper approximation ideal.*

Proof: Suppose that $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation ideal. Then we have $S\overline{\Phi}(X;\iota) \subseteq \overline{\Phi}(X;\iota)$. Whence we have $f(S\overline{\Phi}(X;\iota)) \subseteq f(\overline{\Phi}(X;\iota))$. By Proposition 10 (2), we obtain that

$$T\overline{\Omega}(f(X);\iota) = f(S\overline{\Phi}(X;\iota)) \subseteq f(\overline{\Phi}(X;\iota)) = \overline{\Omega}(f(X);\iota).$$

Hence $\overline{\Omega}(f(X);\iota)$ is a left ideal of T. Similarly, we can prove that $\overline{\Omega}(f(X);\iota)$ is a right ideal of T. Thus $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_\Omega(T;\iota)$-upper approximation ideal.

Conversely, we suppose that $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_\Omega(T;\iota)$-upper approximation ideal. Then we have $T\overline{\Omega}(f(X);\iota) \subseteq \overline{\Omega}(f(X);\iota)$. Let now $s_1 \in S\overline{\Phi}(X;\iota)$. From Proposition 10 (2), it follows that

$$\begin{aligned} f(s_1) \in & f(S\overline{\Phi}(X;\iota)) \\ = & T\overline{\Omega}(f(X);\iota) \\ \subseteq & \overline{\Omega}(f(X);\iota) \\ = & f(\overline{\Phi}(X;\iota)). \end{aligned}$$

Thus there exists $s_2 \in \overline{\Phi}(X;\iota)$ such that $f(s_1) = f(s_2)$, and so $OS_\Phi(s_2;\iota) \cap X \neq \emptyset$. Since f is injective, we have $s_1 = s_2$. Hence $OS_\Phi(s_1;\iota) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Phi}(X;\iota)$. Thus $S\overline{\Phi}(X;\iota) \subseteq \overline{\Phi}(X;\iota)$. Whence $\overline{\Phi}(X;\iota)$ is a left ideal of S. Similarly, we can prove that $\overline{\Phi}(X;\iota)$ is a right ideal of S. Therefore, $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_\Phi(S;\iota)$-upper approximation ideal. ■

Theorem 10. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Phi(S;\iota))$ to T in $(T, \mathcal{CS}_\Omega(T;\iota))$ type TCF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_\Phi(S;\iota)$-lower approximation ideal if and only if $\underline{\Omega}(f(X);\iota)$ is a $\mathcal{OS}_\Omega(T;\iota)$-lower approximation ideal.*

Proof: By Proposition 10 (4) and using the similar method in the proof of Theorem 9, we can prove that the statement holds. ■

The following corollary is immediate consequences of Theorems 9 and 10.

Corollary 5. *Let f be an isomorphism from S in $(S, \mathcal{CS}_{\Phi}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Omega}(T;\iota))$ type TCF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\Phi(X;\iota)$ is a $\mathcal{OS}_{\Phi}(S;\iota)$-rough ideal if and only if $\Omega(f(X);\iota)$ is a $\mathcal{OS}_{\Omega}(T;\iota)$-rough ideal.*

Theorem 11. *Let f be an isomorphism from S in $(S, \mathcal{CS}_{\Phi}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Omega}(T;\iota))$ type CPF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_{\Phi}(S;\iota)$-upper approximation completely prime ideal if and only if $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_{\Omega}(T;\iota)$-upper approximation completely prime ideal.*

Proof: Assume that $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_{\Phi}(S;\iota)$-upper approximation completely prime ideal. Let $t_1, t_2 \in T$ be such that $t_1t_2 \in \overline{\Omega}(f(X);\iota)$. Thus there exist $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$. Hence $OS_{\Omega}(f(s_1)f(s_2);\iota) \cap f(X) \neq \emptyset$. Since Ω is complete, we have

$$\begin{aligned}&(OS_{\Omega}(f(s_1);\iota))(OS_{\Omega}(f(s_2);\iota)) \cap f(X)\\ &= OS_{\Omega}(f(s_1)f(s_2);\iota) \cap f(X) \neq \emptyset.\end{aligned}$$

Then there exist $f(s_3) \in OS_{\Omega}(f(s_1);\iota)$ and $f(s_4) \in OS_{\Omega}(f(s_2);\iota)$ such that $f(s_3)f(s_4) \in f(X)$, and so $f(s_3s_4) \in f(X)$. Then there exists $s_5 \in X$ such that $f(s_3s_4) = f(s_5)$. Since f is injective, we have $s_3s_4 = s_5$. By Proposition 10 (1), we obtain that $s_3 \in OS_{\Phi}(s_1;\iota)$ and $s_4 \in OS_{\Phi}(s_2;\iota)$. From Proposition 7 and Proposition 10 (5), we get that $s_5 = s_3s_4 \in OS_{\Phi}(s_1s_2;\iota)$. Thus we have that $OS_{\Phi}(s_1s_2;\iota) \cap X \neq \emptyset$, and so $s_1s_2 \in \overline{\Phi}(X;\iota)$. Since $\overline{\Phi}(X;\iota)$ is a completely prime ideal of S, we have $s_1 \in \overline{\Phi}(X;\iota)$ or $s_2 \in \overline{\Phi}(X;\iota)$. Hence we have $f(s_1) \in f(\overline{\Phi}(X;\iota))$ or $f(s_2) \in f(\overline{\Phi}(X;\iota))$. From Proposition 10 (2), we get $f(s_1) \in \overline{\Omega}(f(X);\iota)$ or $f(s_2) \in \overline{\Omega}(f(X);\iota)$, which yields $t_1 \in \overline{\Omega}(f(X);\iota)$ or $t_2 \in \overline{\Omega}(f(X);\iota)$. Thus $\overline{\Omega}(f(X);\iota)$ is a completely prime ideal of T. Therefore, $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_{\Omega}(T;\iota)$-upper approximation completely prime ideal.

Conversely, we suppose that $\overline{\Omega}(f(X);\iota)$ is an $\mathcal{OS}_{\Phi}(S;\iota)$-upper approximation completely prime ideal. Let s_6, s_7 be elements in S such that $s_6s_7 \in \overline{\Phi}(X;\iota)$. Then, $f(s_6s_7) \in f(\overline{\Phi}(X;\iota))$. By Proposition 10 (2), we obtain that

$$f(s_6)f(s_7) = f(s_6s_7) \in f(\overline{\Phi}(X;\iota)) = \overline{\Omega}(f(X);\iota).$$

Thus $f(s_6) \in \overline{\Omega}(f(X);\iota)$ or $f(s_7) \in \overline{\Omega}(f(X);\iota)$. Now we consider the following two cases.

Case 1. If $f(s_6) \in \overline{\Omega}(f(X);\iota)$, then we have $f(s_6) \in f(\overline{\Phi}(X;\iota))$ since Proposition 10 (2). Thus there exists $s_8 \in \overline{\Phi}(X;\iota)$ such that $f(s_6) = f(s_8)$. Whence $OS_{\Phi}(s_8;\iota) \cap X \neq \emptyset$. Since f is injective, we have $s_6 = s_8$. Hence we get $OS_{\Phi}(s_6;\iota) \cap X \neq \emptyset$, and so $s_6 \in \overline{\Phi}(X;\iota)$.

Case 2. If $f(s_7) \in \overline{\Omega}(f(X);\iota)$, then $s_7 \in \overline{\Phi}(X;\iota)$ since the proof is similar to that the case above.

As a consequence, $\overline{\Phi}(X;\iota)$ is an $\mathcal{OS}_{\Phi}(S;\iota)$-upper approximation completely prime ideal. ■

Theorem 12. *Let f be an isomorphism from S in $(S, \mathcal{CS}_{\Phi}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Omega}(T;\iota))$ type CPF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\underline{\Phi}(X;\iota)$ is a $\mathcal{OS}_{\Phi}(S;\iota)$-lower approximation completely prime ideal if and only if $\underline{\Omega}(f(X);\iota)$ is a $\mathcal{OS}_{\Omega}(T;\iota)$-lower approximation completely prime ideal.*

Proof: By Proposition 10 (4) and using the similar method in the proof of Theorem 11, we can prove that the statement holds. ■

The following corollary is immediate consequences of Theorems 11 and 12.

Corollary 6. *Let f be an isomorphism from S in $(S, \mathcal{CS}_{\Phi}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Omega}(T;\iota))$ type CPF, where Φ is defined by for all $s_1, s_2 \in S$, $\Phi(s_1, s_2) = \Omega(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\Phi(X;\iota)$ is a $\mathcal{OS}_{\Phi}(S;\iota)$-rough completely prime ideal if and only if $\Omega(f(X);\iota)$ is a $\mathcal{OS}_{\Omega}(T;\iota)$-rough completely prime ideal.*

VI. Conclusions

Under a serial fuzzy relation between two universes, we have proposed a novel rough set in an approximation space based on overlaps of successor classes with respect to level in a closed unit interval and gave a real-world example and proved some interesting properties. Based on this point, we defined a definition of a non-empty rough set and obtained a sufficient condition of such rough set. We introduced concepts of rough semigroups, rough ideals and rough completely prime ideals in approximation spaces under transitive and compatible fuzzy relations on semigroups and derived sufficient conditions of them. Also, we proved relationships between rough semigroups (resp. rough ideals and rough completely prime ideals) and their homomorphic images. The novel rough set in Section III can be used in a semigroup for approximation processings in terms of crisp sets.

Finally, we hope that new definitions and main results in this work may provide a powerful tool for assessment and decision problems in various fields with respect to information sciences, computer sciences, data minings and information engineerings.

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