On the Indentation of a System of Punches into a Layered Foundation

Kirill E. Kazakov and Dmitry A. Parshin

***Abstract*—Contact problem for a system of rigid rough punches and layered foundation with nonuniform upper layer is considered. Mathematical model of the problem is compiled. It's effective solution is constructed by using Manzhirov projection method. Qualitative conclusions are presented.**

***Index Terms*—coating, multibody contact, nonuniformity, Manzhirov projection method, rapidly changing functions, system of punches, roughness.**

INTRODUCTION

PROBLEMS of the contact interaction for elastic and viscoelastic foundations with different coatings have been studied in good number of papers (see, e.g., [1]–[16]). But we have study the case with periodical nonuniformities or shapes and regular system of punches in the multiple contact problems. (A system of punches is said to be regular if the distances between neighboring punches as well as the widths of punches are equal to each other respectively and shapes of punches base are identical.) This paper provide solution for the common case of arbitrary coating nonuniformity and nonregular system of rigid punches.

I. STATEMENT OF THE PROBLEM

Viscoelastic aging layer (the moment of its production is τ_{lower}) of a thickness h_{lower} with a elastic coating of an thickness h lies on a rigid basis. We assume that the coating rigidity $R(x)$ depend on the longitudinal coordinate. Such foundations sometimes are called as surface nonuniform foundations. The strong material nonuniformity of foundations usually results from the process of layer by layer manufacturing (additive manufacturing) of a coating (e.g., laser treatment, ion implantation, etc. [17], [18]) and its further processing. Rigidity of coating is less than the rigidity of the lower layer or they are of the same order of magnitude. There is smooth contact or perfect contact between layers and between the lower layer and the rigid base.

At time $\tau_0 \geq \tau_{\text{lower}}$, the forces $P_i(t)$ with eccentricities $e_i(t)$ starts to indent system of n smooth rigid punches into the surface of such a foundation. The lengths of lines of contact area are equal to punch lengths $\bar{a}_i = b_i - a_i$ and punch base forms describes by functions $g_i(x)$. (Here a_i and b_i are left and right coordinates of ith punch, $i = 1, 2, \ldots, n$.) The coating is assumed to be thin compared with the contact areas, i.e., its thickness satisfies the condition $h \ll \max\limits_{i=1,2,\ldots,n} \bar{a}_i$.

Manuscript received January 5, 2019; revised January 9, 2019. This work was financially supported by the Russian Foundation for Basic Research (under grant No. 18-01-00770-a).

K. E. Kazakov and D. A. Parshin are with the Ishlinsky Institute for Problems in Mechanics RAS, Prospekt Vernadskogo 101-1, Moscow, 119526, Russia; and Bauman Moscow State Technical University, 2-ya Baumanskaya ulitza 5, Moscow, 105005, Russia; e-mails: kazakov-ke@yandex.ru, parshin@ipmnet.ru.

The system of integral equation of this problem can be written in the form

$$\begin{aligned}
&\frac{q_i(x,t)h}{R(x)} + (\mathbf{I}-\mathbf{V})\frac{2(1-\nu_{\text{lower}}^2)}{\pi E_{\text{lower}}(t-\tau_{\text{lower}})} \\
&\qquad \times \sum_{j=1}^{n} \int_{a_j}^{b_j} k_{\text{pl}}\Big(\frac{x-\xi}{h_{\text{lower}}}\Big) q_j(\xi,t)\,d\xi \\
&\qquad = \delta_i(t) + \alpha_i(t)(x-\eta_i) - g_i(x), \qquad (1)\\
&\mathbf{V}f(x,t) = \int_{\tau_0}^{t} K(t-\tau_{\text{lower}}, \tau-\tau_{\text{lower}}) f(x,\tau)\,d\tau, \\
&K(t,\tau) = E_{\text{lower}}(\tau)\tfrac{\partial}{\partial\tau}\big[E_{\text{lower}}^{-1}(\tau) + C_{\text{lower}}(t,\tau)\big], \\
&x \in [a_i, b_i], \quad t \geqslant \tau_0, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

where $\delta_i(t)$ are the punch settlements and $\alpha_i(t)$ are tilt angles, $\eta_i = \frac{1}{2}(a_i + b_i)$ are the punch midpoints; $E_{\text{lower}}(t)$ and ν_{lower} are the Young modulus and Poisson's ratio of the lower layer; $\mathbf{I}$ is the identity operator; $\mathbf{V}$ are the Volterra integral operator with tensile creep kernel $K_{\text{lower}}(t,\tau)$; $C_{\text{lower}}(t,\tau)$ is the tensile creep function; contact rigidity $R(x)$ depend on the contact conditions between coating and lower layer (see, for example, [19]); in the case of a smooth coating-layer contact, we have $R(x) = E(x)/[1-\nu^2(x)]$, and in the case of an perfect contact, $R(x) = E(x)[1-\nu(x)]/[1-\nu(x)-2\nu^2(x)]$, where $E(x)$ and $\nu(x)$ are the Young modulus and Poisson's ratio of the coating; $k_{\text{pl}}[(x-\xi)/h_{\text{lower}}]$ is known kernel of the plane contact problem, which has the form [20], [21]:

$$k_{\text{pl}}(s) = \int_0^\infty \frac{L(u)}{u}\cos(su)\,du,$$

and, in the case of a smooth contact between the lower layer and the rigid base,

$$L(u) = \frac{\cosh(2u)-1}{\sinh(2u)+2u},$$

and in the case of a perfect contact,

$$L(u) = \frac{2\varkappa\sinh(2u)-4u}{2\varkappa\cosh(2u)+4u^2+1+\varkappa^2}, \quad \varkappa = 3-4\nu_{\text{lower}}.$$

We supplement Eq. (1) with the conditions of the punches equilibrium on the foundation

$$\begin{aligned}
&\int_{a_i}^{b_i} q_i(\xi,t)\,d\xi = P_i(t), \quad \int_{a_i}^{b_i} \xi q_i(\xi,t)\,d\xi = M_i(t), \qquad (2)\\
&t \geq \tau_0, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Here $M_i(t) = e_i(t)P_i(t)$ denotes the moments of application of the forces $P_i(t)$.

There exist a lot of versions of mathematical statements for the contact problem for a system of punches in the plane case. It is easy to show that there is one of four types of conditions on each punch: the load force and moment are given, the tilt angle of the punch and the load force are given,

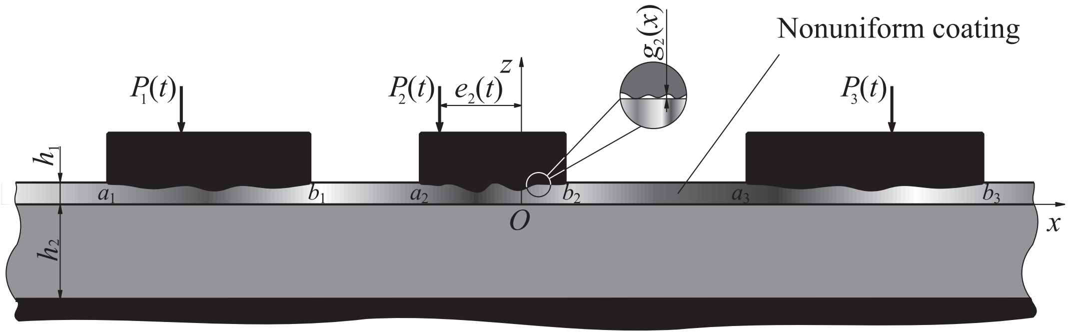

Fig. 1. Multiple contact interaction between foundation with nonuniform coating and system of rigid punches with rough bases.

the punch settlement and the load moment are given, or the settlement and tilt angle of the punch are given. Thus, there are 15 versions of mathematical statements for the plane contact problem for a system of punches.

II. Model representation

Let us make the change of variables in (1) and (2) by the formulas

$$
\begin{aligned}
&x^* = \frac{2(x-\eta_i)}{\bar{a}_1}, \quad \xi^* = \frac{2(\xi-\eta_j)}{\bar{a}_1}, \quad \lambda = \frac{2h_{\text{lower}}}{\bar{a}_1},\\
&\eta^{i*} = \frac{2\eta_i}{\bar{a}_1}, \quad t^* = \frac{t}{\tau_0}, \quad \tau^*_{\text{lower}} = \frac{\tau_{\text{lower}}}{\tau_0}, \quad \alpha^{i*}(t^*) = \alpha_i(t),\\
&\delta^{i*}(t^*) = \frac{2\delta_i(t)}{\bar{a}_1}, \quad c^*(t^*) = \frac{E_{\text{lower}}(t-\tau_{\text{lower}})}{E_\circ},\\
&g^{i*}(x^*) = \frac{2g_i(x)}{\bar{a}_1}, \quad m^i(x^*) = \frac{E_\circ}{R(x)(1-\nu^2_{\text{lower}})}\frac{h}{\bar{a}_1},\\
&q^{i*}(x^*,t^*) = \frac{2(1-\nu^2_{\text{lower}})q_i(x,t)}{E_{\text{lower}}(t-\tau_{\text{lower}})},\\
&P^{i*}(t^*) = \frac{4P_i(t)(1-\nu^2_{\text{lower}})}{E_{\text{lower}}(t-\tau_{\text{lower}})\bar{a}_1},\\
&M^{i*}(t^*) = \frac{8M_i(t)(1-\nu^2_{\text{lower}})}{E_{\text{lower}}(t-\tau_{\text{lower}})\bar{a}_1^2},\\
&\mathbf{F}^{ij*}f(x^*) = \int_{-1}^{1} k^{ij}(x^*,\xi^*)f(\xi^*)d\xi^*,\\
&k^{ij}(x^*,\xi^*) = \frac{1}{\pi}k_{\text{pl}}\left(\frac{x^*+\eta^{i*}-\xi^*-\eta^{j*}}{\lambda}\right) = \frac{1}{\pi}k_{\text{pl}}\left(\frac{x-\xi}{h_{\text{lower}}}\right),\\
&\mathbf{V}^*f(t^*) = \int_{1}^{t^*} K^*(t^*,\tau^*)f(\tau^*)d\tau^*,\\
&K^*(t^*,\tau^*) = K_{\text{lower}}(t-\tau_{\text{lower}},\tau-\tau_{\text{lower}})\tau_0,\\
&i,j = 1,2,\ldots,n.
\end{aligned} \tag{3}
$$

Here $E_\circ$ some dimensional modulus, for example, $E_\circ = \max_{x\in(-\infty,+\infty)} E(x)$.

Then, omitting the asterisks, we obtain a system of mixed integral equations and additional conditions in the dimensionless form

$$
\begin{aligned}
&c(t)m^i(x)q^i(x,t) + (\mathbf{I}-\mathbf{V})\sum_{j=1}^{n}\mathbf{F}^{ij}q^j(x,t)\\
&\qquad = \delta^i(t) + \alpha^i(t)x - g^i(x),\\
&\int_{-1}^{1} q^i(\xi,t)\,d\xi = P^i(t), \quad \int_{-1}^{1} \xi q^i(\xi,t)\,d\xi = M^i(t),\\
&x\in[-1,1], \quad t\geq 1, \quad i = 1,2,\ldots,n.
\end{aligned} \tag{4}
$$

Assuming that

$$
\begin{aligned}
&\mathbf{q}(x,t) = q^i(x,t)\mathbf{i}^i, \quad \boldsymbol{\delta}(t) = \delta^i(t)\mathbf{i}^i, \quad \boldsymbol{\alpha}(t) = \alpha^i(t)\mathbf{i}^i,\\
&\mathbf{g}(x) = g^i(x)\mathbf{i}^i, \quad \mathbf{P}(t) = P^i(t)\mathbf{i}^i, \quad \mathbf{M}(t) = M^i(t)\mathbf{i}^i,\\
&\mathbf{D}(x) = D^{ij}(x)\mathbf{i}^i\mathbf{i}^j,\\
&D^{ij}(x) = m^i(x)\delta^{ij} = \begin{cases} m^i(x), & i=j,\\ 0, & i\neq j,\end{cases}\\
&\mathbf{k}(x,\xi) = k^{ij}(x,\xi)\mathbf{i}^i\mathbf{i}^j, \quad \mathbf{F}\mathbf{f}(x) = \int_{-1}^{1}\mathbf{k}(x,\xi)\cdot\mathbf{f}(\xi)\,d\xi,
\end{aligned} \tag{5}
$$

we can represent system with additional conditions (4) as

$$
\begin{aligned}
&c(t)\mathbf{D}(x)\cdot\mathbf{q}(x,t)+(\mathbf{I}-\mathbf{V})\mathbf{F}\mathbf{q}(x,t)=\boldsymbol{\delta}(t)+\boldsymbol{\alpha}(t)x-\mathbf{g}(x),\\
&\int_{-1}^{1}\mathbf{q}(\xi,t)\,d\xi = \mathbf{P}(t), \quad \int_{-1}^{1}\xi\mathbf{q}(\xi,t)\,d\xi = \mathbf{M}(t),\\
&x\in[-1,1], \quad t\geqslant 1.
\end{aligned} \tag{6}
$$

Hereinafter, it will be the summation over repeated upper indices i and j from 1 to n if the left side of the formula is independent of the index.

In what follows, we construct the solution of the system of two-dimensional equations with the system of auxiliary conditions (4), which contains integral operators with constant as well as variable limits of integration and $2n$ different rapidly changing functions ($m^i(x)$ and $g^i(x)$, $i = 1,2,\ldots,n$).

III. Analytical solution

Now we introduce the notation

$$
\mathbf{q}(x,t) = \tilde{\mathbf{q}}(x,t) - \frac{1}{c(t)}\mathbf{D}^{-1}(x)\cdot\mathbf{g}(x), \tag{7}
$$

where $\mathbf{D}^{-1}(x) = [D^{ij}(x)]^{-1}\mathbf{i}^i\mathbf{i}^j$ (because $\mathbf{D}(x)$ is diagonal matrix) and $\tilde{\mathbf{q}}(x,t)$ is new unknown function. Then operator equation and auxiliary conditions (6) can be reduced to the following equations:

$$
\begin{aligned}
&c(t)\mathbf{D}(x)\cdot\tilde{\mathbf{q}}(x,t) + (\mathbf{I}-\mathbf{V})\mathbf{F}\tilde{\mathbf{q}}(x,t)\\
&\qquad = \boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t)x - \tilde{c}(t)\tilde{\mathbf{g}}(x),\\
&\int_{-1}^{1}\tilde{\mathbf{q}}(\xi,t)\,d\xi = \tilde{\mathbf{P}}(t), \quad \int_{-1}^{1}\xi\tilde{\mathbf{q}}(\xi,t)\,d\xi = \tilde{\mathbf{M}}(t),\\
&x\in[-1,1], \quad t\geqslant 1,
\end{aligned} \tag{8}
$$

where

$$\begin{aligned} \tilde{\mathbf{g}}(x) &= \int_{-1}^{1} \mathbf{k}(x,\xi) \cdot [\mathbf{D}^{-1}(\xi) \cdot \mathbf{g}(\xi)]\, d\xi, \\ \tilde{c}(t) &= -(\mathbf{I}-\mathbf{V})\frac{1}{c(t)}, \\ \tilde{\mathbf{P}}(t) &= \mathbf{P}(t) + \frac{1}{c(t)}\int_{-1}^{1} \mathbf{D}^{-1}(\xi) \cdot \mathbf{g}(\xi)\, d\xi, \\ \tilde{\mathbf{M}}(t) &= \mathbf{M}(t) + \frac{1}{c(t)}\int_{-1}^{1} \mathbf{D}^{-1}(\xi) \cdot \mathbf{g}(\xi)\xi\, d\xi. \end{aligned} \tag{9}$$

We obtain new mixed operator equation with $2n$ rapidly changing function $m^i(x)$ $(i = 1, 2, \dots, n)$ supplemented by two vector conditions. Last term in right-hand side of operator equation (8) is "good": its smoothness defined by kernel $\mathbf{k}(x, \xi)$. Obtained operator equation with additional conditions has the same form as the main equation and additional conditions in [12]. (Only first x-depend factor $\mathbf{D}(x)$ is matrix-function and the last known term in the right-hand side contain t-dependent factor.) But we show that the solution method will be similar.

We will construct the solution of equations (8) for the version when all forces and moments are known.

A. Solution form and special basis

By introducing notations in (8)

$$\begin{aligned} \tilde{\mathbf{q}}(x,t) &= \mathbf{D}^{-1/2}(x) \cdot \mathbf{Q}(x,t), \\ \mathbf{k}(x,\xi) &= \mathbf{D}^{1/2}(x) \cdot \mathbf{K}(x,\xi) \cdot \mathbf{D}^{1/2}(\xi), \\ \mathbf{G}\mathbf{f}(x) &= \int_{-1}^{1} \mathbf{K}(x,\xi) \cdot \mathbf{f}(\xi)\, d\xi, \end{aligned} \tag{10}$$

we obtain new operator equation and additional conditions

$$\begin{aligned} & c(t)\mathbf{Q}(x,t) + (\mathbf{I}-\mathbf{V})\mathbf{G}\mathbf{Q}(x,t) \\ &\qquad = \mathbf{D}^{1/2}(x) \cdot [\boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t)x - \tilde{c}(t)\tilde{\mathbf{g}}(x)], \\ & \int_{-1}^{1} \mathbf{D}^{-1/2}(x) \cdot \mathbf{Q}(\xi,t)\, d\xi = \tilde{\mathbf{P}}(t), \\ & \int_{-1}^{1} \mathbf{D}^{-1/2}(x) \cdot \mathbf{Q}(\xi,t)\xi\, d\xi = \tilde{\mathbf{M}}(t), \\ & x \in [-1,1], \quad t \ge 1. \end{aligned} \tag{11}$$

Here $\mathbf{D}^{1/2}(x) = \sqrt{D^{ij}(x)}\mathbf{i}^i\mathbf{i}^j$, $\mathbf{D}^{-1/2}(x) = \dfrac{1}{\sqrt{D^{ij}(x)}}\mathbf{i}^i\mathbf{i}^j$; $\mathbf{Q}(x,t)$ is new unknown function, and $\mathbf{K}(x,\xi)$ is new kernel.

We seek the solution of Eq. (11) in the class of vector functions continuous in time t in the Hilbert space $L_2([-1,1],V)$ (see [22]). To this end, we at first construct an orthonormal system of vector functions in $L_2([-1,1],V)$ which contains the factors $1/\sqrt{m^i(x)}$ and remaining basis functions can be written as the products of vector functions depending on x and weight functions $1/\sqrt{m^i(x)}$. To this end we will orthonormal on $[-1,1]$ the linearly independent system of vector-functions

$$\left\{ \frac{\mathbf{i}^1}{\sqrt{m^1(x)}}, \frac{x\mathbf{i}^1}{\sqrt{m^1(x)}}, \frac{x^2\mathbf{i}^1}{\sqrt{m^1(x)}}, \dots, \frac{\mathbf{i}^2}{\sqrt{m^2(x)}}, \frac{x\mathbf{i}^2}{\sqrt{m^2(x)}}, \frac{x^2\mathbf{i}^2}{\sqrt{m^2(x)}}, \dots\dots, \frac{\mathbf{i}^n}{\sqrt{m^n(x)}}, \frac{x\mathbf{i}^n}{\sqrt{m^n(x)}}, \frac{x^2\mathbf{i}^n}{\sqrt{m^n(x)}}, \dots \right\}$$

by the formulas:

$$\begin{aligned} & \mathbf{p}_k^i(x) = \mathbf{D}^{-1/2}(x) \cdot \mathbf{p}_k^{i\circ}(x), \quad \mathbf{p}_k^{i\circ}(x) = p_k^{i\circ}(x)\mathbf{i}^i, \\ & d_{-1,i} = 1, \\ & J_{k,i} = \int_{-1}^{1} \frac{\xi^k}{m^i(\xi)}\, d\xi, \quad d_{k,i} = \begin{vmatrix} J_{0,i} & \cdots & J_{k,i} \\ \vdots & \ddots & \vdots \\ J_{k,i} & \cdots & J_{2k,i} \end{vmatrix}, \\ & p_k^{i*}(x) = \frac{1}{\sqrt{d_{k-1,i}d_{k,i}}} \begin{vmatrix} J_{0,i} & J_{1,i} & \cdots & J_{k,i} \\ J_{1,i} & J_{2,i} & \cdots & J_{k+1,i} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x & \cdots & x^k \end{vmatrix}, \\ & i = 1, 2, \dots, n, \quad k = 0, 1, 2, \dots, \quad x \in [-1,1]. \end{aligned} \tag{12}$$

Thus, $\{\mathbf{p}_k^i(x)\}$ $(i = 1, 2, \dots, n,\ k = 0, 1, 2, \dots)$ is a basis in $L_2([-1,1],V)$.

B. Orthoprojectors, spectral problem, and solution

The Hilbert space $L_2([-1,1],V)$ can be presented as the direct sum of orthogonal subspaces $L_2([-1,1],V) = L_2^{(1)}([-1,1],V) \oplus L_2^{(2)}([-1,1],V)$, where $L_2^{(1)}([-1,1],V)$ is the Euclidean space with the basis $\{\mathbf{p}_0^i(x), \mathbf{p}_1^i(x)\}_{i=1,2,\dots,n}$ and $L_2^{(2)}([-1,1],V)$ is the Hilbert space with the basis $\{\mathbf{p}_2^i(x), \mathbf{p}_3^i(x), \dots\}_{i=1,2,\dots,n}$. The integrand and the right-hand side of integral equation (11) can also be presented in the form of the algebraic sum of functions continuous in time t and ranging in $L_2^{(1)}([-1,1],V)$ and $L_2^{(2)}([-1,1],V)$, respectively, i.e.,

$$\begin{aligned} & \mathbf{Q}(x,t) = \mathbf{Q}_1(x,t) + \mathbf{Q}_2(x,t), \\ & \mathbf{D}^{-1/2}(x) \cdot [\boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t) - \tilde{c}(t)\tilde{\mathbf{g}}(x)] = \boldsymbol{\Delta}_1(x,t) + \boldsymbol{\Delta}_2(x,t), \\ & \mathbf{Q}_1(x,t) = z_0^i(t)\mathbf{p}_0^i(x) + z_1^i(t)\mathbf{p}_1^i(x), \\ & \boldsymbol{\Delta}_1(x,t) = \mathbf{D}^{-1/2}(x) \cdot [\boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t)x - \tilde{c}(t)\tilde{\mathbf{g}}_1(x)] \\ &\qquad = \Delta_0^i(t)\mathbf{p}_0^i(x) + \Delta_1^i(t)\mathbf{p}_1^i(x), \\ & \boldsymbol{\Delta}_2(x,t) = \tilde{c}(t)\mathbf{D}^{-1/2}(x) \cdot \tilde{\mathbf{g}}_2(x) = \sum_{m=2}^{\infty} \Delta_m^i(t)\mathbf{p}_m^i(x), \\ & \Delta_0^i(t) = \sqrt{J_{0,i}}\delta^i(t) + \frac{J_{1,i}}{\sqrt{J_{0,i}}}\alpha^i(t) - \tilde{g}_0^i\tilde{c}(t), \\ & \Delta_1^i(t) = \sqrt{\frac{J_{0,i}J_{2,i} - J_{1,i}^2}{J_{0,i}}}\alpha^i(t) - \tilde{g}_1^i\tilde{c}(t), \\ & \Delta_m^i(t) = -\tilde{g}_m^i\tilde{c}(t), \quad i = 1, 2, \dots, n, \quad m = 2, 3, 4, \dots \end{aligned} \tag{13}$$

Here $\mathbf{Q}_1(x,t), \mathbf{f}_1(x,t) \in L_2^{(1)}([-1,1],V)$, $\mathbf{Q}_2(x,t), \mathbf{f}_2(x,t) \in L_2^{(2)}([-1,1],V)$, $\tilde{\mathbf{g}}(x) = \tilde{\mathbf{g}}_1(x) + \tilde{\mathbf{g}}_2(x)$, $\mathbf{D}^{-1/2}(x) \cdot \tilde{\mathbf{g}}_1(x) = \tilde{g}_0^i\mathbf{p}_0^i(x) + \tilde{g}_1^i\mathbf{p}_1^i(x) \in L_2^{(1)}([-1,1],V)$, $\mathbf{D}^{-1/2}(x) \cdot \tilde{\mathbf{g}}_2(x) = \sum_{m=2}^{\infty} \tilde{g}_m^i\mathbf{p}_m^i(x) \in L_2^{(2)}([-1,1],V)$. Using formulas (9), (10), and (12) we can determine coefficients $\tilde{g}_m^i$

$$\begin{aligned} & \tilde{g}_m^i = \sum_{l=0}^{\infty} K_{ml}^{ij} \int_{-1}^{1} \frac{p_l^{i\circ}(x)g^j(x)}{m^i(x)}\, dx, \\ & i = 1, 2, \dots, n, \quad m = 0, 1, 2, \dots, \end{aligned} \tag{14}$$

where K_{mn}^{ij} are expansion coefficients of the kernel $\mathbf{K}(x,\xi)$:

$$\begin{aligned} &\mathbf{K}(x,\xi)=\sum_{m,l=0}^{\infty}K_{ml}^{ij}\mathbf{p}_m^i(x)\mathbf{p}_l^j(\xi),\\ &K_{ml}^{ij}=\int_{-1}^{1}\int_{-1}^{1}\frac{k^{ij}(x,\xi)p_m^{i\circ}(x)p_l^{j\circ}(\xi)}{m^i(x)m^j(\xi)}\,dx\,d\xi,\\ &m,l=0,1,\ldots,\quad i,j=1,2,\ldots,n \end{aligned} \tag{15}$$

The formula for $\mathbf{Q}(x,t)$ contains known term $\mathbf{Q}_1(x,t)$ which is determined by the auxiliary conditions (11)

$$z_0^i(t)=\frac{\tilde{P}^i(t)}{\sqrt{J_{0,i}}},\quad z_1^i(t)=\frac{J_{0,i}\tilde{P}^i(t)+J_{1,i}\tilde{M}^i(t)}{\sqrt{J_{0,i}(J_{0,i}J_{2,i}-J_{1,i}^2)}}. \tag{16}$$

To this end we should use representations for $\mathbf{Q}(x,t)$ and $\mathbf{Q}_1(x,t)$ from (13) and formulas (12). The term $\mathbf{Q}_2(x,t)$ must be found. Conversely, for the right-hand side, one should find $\mathbf{\Delta}_1(x,t)$, while $\mathbf{\Delta}_2(x,t)$ is known and determined be the function $\tilde{\mathbf{g}}_2(x)$. These peculiarities permit one to class the resulting problem as a specific case of the generalized projection problem stated in [22]–[24].

We can introduce the orthogonal projection operator mapping the space $L_2([-1,1],V)$ onto subspace $L_2^{(1)}([-1,1],V)$

$$\mathbf{P}_1\mathbf{f}(x)=\int_{-1}^{1}\mathbf{f}(\xi)\cdot[\mathbf{p}_0^i(\xi)\mathbf{p}_0^i(x)+\mathbf{p}_1^i(\xi)\mathbf{p}_1^i(x)]\,d\xi. \tag{17}$$

Obviously, the orthoprojector $\mathbf{P}_2$ maps the space $L_2([-1,1],V)$ onto $L_2^{(2)}([-1,1],V)$:

$$\begin{aligned} \mathbf{P}_2\mathbf{f}(x)&=\int_{-1}^{1}\mathbf{f}(\xi)\cdot\sum_{m=2}^{\infty}[\mathbf{p}_m^i(\xi)\mathbf{p}_m^i(x)]\,d\xi\\ &=(\mathbf{I}-\mathbf{P}_1)\mathbf{f}(x). \end{aligned} \tag{18}$$

Following [22], we apply the orthogonal projection operator $\mathbf{P}_2$ to operator equation (11). As a result, we obtain the equation for determining $\mathbf{Q}_2(x,t)$ with a known right-hand side

$$\begin{aligned} &c(t)\mathbf{Q}_2(x,t)+(\mathbf{I}-\mathbf{V})\mathbf{P}_2\mathbf{G}\mathbf{Q}_2(x,t)\\ &\quad=-(\mathbf{I}-\mathbf{V})\mathbf{P}_2\mathbf{G}\mathbf{Q}_1(x,t)+\mathbf{\Delta}_2(x,t). \end{aligned} \tag{19}$$

It is necessary to construct its solution in the form of an expansion in the eigenfunctions of the operator $\mathbf{P}_2\mathbf{G}$ which is a compact, strong positive, and self-adjoint operator from $L_2^{(2)}([-1,1],V)$ into $L_2^{(2)}([-1,1],V)$. The system of eigenfunctions of such an operator is a basis in the space $L_2^{(2)}([-1,1],V)$. The spectral problem for the operator $\mathbf{P}_2\mathbf{G}$ can be written in the form

$$\begin{aligned} &\mathbf{P}_2\mathbf{G}\boldsymbol{\varphi}_k(x)=\gamma_k\boldsymbol{\varphi}_k(x),\\ &\boldsymbol{\varphi}_k(x)=\sum_{m=2}^{\infty}\psi_{km}^i\mathbf{p}_m^i(x),\quad k=2,3,\ldots. \end{aligned} \tag{20}$$

According to (10), (17), and (18) this problem leads to the search for solutions of spectral problem about coefficients γ_k and ψ_{km}^i ($k,m=2,3,\ldots$, $i=1,2,\ldots,n$):

$$\sum_{l=2}^{\infty}K_{ml}^{ij}\psi_{kl}^j=\gamma_k\psi_{km}^i,\quad k,m=2,3,\ldots,$$

where coefficients K_{ml}^{ij} determined by (15).

We expand the functions $\mathbf{Q}_2(x,t)$, $\mathbf{P}_2\mathbf{G}\mathbf{Q}_1(x,t)$, and $\mathbf{\Delta}_2(x,t)$ with respect to the new basis functions $\boldsymbol{\varphi}_k(x)$ ($k=2,3,\ldots$) in $L_2^{(2)}([-1,1],V)$, i.e.,

$$\begin{aligned} &\mathbf{Q}_2(x,t)=\sum_{k=2}^{\infty}z_k(t)\boldsymbol{\varphi}_k(x),\\ &\mathbf{P}_2\mathbf{G}\mathbf{Q}_1(x,t)=\sum_{k=2}^{\infty}\sigma_k(t)\boldsymbol{\varphi}_k(x),\\ &\mathbf{\Delta}_2(x,t)=\sum_{k=2}^{\infty}\Delta_k(t)\boldsymbol{\varphi}_k(x). \end{aligned} \tag{21}$$

Using (10), (12), (13), (15), (17), (18), and (20) we can obtain functions $\sigma_k(t)$ and $\Delta_k(t)$

$$\begin{aligned} &\sigma_k(t)=\sum_{m=2}^{\infty}K_{m0}^{ij}\psi_{km}^i z_0^j(t)+\sum_{m=2}^{\infty}K_{m1}^{ij}\psi_{km}^i z_1^j(t),\\ &\Delta_k(t)=\sum_{m=2}^{\infty}\Delta_m^i(t)\psi_{km}^i,\quad k=2,3,4,\ldots. \end{aligned}$$

Substituting equations (21) into (19) and using representation for operator $\mathbf{V}$ for (3) we obtain formula for the unknown expansion functions $z_k(t)$ ($k=2,3,\ldots$)

$$\begin{aligned} &z_k(t)=(\mathbf{I}+\mathbf{W}_k)\frac{-(\mathbf{I}-\mathbf{V})\sigma_k(t)+\Delta_k(t)}{c(t)+\gamma_k},\\ &\mathbf{W}_k f(t)=\int_1^t R_k^*(t,\tau)f(\tau)\,d\tau, \end{aligned} \tag{22}$$

where $R_k^*(t,\tau)$ ($k=1,2,\ldots$) is the resolvent of the kernel

$$K_k^*(t,\tau)=\frac{\gamma_k K(t,\tau)}{c(t)+\gamma_k}.$$

According (7), (10), (12), (13), (20), and (21) the final solution has the following structure

$$\begin{aligned} \mathbf{q}(x,t)=\mathbf{D}^{-1}(x)\cdot\Big[&z_0^i(t)\mathbf{p}_0^{i\circ}(x)+z_1^i(t)\mathbf{p}_1^{i\circ}(x)\\ &+\sum_{k=2}^{\infty}z_k(t)\sum_{m=2}^{\infty}\psi_{km}^i\mathbf{p}_m^{i\circ}(x)\Big]-\frac{1}{c(t)}\mathbf{D}^{-1}(x)\cdot\mathbf{g}(x), \end{aligned} \tag{23}$$

or in coordinate form ($i=1,2,\ldots,n$)

$$\begin{aligned} q^i(x,t)=\frac{1}{m^i(x)}\Big[&z_0^i(t)p_0^{i\circ}(x)+z_1^i(t)p_1^{i\circ}(x)\\ &+\sum_{k=2}^{\infty}z_k(t)\sum_{m=2}^{\infty}\psi_{km}^i p_m^{i\circ}(x)\Big]-\frac{g^i(x)}{c(t)m^i(x)}, \end{aligned}$$

i.e., one can explicitly write out the weight functions $m^i(x)$ and $g^i(x)$ in the solution. Note that the coating nonuniformities are related to functions $m^i(x)$ and punches base forms functions are related to functions $g^i(x)$ in the relations of change of variables (3). These parameters can described by complicated and rapidly changing functions. The formulas obtained permit obtaining efficient analytical solutions.

In order to find the unknown punch settlements and tilt angles we act operator equation (11) by operator $\mathbf{P}_1$

$$\alpha^i(t) = \sqrt{\frac{J_{0,i}}{J_{0,i}J_{2,i} - J_{1,i}^2}}\Bigg\{\tilde{g}_1^i\tilde{c}(t) + c(t)z_1^i(t) + (\mathbf{I} - \mathbf{V})\Bigg[K_{00}^{ij}z_0^j(t) + K_{01}^{ij}z_1^j(t) + \sum_{k=2}^{\infty}\Bigg(\sum_{l=2}^{\infty}K_{0l}^{i_1 j}\psi_{kl}^j\Bigg)z_k(t)\Bigg]\Bigg\},$$

$$\delta^i(t) = -\frac{J_{1,i}}{J_{0,i}}\alpha^{i_1}(t) + \frac{1}{\sqrt{J_{0,i}}}\Bigg\{\tilde{g}_0^i\tilde{c}(t) + c(t)z_0^{i_1}(t) + (\mathbf{I} - \mathbf{V})\Bigg[K_{10}^{ij}z_0^j(t) + K_{11}^{ij}z_1^j(t) + \sum_{k=2}^{\infty}z_k(t)\Bigg(\sum_{l=2}^{\infty}K_{1l}^{ij}\psi_{kl}^j\Bigg)\Bigg]\Bigg\}, \quad i = 1, 2, \ldots, n.$$

IV. Conclusions

- Plane problem of multiple contact interaction for base with coating and arbitrary system of punches is posed. The corresponding mathematical model is given and analyzed. Possible variants of the problem statement are formulated.
- The analytical solution for the version of known forces and moments is obtained using Manzhirov generalized projection method. In relation of contact stresses (23) contact stiffness and punch base functions are represented by separate terms and factors. It is allows one to perform effective computations for actual nonuniformities and punch shapes.
- Analytical representation of the solution allow one to analyze carefully the behavior of the punches on the layer, taking into account the complex properties of the coatings and the mutual influence of the punches.

References

[1] A. V. Manzhirov, K. E. Kazakov, and D. A. Parshin, "The interaction between a surface nonuniform foundation and a regular system of punches with rough bases." Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2018, 4–6 July, 2018, London, U.K., pp. 563–566.

[2] A. V. Manzhirov, "Axisymmetric contact problems for non-uniformly aging layered viscoelastic foundations." *J Appl Math Mech*, vol. 47, no. 4, pp. 558–566, 1983.

[3] A. V. Manzhirov, "On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology." *J Appl Math Mech*, vol. 49, no. 6, pp. 777–782, 1985.

[4] V. M. Aleksandrov and A. V. Manzhirov, "Two-dimensional integral equations in applied mechanics of deformable solids." *J Appl Mech Techn Phys*, vol. 28, no. 5, pp. 781–786, 1987.

[5] Y. F. Chen and F. Erdogan, "The interface crack problem for a nonhomogeneous coating bonded to a homogeneous substrate." *J Mech Phys Solids*, vol. 44, no. 5, pp. 771–787, 1996.

[6] V. Pauk and B. W. Zastrau, "Plane contact problems with partial slip for rough half-space," *J Theoret Appl Mech*, vol. 42, no. 1, pp. 107–124, 2004.

[7] R. Kul'chyts'kyi-Zhyhailo and G. Rogowski, "Axially symmetric contact problem of pressing of an absolutely rigid ball into an elastic half space with inhomogeneous coating." *Mater Sci*, vol. 45, no. 6, pp. 845–858, 2009.

[8] K. E. Kazakov, "Modeling of contact interaction for solids with inhomogeneous coatings." *J Phys Conf Ser*, vol. 181, p. 012013, 2009.

[9] A. Vasiliev, S. Volkov, S. Aizikovich, and Y.-R. Jeng, "Axisymmetric contact problems of the theory of elasticity for inhomogeneous layers," *ZAMM*, vol. 94, no. 9, pp. 705–712, 2014.

[10] S. M. Aizikovich, A. S. Vasil'ev, and S. S. Volkov, "The axisymmetric contact problem of the indentation of a conical punch into a half-space with a coating inhomogeneous in depth," *J Appl Math Mech*, vol. 79, no. 5, pp. 500–505, 2015.

[11] R. Ledesma-Alonso, E. Raphaël, L. Léger, et al., "Stress concentration in periodically rough Hertzian contact: Hertz to soft-flat-punch transition." *Proc Math Phys Eng Sci*, vol. 472(2193), p. 20160235, 2016.

[12] K. E. Kazakov, S. P. Kurdina, and A. V. Manzhirov, "Contact interaction between surface nonuniform bases and regular systems of rigid punches." *Proc IUTAM*, vol. 23, pp. 201–209, 2017.

[13] K. E. Kazakov and A. V. Manzhirov, "Axisymmetric contact between an annular rough punch and a surface nonuniform foundation." *Mech Solids*, vol. 52, no. 4, 2017.

[14] A. V. Manzhirov and K. E. Kazakov, "Modeling the contact interaction between a nonuniform foundation and a rough punch." *Math Models Comput Simul*, vol. 10, no. 3, pp. 314–321, 2018.

[15] K. E. Kazakov, "Contact between a regular system of punches and a layered foundation with consideration of complex surface shapes." *AIP Conf Proc*, vol. 2053, p. 949941, 2018.

[16] K. E. Kazakov and S. P. Kurdina, "Contact interaction between a layered foundation and a system of annular punches with complex base shapes." *J Phys Conf Ser*, vol. 991, p. 012040, 2018.

[17] I. Gibson, D. W. Rosen, and B. Stucker, *Additive Manufacturing Technologies: Rapid Prototyping to Direct Digital Manufacturing* Springer Verlag, LLC, New York, 2010.

[18] A. Gebhardt and J. S. Hoetter, *Additive Manufacturing: 3D Printing for Prototyping and Manufacturing* Carl Hanser Verlag, Munich, 2016.

[19] V. M. Alexandrov and E. V. Kovalenko, *Problems in Mechanics of Continuous Media with Mixed Boundary Conditions [in Russian].* Nauka Publ., Moscow, 1986.

[20] I. I. Vorovich, V. M. Alexandrov, and V. A. Babeshko, *Nonclassical Mixed Problems in the Theory of Elasticity [in Russian]*. Nauka Publ., Moscow, 1974.

[21] V. M. Alexandrov and S. M. Mkhitaryan, *Contact Problems for Bodies with Thin Coatings and Interlayers [in Russian]*. Nauka Publ., Moscow, 1983.

[22] A. V. Manzhirov, "A mixed integral equation of mechanics and a generalized projection method of its solution." *Dokl Phys*, vol. 61, no. 10, pp. 489–493, 2016.

[23] A. V. Manzhirov, "Integral equations with several different operators and their application to mechanics." *Lecture Notes in Engineering and Computer Science*, vol. 2224, no. 1, pp. 10–15, 2016.

[24] A. V. Manzhirov, "Mixed integral equations and their application to mechanics." in *IAENG Transactions on Engineering Sciences*. World Scientific Pub Singapore, Netherlands, 8–21.