Compact Difference Schemes for a Class of Space-time Fractional Differential Equations

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Abstract—**In this paper, finite difference schemes for solving a class of space-time fractional differential equations with the order of the spatial fractional derivative more than two are investigated. First the time fractional derivative is approximated by the $L1$ interpolation formula, while the spatial fractional derivative is approximated by the fourth order weighted shifted Grünwald-Letnikov derivative approximation formula. Then based on the concepts of the order reduction method and construction of compact schemes, two compact finite difference schemes are developed. Theoretical analysis of unique solvability, stability and convergence of the present finite difference schemes are discussed. Numerical experiments are also carried out, and the numerical results show their good agreement with the theoretical analysis.**

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Index Terms—**Space-time fractional differential equation; High order spatial fractional derivative; Compact finite difference scheme; Unconditionally stable**

I. Introduction

Fractional derivative is the generalization of the derivative of integer order. Recently, fractional calculus has played an important role in many researching domains such as physics [1-4], fluid mechanics [5], bioengineering [6], finance [7-11] and so on. The most significant advantage of the fractional-order models in comparison with integer-order models lies in that fractional derivatives and integrals are more suitable for the description of the memory and hereditary properties of different substances.

For the basic theory of fractional differential equations, readers can refer to the works [12,13]. One of the most important applications of fractional differential equations is to model the process of subdiffusion and superdiffusion of particles in physics, where the fractional diffusion equation is usually used for modeling this movement [14-16].

In the research of fractional differential equations, seeking solutions has attracted much attention by a lot of researchers. Many authors proposed various valid methods for solving fractional differential equations including the coupled fractional reduced differential transform method [17], the Bernstein polynomials method [18], the residual power series method [19], the Jacobi elliptic function method [20] and so on Unfortunately, it is usually difficult to obtain exact solutions for fractional differential equations in that the fractional derivative operators are quasi-differential operators with singularity. So it becomes important to develop valid numerical methods with good characters for solving fractional differential equations. So far many valid numerical methods have been developed. For example, in [21], Zhou et al. proposed a spatial sixth order finite difference scheme for time fractional sub-diffusion equation with variable coefficient. In [22], Feng proposed a Crank-Nicolson difference scheme for a class of space fractional differential equations with high order spatial fractional derivative. In [23], Feng et al. applied the finite element method with two different time discretization schemes for solving two types of space-time fractional diffusion equations, while in [24], Bu et al. presented a Galerkin finite element method for two-dimensional Riesz space fractional diffusion equations. In [25], Liu et al. proposed an implicit radial basis function meshless approximation method for a class of time fractional diffusion equations. In [26], Huang and Liu considered a class of space-time fractional advection-dispersion equation, and obtained the solution in terms of Green functions and representations of the Green function by applying the Fourier-Laplace transforms. In [27], Yuste established a weighted averaged finite difference scheme for fractional diffusion equations, while in [28], Meerschaert and Tadjeran proposed finite difference approximations for fractional advection-dispersion flow equations, where the fractional derivatives were both approximated by use of the Grünwald-Letnikov approximation formula. Afterwards, many authors applied the finite difference method to solve various time, space, and space-time fractional differential equations (see [29-35] and the references therein for example). We notice that in the current research on numerical methods for solving fractional differential equations, the orders of the fractional derivative are usually less than two, while little attention has been paid so far on developing finite difference schemes for fractional differential equations with the orders of fractional derivatives more than two.

Motivated by the above works, in this paper, we consider the following initial boundary value problem for space-time fractional differential equation as follows:

$$\begin{cases} u_t(x,t) +_0^C D_t^\gamma u(x,t) = \kappa(x)({}_aD_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)) \\ \quad +f(x,t),\ 0<\gamma<1, \\ u(x,0)=h(x),\ x\in[a,b], \\ u(a,t)=u(b,t)=0,\ t\in[0,T], \end{cases} \tag{1}$$

where $\alpha\in(2,3)$ or $\alpha\in[3+\eta,4)$, η is a sufficiently small fixed positive number, $\kappa(x)>0$ for $x\in(a,b)$, ${}_0^C D_t^\gamma u(x,t)$, ${}_aD_x^\alpha u(x,t)$ and ${}_xD_b^\alpha u(x,t)$ denote the Caputo fractional derivative, the left-side Riemann-Liouville fractional derivative and the right-side Riemann-Liouville fractional derivative respectively, and

$$\begin{cases} {}_0^C D_t^\gamma u(x,t) = \dfrac{1}{\Gamma(1-\gamma)}\int_0^t \dfrac{u_t'(x,s)}{(t-s)^\gamma}ds, \\ {}_aD_x^\alpha u(x,t) = \dfrac{d^n}{dx^n}(\dfrac{1}{\Gamma(n-\alpha)}\int_a^x (x-\sigma)^{n-1-\alpha}u(\sigma,t)d\sigma), \\ {}_xD_b^\alpha u(x,t) = \\ \quad (-1)^n\dfrac{d^n}{dx^n}(\dfrac{1}{\Gamma(n-\alpha)}\int_x^b (\sigma-x)^{n-1-\alpha}u(\sigma,t)d\sigma), \end{cases} \tag{2}$$

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where $n-1 \le \alpha < n,\ n \in \mathbf{N}$. For further use, we extend the definition domain of the function $u(x,t)$ to $\mathbf{R} \times [0,T]$, and satisfies $u(x,t) \equiv 0$ for $x \in (-\infty,a] \bigcup [b,\infty)$.

For the approximation of the Riemann-Liouville fractional derivative, the Grünwald-Letnikov approximation formula is the most popularly used so far. Yet as difference schemes generated by use of the standard Grünwald-Letnikov approximation formula is usually unstable, so the shifted or weighted shifted Grünwald-Letnikov approximation formulas are widely used instead [27,28,36]. For the approximation of the Caputo fractional derivative, various L interpolation formulas are widely used [37-39].

We organize the rest of this paper as follows. In Section 2, we propose two compact finite difference schemes for the problem (1) with $\alpha \in (2,3)$ and $[3+\eta,4)$ respectively. Afterwards, in Section 3, theoretical analysis of unique solvability, stability and convergence for the present two difference schemes are discussed. In Section 4, numerical experiments are carried out for testifying the present difference schemes. Some conclusions are proposed at the end of this paper.

II. Construction of the finite difference schemes

Let M, N be two positive integers, and $h = \frac{b-a}{M},\ \tau = \frac{T}{N}$ denote the spatial and temporal step size respectively. Define $x_i = a + i*h (0 \le i \le M),\ t_n = n\tau (0 \le n \le N),\ \Omega_h = \{x_i | 0 \le i \le M\},\ \Omega_\tau = \{t_n | 0 \le n \le N\}$, $(i,n) = (x_i, t^n)$, and then the domain $[a,b]\times[0,T]$ is covered by $\Omega_h \times \Omega_\tau$. Let $V_h = \{u_i^n | 0 \le i \le M,\ 0 \le n \le N\}$ be the grid function on the mesh $\Omega_h \times \Omega_\tau$. $U_i^n = u(x_i, t^n)$ and u_i^n denote the exact solution and numerical solution at the point (i,n) respectively. $U^n = (U_1^n,\ U_2^n, ...,\ U_M^n)^T,\ u^n = (u_1^n,\ u_2^n, ...,\ u_M^n)^T$. For further use, Denote

$$\delta_t u_i^n = \frac{u_i^n - u_i^{n-1}}{\tau},\ \delta_x u_{i-\frac{1}{2}}^n = \frac{u_i^n - u_i^{n-1}}{h},$$
$$\delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

Property 1. For the left-side Riemann-Liouville derivative and the right-side Riemann-Liouville derivative, it holds that for some $k \in \mathbf{N}$

$$\begin{aligned} {}_aD_x^{\alpha+k} u(x,t) &= \frac{d^k}{dx^k}({}_aD_x^\alpha u(x,t)), \\ {}_xD_b^{\alpha+k} u(x,t) &= (-1)^k \frac{d^k}{dx^k}({}_xD_b^\alpha u(x,t)). \end{aligned} \tag{3}$$

Property 2. The first order shifted Grünwald-Letnikov approximation formulas approximating the Riemann-Liouville derivatives can be denoted as follows

$$\begin{cases} \frac{1}{h^\alpha} \sum\limits_{k=0}^{\infty} g_k^{(\alpha)} u(x-(k-p)h) = {}_{-\infty}D_x^\alpha u(x) + O(h), \\ \frac{1}{h^\alpha} \sum\limits_{k=0}^{\infty} g_k^{(\alpha)} u(x+(k-p)h) = {}_xD_\infty^\alpha u(x) + O(h), \end{cases}$$

where p is an integer, and $g_0^{(\alpha)} = 1,\ g_k^{(\alpha)} = (1 - \frac{\alpha+1}{k}) g_{k-1}^{(\alpha)},\ k = 1, 2, ...$

Especially, when $u \in C(\mathbf{R})$, and $u(x) \equiv 0,\ x \in (-\infty,a] \bigcup [b,\infty)$, it holds that

$$\begin{cases} \frac{1}{h^\alpha} \sum\limits_{k=0}^{[(x-a)/h]+p} g_k^{(\alpha)} u(x-(k-p)h) = {}_aD_x^\alpha u(x) + O(h), \\ \frac{1}{h^\alpha} \sum\limits_{k=0}^{[(b-x)/h]+p} g_k^{(\alpha)} u(x+(k-p)h) = {}_xD_b^\alpha u(x) + O(h), \end{cases} \tag{4}$$

Lemma 1 [40]. Suppose $\alpha \in (1,2)$. Define the averaging difference operator $\mathcal{A}_1 v(x) = (1 + c_2^\alpha h^2 \delta_x^2) v(x)$, where $c_2^\alpha = \frac{-\alpha^2+\alpha+4}{24}$. Suppose $u \in C(\mathbf{R})$ and $u \in \wp^{4+\alpha}(\mathbf{R})$, where $\wp^{n+\alpha}(\mathbf{R}) = \{f | \int_{-\infty}^{\infty} (1+|\omega|)^{n+\alpha} \hat{f}(\omega) d\omega < \infty\}$, and $\hat{f}(\omega)$ is the Fourier transformation of $f(x)$. Then for $u(x) \equiv 0,\ x \in (-\infty,a] \bigcup [b,\infty)$, the following fourth order weighted shifted Grünwald-Letnikov approximation formulas hold

$$\begin{cases} \mathcal{A}_1({}_aD_x^\alpha u(x)) = \mathcal{A}_1({}_{-\infty}D_x^\alpha u(x)) \\ = \frac{1}{h^\alpha} \sum\limits_{k=0}^{\infty} w_k^{(\alpha)} u(x-(k-1)h) + O(h^4) \\ = \frac{1}{h^\alpha} \sum\limits_{k=0}^{[(x-a)/h]+1} w_k^{(\alpha)} u(x-(k-1)h) + O(h^4), \\ \mathcal{A}_1({}_xD_b^\alpha u(x)) = \mathcal{A}_1({}_xD_\infty^\alpha u(x)) \\ = \frac{1}{h^\alpha} \sum\limits_{k=0}^{\infty} w_k^{(\alpha)} u(x+(k-1)h) + O(h^4) \\ = \frac{1}{h^\alpha} \sum\limits_{k=0}^{[(b-x)/h]+1} w_k^{(\alpha)} u(x+(k-1)h) + O(h^4), \end{cases} \tag{5}$$

where

$$\begin{cases} w_0^{(\alpha)} = \frac{\alpha^2+3\alpha+2}{12} g_0^{(\alpha)} = \frac{\alpha^2+3\alpha+2}{12}, \\ w_1^{(\alpha)} = \frac{\alpha^2+3\alpha+2}{12} g_1^{(\alpha)} + \frac{4-\alpha^2}{6} g_0^{(\alpha)}, \\ w_k^{(\alpha)} = \frac{\alpha^2+3\alpha+2}{12} g_k^{(\alpha)} + \frac{4-\alpha^2}{6} g_{k-1}^{(\alpha)} \\ + \frac{\alpha^2-3\alpha+2}{12} g_{k-2}^{(\alpha)},\ k = 2, 3, ..., \end{cases} \tag{6}$$

and $g_k^{(\alpha)},\ k = 0, 1, 2, ...$ are defined as in Property 2.

Remark 1. The shifted or weighted shifted Grünwald-Letnikov approximation formulas listed above are widely used to approximate spatial Riemann-Liouville fractional derivative, and furthermore are applied to construct unconditionally stable difference schemes for spatial fractional differential equations with the spatial fractional order $\alpha < 2$. However, for those difference schemes constructed by direct use of the shifted or weighted shifted Grünwald-Letnikov approximation formulas with $\alpha > 2$, the analysis of stability and convergence is difficult to fulfil.

Lemma 2 [37, Lem. 2.1](The $L1$ formula). Suppose $0 < \gamma < 1$, and $u(t) \in C^2[0, t_n]$. Then it holds that

$$\begin{aligned} &|{}_0^C D_t^\gamma u(t) - \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}[a_0^{(\gamma)} u(t_n) \\ &- \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) u(t_k) - a_{n-1}^{(\gamma)} u(t_0)]| \\ &\le \frac{1}{\Gamma(2-\gamma)}[\frac{1-\gamma}{12} + \frac{2^{2-\gamma}}{2-\gamma} - (1+2^{-\gamma})] \\ &\max_{t_0 \le t \le t_n} |u''(t)| \tau^{2-\gamma}, \end{aligned} \tag{7}$$

where $a_k^{(\gamma)} = (k+1)^{1-\gamma} - k^{1-\gamma},\ k \ge 0$, and satisfies $(1-\gamma)(k+1)^{-\gamma} < a_k^{(\gamma)} < (1-\gamma) k^{-\gamma}$.

In order to derive difference schemes for Eqs. (1) by use of the weighted shifted Grünwald-Letnikov approximation formulas, it is feasible to use the order reduction method. Next we will construct difference schemes in two subsections. Define the operators $\mathcal{A}_1,\ \mathcal{A}_2$ by

$$\begin{cases} \mathcal{A}_1 v(x) = (1 + c_2^\beta h^2 \delta_x^2) v(x), \\ \mathcal{A}_2 v(x) = (1 + \frac{1}{12} h^2 \delta_x^2) v(x), \end{cases}$$

where $c_2^\beta = \dfrac{-\beta^2 + \beta + 4}{24},\ \beta \in (1,2)$.

A. Finite difference scheme with $\alpha \in [3+\eta, 4)$

Set $\beta = \alpha - 2$. Then $\beta \in [1+\eta, 2)$ for $\alpha \in [3+\eta, 4)$. By use of Lemma 1 one can obtain the following approximation at the grid point (i,n)

$$\begin{aligned} &\mathcal{A}_1[{}_aD_x^\beta u(x,t) -{}_x D_b^\beta u(x,t)]_{(i,n)} \\ &= \frac{1}{h^\beta}\sum_{k=0}^{i+1} w_k^{(\beta)} U_{i-k+1}^n - \frac{1}{h^\beta}\sum_{k=0}^{M-i+1} w_k^{(\beta)} U_{i+k-1}^n + O(h^4) \\ &= \frac{1}{h^\beta}\sum_{k=-M+i}^{i} r_k^{(\beta)} U_{i-k}^n + O(h^4),\ 1 \le i \le M-1, \end{aligned} \quad (8)$$

where $w_k^{(\beta)},\ k = 0,1,...$ are defined as in (6), and

$$\begin{cases} r_0^{(\beta)} = w_1^{(\beta)} - w_1^{(\beta)} = 0, \\ r_1^{(\beta)} = w_2^{(\beta)} - w_0^{(\beta)}, \\ r_k^{(\beta)} = w_{k+1}^{(\beta)},\ k = 2,3,..., \\ r_{-k}^{(\beta)} = -r_k^{(\beta)},\ k = 1,2,.... \end{cases}$$

Define $m(x,t) ={}_a D_x^\beta u(x,t) -{}_x D_b^\beta u(x,t)$. Then it holds that $m_x''(x,t) ={}_a D_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)$, and the first equation of (1) can be rewritten as follows

$$\frac{1}{\kappa(x)}[u_t(x,t) + {}_0^C D_t^\gamma u(x,t)] = m_x''(x,t) + \frac{1}{\kappa(x)} f(x,t). \quad (9)$$

On the other hand, the following approximation formula holds provided that $m(x,t) \in C^{(6,1)}(\mathbf{R} \times [0,T])$:

$$\begin{aligned} &\frac{m_{i+1}^n - 2m_i^n + m_{i-1}^n}{h^2} \\ &= m_x''(x_i,t^n) + \frac{h^2}{12} m_x^{(4)}(x_i,t^n) + O(h^4) \\ &= m_x''(x_i,t^n) + \frac{1}{12}[m_x''(x_{i+1},t^n) - 2m_x''(x_i,t^n) \\ &\quad + m_x''(x_{i-1},t^n)] + O(h^4) \\ &= (\mathcal{A}_2 m_x'')_{(i,n)} + O(h^4) \\ &= \mathcal{A}_2[{}_aD_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)]_{(i,n)} + O(h^4). \end{aligned} \quad (10)$$

Applying the operator $\mathcal{A}_1$ on both sides of (10), and by use of (8) one can obtain that

$$\begin{aligned} &\mathcal{A}_1\mathcal{A}_2[{}_aD_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)]_{(i,n)} \\ &= \mathcal{A}_1(\frac{m_{i+1}^n - 2m_i^n + m_{i-1}^n}{h^2}) + O(h^4) \\ &= \frac{1}{h^2}[\frac{1}{h^\beta}\sum_{k=-M+i+1}^{i+1} r_k^{(\beta)} U_{i+1-k}^n - \frac{2}{h^\beta}\sum_{k=-M+i}^{i} r_k^{(\beta)} U_{i-k}^n \\ &+ \frac{1}{h^\beta}\sum_{k=-M+i-1}^{i-1} r_k^{(\beta)} U_{i-1-k}^n] + O(h^2) \\ &= \frac{1}{h^\alpha}\sum_{k=0}^{M} \lambda_{i-k}^{(\alpha)} U_k^n + O(h^2),\ 1 \le i \le M-1. \end{aligned} \quad (11)$$

where

$$\begin{cases} \lambda_0^{(\alpha)} = r_1^{(\beta)} - 2r_0^{(\beta)} + r_{-1}^{(\beta)} = 0, \\ \lambda_k^{(\alpha)} = r_{k+1}^{(\beta)} - 2r_k^{(\beta)} + r_{k-1}^{(\beta)},\ k = 1,2,..., \\ \lambda_{-k}^{(\alpha)} = -\lambda_k^{(\alpha)},\ k = 1,2,.... \end{cases}$$

So if we put the operators $\mathcal{A}_2$ and $\mathcal{A}_1$ on both sides of (9) at the point (i,n), then together with the use of Lemma 2 and the backward difference formula one can deduce that

$$\begin{aligned} &\mathcal{A}_1\mathcal{A}_2(\frac{\delta_t U_i^n + \delta_t^{(\gamma)} U_i^n}{\kappa_i}) = \frac{1}{h^\alpha}\sum_{k=0}^{M} \hat{r}_{i-k}^{(\alpha)} U_k^n + \mathcal{A}_1\mathcal{A}_2(\frac{f_i^n}{\kappa_i}) \\ &+ O(\tau + \tau^{2-\gamma} + h^2),\ 1 \le i \le M-1. \end{aligned} \quad (12)$$

where $\delta_t^{(\gamma)} U_i^n = \dfrac{\tau^{-\gamma}}{\Gamma(2-\gamma)}[a_0^{(\gamma)} U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) U_i^k - a_{n-1}^{(\gamma)} U_i^0]$. Then the compact finite difference scheme approximating the Eqs. (1) can be denoted as follows:

$$\begin{cases} \mathcal{A}_1\mathcal{A}_2(\frac{\delta_t u_i^n + \delta_t^{(\gamma)} u_i^n}{\kappa_i}) = \frac{1}{h^\alpha}\sum_{k=0}^{M} \hat{r}_{i-k}^{(\alpha)} u_k^n + \mathcal{A}_1\mathcal{A}_2(\frac{f_i^n}{\kappa_i}), \\ \quad 1 \le n \le N,\ i = 1,2,...,M-1, \\ u_i^0 = h(x_i),\ i = 1,2,...,M-1. \end{cases} \quad (13)$$

Remark 2. The reason for $\alpha \in [3+\eta, 4)$ instead of $(3,4)$ lies in that the unconditional stability of the difference scheme established can be ensured.

B. Finite difference scheme with $\alpha \in (2,3)$

Set $\beta = \alpha - 1$. Then $\beta \in (1,2)$ for $\alpha \in (2,3)$, and similarly from Lemma 1 we have

$$\begin{aligned} &\mathcal{A}_1[{}_aD_x^\beta u(x,t) +{}_x D_b^\beta u(x,t)]_{(i,n)} \\ &= \frac{1}{h^\beta}\sum_{k=0}^{i+1} w_k^{(\beta)} U_{i-k+1}^n + \frac{1}{h^\beta}\sum_{k=0}^{M-i+1} w_k^{(\beta)} U_{i+k-1}^n + O(h^4) \\ &= \frac{1}{h^\beta}\sum_{k=-M+i}^{i} \widehat{r}_k^{(\beta)} U_{i-k}^n + O(h^4),\ 1 \le i \le M-1, \end{aligned} \quad (14)$$

where $w_k^{(\beta)},\ k = 0,1,...$ are defined as in (6), and

$$\begin{cases} \widehat{r}_0^{(\beta)} = 2w_1^{(\beta)}, \\ \widehat{r}_1^{(\beta)} = w_2^{(\beta)} + w_0^{(\beta)}, \\ \widehat{r}_k^{(\beta)} = w_{k+1}^{(\beta)},\ k = 2,3,..., \\ \widehat{r}_{-k}^{(\beta)} = r_k^{(\beta)},\ k = 1,2,.... \end{cases}$$

Let $p(x,t) ={}_a D_x^\beta u(x,t) +{}_x D_b^\beta u(x,t)$. Then $p_x'(x,t) = {}_aD_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)$, and the first equation of (1) can be rewritten as follows

$$\frac{1}{\kappa(x)}[u_t(x,t) + {}_0^C D_t^\gamma u(x,t)] = p_x'(x,t) + \frac{1}{\kappa(x)} f(x,t). \quad (15)$$

As the following center difference formula holds provided that $p \in C^{(5,1)}(\mathbf{R} \times [0,T])$

$$\frac{p_{i+1}^n - p_{i-1}^n}{2h} = p_x'(x_i,t^n) + O(h^2). \quad (16)$$

Applying the operator $\mathcal{A}_1$ on both sides of (16), and by use of (14) one can obtain that

$$\begin{aligned} &\mathcal{A}_1[{}_aD_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)]_{(i,n)} \\ &= \mathcal{A}_1(\frac{m_{i+1}^n - m_{i-1}^n}{2h}) + O(h^2) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2h}[\frac{1}{h^\beta}\sum_{k=-M+i+1}^{i+1}\widehat{r}_k^{(\beta)}U_{i+1-k}^n \\
&\quad -\frac{1}{h^\beta}\sum_{k=-M+i-1}^{i-1}\widehat{r}_k^{(\beta)}U_{i-1-k}^n] + O(h^2) \\
&= \frac{1}{2h^\alpha}\sum_{k=0}^{M}\widehat{\lambda}_{i-k}^{(\alpha)}U_k^n + O(h^2),\ 1\le i\le M-1. \qquad (17)
\end{aligned}
$$

where

$$
\begin{cases}
\widehat{\lambda}_0^{(\alpha)} = \widehat{r}_1^{(\beta)} - \widehat{r}_{-1}^{(\beta)} = 0,\\
\widehat{\lambda}_k^{(\alpha)} = \widehat{r}_{k+1}^{(\beta)} - \widehat{r}_{k-1}^{(\beta)},\ k=1,2,...,\\
\widehat{\lambda}_{-k}^{(\alpha)} = -\widehat{\lambda}_k^{(\alpha)},\ k=1,2,....
\end{cases}
$$

So applying the operator $\mathcal{A}_1$ on both sides of (15) at the point (i,n), together with the use of Lemma 2 and the backward difference formula one can get that

$$
\mathcal{A}_1(\frac{\delta_t U_i^n + \delta_t^{(\gamma)}U_i^n}{\kappa_i}) = \frac{1}{2h^\alpha}\sum_{k=0}^{M}\widehat{\lambda}_{i-k}^{(\alpha)}U_k^n + \mathcal{A}_1(\frac{f_i^n}{\kappa_i}) + O(\tau+\tau^{2-\gamma}+h^2),\ 1\le i\le M-1. \qquad (18)
$$

where $\delta_t^{(\gamma)}U_i^n$ is defined as in (12). Then the compact finite difference scheme approximating the Eqs. (1) can be denoted as follows:

$$
\begin{cases}
\mathcal{A}_1(\frac{\delta_t u_i^n + \delta_t^{(\gamma)}u_i^n}{\kappa_i}) = \frac{1}{2h^\alpha}\sum_{k=0}^{M}\widehat{\lambda}_{i-k}^{(\alpha)}u_k^n + \mathcal{A}_1(\frac{f_i^n}{\kappa_i}),\\
\quad 1\le n\le N,\ i=1,2,...,M-1,\\
u_i^0 = h(x_i),\ i=1,2,...,M-1.
\end{cases} \qquad (19)
$$

Remark 3. The construction of the difference scheme (19) is different from that of (13) in that only one compact operator $\mathcal{A}_1$ is applied in the derivation of (19), while two compact operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are applied in the derivation of (13). We note that if two operators are applied in the derivation of the latter, then the obtained difference scheme may be unstable.

III. Theoretical analysis of the difference scheme

In this section, we discuss the unique solvability, stability and convergence for the finite difference schemes (13) and (19). Define the grid functions spaces $U_h = \{u|u=(...,u_{-2},\ u_{-1},\ u_0,\ u_1,\ u_2,...)\}$ and $U_h^0 = \{u|u\in V_h,\ \lim\limits_{|i|\to\infty}u_i = 0,\ \lim\limits_{|i|\to\infty}\delta_x u_{i-\frac{1}{2}} = 0\}$. For $u,\ v\in U_h^0$, define two discrete inner products as $(u,v) = h\sum\limits_{i=-\infty}^{\infty}u_i v_i$ and $(u,v)_{\widehat{\kappa}} = h\sum\limits_{i=-\infty}^{\infty}\widehat{\kappa}_i u_i v_i$, while the discrete L_2 norms are defined by $\|u\| = \sqrt{(u,u)} = (\sum\limits_{i=-\infty}^{\infty}h|u_i|^2)^{\frac{1}{2}}$ and $\|u\|_{\widehat{\kappa}} = \sqrt{(u,u)} = (\sum\limits_{i=-\infty}^{\infty}\widehat{\kappa}_i|u_i|^2)^{\frac{1}{2}}$ respectively.

If we set $\mu = \frac{\tau\Gamma(2-\gamma)}{2h^\alpha}$,

$$
\begin{cases}
\widehat{\kappa}(x) = \kappa(x),\ x\in(a,b),\\
\widehat{\kappa}(x) = 0,\ x\in(-\infty,a]\bigcup[b,\infty),
\end{cases}
$$

$$
\begin{cases}
\widehat{u}(x,t) = \frac{u(x,t)}{\kappa(x)},\ x\in(a,b),\\
\widehat{u}(x) = 0,\ x\in(-\infty,a]\bigcup[b,\infty),
\end{cases}
$$

$$
\begin{cases}
\widehat{f}(x,t) = \frac{f(x,t)}{\kappa(x)},\ x\in[a,b],\\
\widehat{f}(x) = 0,\ x\in(-\infty,a]\bigcup[b,\infty),
\end{cases}
$$

then the first equation of (13) can be rewritten as

$$
\begin{aligned}
&\mathcal{A}_1\mathcal{A}_2[(\Gamma(2-\gamma)+\tau^{1-\gamma}a_0^{(\gamma)})\widehat{u}_i^n - \Gamma(2-\gamma)\widehat{u}_i^{n-1} \\
&-\sum_{k=1}^{n-1}\tau^{1-\gamma}(a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})\widehat{u}_i^k - \tau^{1-\gamma}a_{n-1}^{(\gamma)}\widehat{u}_i^0] \\
&= \mu\sum_{k=-\infty}^{\infty}\lambda_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n + \tau\Gamma(2-\gamma)\mathcal{A}_1\mathcal{A}_2\widehat{f}_i^n, \\
&1\le n\le N,\ i=0,\pm1,\pm2,..., \qquad (20)
\end{aligned}
$$

Similarly, the first equation of (19) can also be rewritten as

$$
\begin{aligned}
&\mathcal{A}_1[(\Gamma(2-\gamma)+\tau^{1-\gamma}a_0^{(\gamma)})\widehat{u}_i^n - \Gamma(2-\gamma)\widehat{u}_i^{n-1} \\
&-\sum_{k=1}^{n-1}\tau^{1-\gamma}(a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})\widehat{u}_i^k - \tau^{1-\gamma}a_{n-1}^{(\gamma)}\widehat{u}_i^0] \\
&= \mu\sum_{k=-\infty}^{\infty}\widehat{\lambda}_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n + \tau^\gamma\Gamma(2-\gamma)\mathcal{A}_1\widehat{f}_i^n, \\
&1\le n\le N,\ i=0,\pm1,\pm2,..., \qquad (21)
\end{aligned}
$$

For the solutions of the difference schemes (13) and (19), as the function u is defined on the whole $\mathbf{R}$, and $u(x,t)\equiv 0$ for $x\in(-\infty,a]\bigcup[b,\infty)$, then $\|\widehat{u}\|$ and $\|\widehat{u}\|_{\widehat{\kappa}}$ exist, and furthermore we have the following lemmas.

Lemma 3 [41, Lemma 2.1.1]. For the solutions of the difference schemes (20) and (21), it holds that

$$
\begin{cases}
\frac{\sqrt{6}}{(b-a)}\|\widehat{u}^n\| \le \|\delta_x\widehat{u}^n\| \le \frac{2}{h}\|\widehat{u}^n\|,\\
\frac{6}{(b-a)^2}\|\widehat{u}^n\| \le \frac{\sqrt{6}}{(b-a)}\|\delta_x\widehat{u}^n\| \le \|\delta_x^2\widehat{u}^n\|\\
\qquad = \|\delta_x\delta_x\widehat{u}^n\| \le \frac{2}{h}\|\delta_x\widehat{u}^n\| \le \frac{4}{h^2}\|\widehat{u}^n\|,
\end{cases}
$$

Lemma 4. Let the operators $\mathcal{A}_1$, $\mathcal{A}_2$ are defined as above, then for the solutions of the difference schemes (20) and (21), we have

$$
\begin{cases}
[\frac{(1+\eta)\eta}{6} + \frac{36c_2^\beta h^4}{12(b-a)^4}]\|\widehat{u}^n\|^2\\
\qquad \le (\mathcal{A}_1\mathcal{A}_2\widehat{u}^n,\widehat{u}^n) \le (1+\frac{4c_2^\beta}{3})\|\widehat{u}^n\|^2,\\
\frac{1}{3}\|\widehat{u}^n\|^2 \le \frac{\eta^2+\eta+2}{6}\|\widehat{u}^n\|^2 \le (1-4c_2^\beta)\|\widehat{u}^n\|^2\\
\qquad \le (\mathcal{A}_1\widehat{u}^n,\widehat{u}^n) \le \|\widehat{u}^n\|^2.
\end{cases} \qquad (22)
$$

Proof. Since $\mathcal{A}_1\mathcal{A}_2 = (1+c_2^\beta h^2\delta_x^2)(1+\frac{1}{12}h^2\delta_x^2) = 1 + c_2^\beta h^2\delta_x^2 + \frac{1}{12}h^2\delta_x^2 + \frac{c_2^\beta h^4}{12}\delta_x^2\delta_x^2$, by use of the discrete Green formula one can obtain that

$$
\begin{aligned}
&(\mathcal{A}_1\mathcal{A}_2\widehat{u}^n,\widehat{u}^n) = (\widehat{u}^n,\widehat{u}^n) + c_2^\beta h^2(\delta_x^2\widehat{u}^n,\widehat{u}^n) \\
&\quad + \frac{1}{12}h^2(\delta_x^2\widehat{u}^n,\widehat{u}^n) + \frac{c_2^\beta h^4}{12}(\delta_x^2\delta_x^2\widehat{u}^n,\widehat{u}^n) \\
&= \|\widehat{u}^n\|^2 - (c_2^\beta+\frac{1}{12})h^2(\delta_x\widehat{u}^n,\delta_x\widehat{u}^n) + \frac{c_2^\beta h^4}{12}(\delta_x^2\widehat{u}^n,\delta_x^2\widehat{u}^n) \\
&= \|\widehat{u}^n\|^2 - (c_2^\beta+\frac{1}{12})h^2\|\delta_x\widehat{u}^n\|^2 + \frac{c_2^\beta h^4}{12}\|\delta_x^2\widehat{u}^n\|^2
\end{aligned}
$$

and

$$(\mathcal{A}_1\widehat{u}^n,\widehat{u}^n)=(\widehat{u}^n,\widehat{u}^n)+c_2^\beta h^2(\delta_x^2\widehat{u}^n,\widehat{u}^n)$$

$$=\|\widehat{u}^n\|^2-c_2^\beta h^2(\delta_x\widehat{u}^n,\delta_x\widehat{u}^n)$$
$$=\|\widehat{u}^n\|^2-c_2^\beta h^2\|\delta_x\widehat{u}^n\|^2.$$

Considering $c_2^\beta \in (\frac{1}{12},\frac{-(1+\eta)^2+(1+\eta)+4}{24}]$, by use of Lemma 3 we can obtain the desired results.

Remark 4. According to Lemma 4, for $u,\ v\in U_h^0$, we can define another two discrete inner products as $(u,v)_{\mathcal{A}_1\mathcal{A}_2}=h\sum\limits_{i=-\infty}^{\infty}(\mathcal{A}_1\mathcal{A}_2u_i)v_i$ and $(u,v)_{\mathcal{A}_1}=h\sum\limits_{i=-\infty}^{\infty}(\mathcal{A}_1u_i)v_i$, while the discrete norms are defined by $\|u\|_{\mathcal{A}_1\mathcal{A}_2}=(\mathcal{A}_1\mathcal{A}_2u,u)$ and $\|u\|_{\mathcal{A}_1}=(\mathcal{A}_1u,u)$ respectively. Furthermore, $\|u\|_{\mathcal{A}_1\mathcal{A}_2}$ and $\|u\|_{\mathcal{A}_1}$ are all equivalent to $\|u\|$.

Lemma 5. If $u\in U_h^0$, then for any integer k, it holds that

$$\sum_{i=-\infty}^{\infty}u_{i-k}u_i=\sum_{i=-\infty}^{\infty}u_{i+k}u_i.$$

Proof. Setting $j=i-k$, we have $\sum\limits_{i=-\infty}^{\infty}v_{i-k}^nv_i^n=\sum\limits_{j=-\infty}^{\infty}v_j^nv_{j+k}^n=\sum\limits_{i=-\infty}^{\infty}v_{i+k}^nv_i^n$, and the proof is complete.

Lemma 6. For the solutions difference schemes (20) and (21), it holds that $\sum\limits_{i=-\infty}^{\infty}[\sum\limits_{k=-\infty}^{\infty}\lambda_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n\widehat{u}_i^n]=0$ and $\sum\limits_{i=-\infty}^{\infty}[\sum\limits_{k=-\infty}^{\infty}\widehat{\lambda}_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n\widehat{u}_i^n]=0$.

Proof. By use of lemma 5 one can deduce that $\sum\limits_{i=-\infty}^{\infty}[\sum\limits_{k=-\infty}^{\infty}\lambda_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n\widehat{u}_i^n]=\sum\limits_{k=-\infty}^{\infty}[\sum\limits_{i=-\infty}^{\infty}\lambda_{k}^{(\alpha)}\widehat{\kappa}_{i-k}^n\widehat{u}_{i-k}^n\widehat{u}_i^n]$

$$=\sum_{k=-\infty}^{-1}[\sum_{i=-\infty}^{\infty}\lambda_k^{(\alpha)}\widehat{\kappa}_{i-k}^n\widehat{u}_{i-k}^n\widehat{u}_i^n]$$
$$+\sum_{k=1}^{\infty}[\sum_{i=-\infty}^{\infty}\lambda_k^{(\alpha)}\widehat{\kappa}_{i-k}^n\widehat{u}_{i-k}^n\widehat{u}_i^n]+\sum_{i=-\infty}^{\infty}\lambda_0^{(\alpha)}\widehat{\kappa}_i^n\widehat{u}_i^n\widehat{u}_i^n$$
$$=\sum_{k=-\infty}^{-1}[\sum_{i=-\infty}^{\infty}\lambda_k^{(\alpha)}\widehat{\kappa}_{i+k}^n\widehat{u}_{i+k}^n\widehat{u}_i^n]$$
$$+\sum_{k=1}^{\infty}[\sum_{i=-\infty}^{\infty}\lambda_k^{(\alpha)}\widehat{\kappa}_{i-k}^n\widehat{u}_{i-k}^n\widehat{u}_i^n]$$
$$=\sum_{k=1}^{\infty}[\sum_{i=-\infty}^{\infty}\lambda_{-k}^{(\alpha)}\widehat{\kappa}_{i-k}^n\widehat{u}_{i-k}^n\widehat{u}_i^n]$$
$$+\sum_{k=1}^{\infty}[\sum_{i=-\infty}^{\infty}\lambda_k^{(\alpha)}\widehat{\kappa}_{i-k}^n\widehat{u}_{i-k}^n\widehat{u}_i^n]$$
$$=0.$$

Similarly we also have $\sum\limits_{i=-\infty}^{\infty}[\sum\limits_{k=-\infty}^{\infty}\widehat{\lambda}_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n\widehat{u}_i^n]=0$. The proof is complete.

A. Unique solvability

First we analyze the unique solvability of the difference scheme (13). For the sake of proving the unique solvability, we only need to prove that there is only zero solution for the corresponding homogeneous difference equation of (20), which is denoted as follows due to $a_0^\gamma=0$

$$\mathcal{A}_1\mathcal{A}_2[(\Gamma(2-\gamma)+\tau^{1-\gamma})\widehat{u}_i^n]=\mu\sum_{k=-\infty}^{\infty}\lambda_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n. \quad (23)$$

Theorem 1. The difference scheme denoted by (13) is uniquely solvable.

proof. Multiplying $h\widehat{u}_i^n$ on both sides of Eq. (23) and a summation with respect to i from $-\infty$ to ∞ yields that

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^n\|_{\mathcal{A}_1\mathcal{A}_2}^2=$$
$$\mu\sum_{i=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty}\lambda_{i-k}^{(\alpha)}\widehat{\kappa}_k\widehat{u}_k^n\widehat{u}_i^n]=0,$$

where Lemma 6 is used in the deduction above. Therefore, $\|\widehat{u}^n\|_{\mathcal{A}_1\mathcal{A}_2}=0$, and according to Lemma 4 and Remark 4 one has $\|\widehat{u}^n\|=0$. So $\widehat{u}_i^n=0,\ i=1,2,...,M-1$, which implies that $u_i^n=0,\ i=1,2,...,M-1$. Then there is only zero solution for (23), which implies (13) is uniquely solvable. The proof is complete.

Following in a similar proof process one can obtain the following theorem:

Theorem 2. The difference scheme denoted by (19) is also uniquely solvable.

B. Stability

Theorem 3. The difference scheme denoted by (13) is unconditionally stable on the initial value and the the right term f.

Proof. Multiplying $h\widehat{u}_i^n$ on both sides (20) and a summation with respect to i from $-\infty$ to ∞, together with use of Lemma 6 one can deduce that

$$h\sum_{i=-\infty}^{\infty}\{\mathcal{A}_1\mathcal{A}_2[(\Gamma(2-\gamma)+\tau^{1-\gamma}a_0^{(\gamma)})\widehat{u}_i^n-\Gamma(2-\gamma)\widehat{u}_i^{n-1}$$
$$-\sum_{k=1}^{n-1}\tau^{1-\gamma}(a_{n-k-1}^{(\gamma)}-a_{n-k}^{(\gamma)})\widehat{u}_i^k-\tau^{1-\gamma}a_{n-1}^{(\gamma)}\widehat{u}_i^0]\widehat{u}_i^n\}$$
$$=\tau\Gamma(2-\gamma)h\sum_{i=-\infty}^{\infty}(\mathcal{A}_1\mathcal{A}_2\widehat{f}_i^n)\widehat{u}_i^n,$$

which implies that

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^n\|_{\mathcal{A}_1\mathcal{A}_2}^2$$
$$=\sum_{k=1}^{n-1}[\tau^{1-\gamma}(a_{n-k-1}^{(\gamma)}-a_{n-k}^{(\gamma)})(\mathcal{A}_1\mathcal{A}_2\widehat{u}^k,\widehat{u}^n)]$$
$$+\tau^{1-\gamma}a_{n-1}^{(\gamma)}(\mathcal{A}_1\mathcal{A}_2\widehat{u}^0,\widehat{u}_i^n)$$
$$+\Gamma(2-\gamma)(\mathcal{A}_1\mathcal{A}_2\widehat{u}^{n-1},\widehat{u}_i^n)+\tau\Gamma(2-\gamma)(\mathcal{A}_1\mathcal{A}_2\widehat{f}^n,\widehat{u}^n)$$
$$\le\frac{1}{2}\sum_{k=1}^{n-1}[\tau^{1-\gamma}(a_{n-k-1}^{(\gamma)}-a_{n-k}^{(\gamma)})(\|\widehat{u}^k\|_{\mathcal{A}_1\mathcal{A}_2}^2+\|\widehat{u}^n\|_{\mathcal{A}_1\mathcal{A}_2}^2)]$$
$$+\frac{1}{2}\tau^{1-\gamma}a_{n-1}^{(\gamma)}(\|\widehat{u}^0\|_{\mathcal{A}_1\mathcal{A}_2}^2+\|\widehat{u}^n\|_{\mathcal{A}_1\mathcal{A}_2}^2)$$

$$+\frac{\Gamma(2-\gamma)}{2}(\|\widehat{u}^{n-1}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2})$$
$$+\tau\Gamma(2-\gamma)[\frac{\tau(1+\tau^{1-\gamma})}{2\tau\Gamma(2-\gamma)(1+\tau)}\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\frac{2\tau\Gamma(2-\gamma)(1+\tau)}{4\tau(1+\tau^{1-\gamma})}\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}].$$

Furthermore, one can deduce that from above

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$\leq\sum_{k=1}^{n-1}\tau^{1-\gamma}(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})\|\widehat{u}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\tau^{1-\gamma}a^{(\gamma)}_{n-1}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\Gamma(2-\gamma)\|\widehat{u}^{n-1}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+2\tau\Gamma(2-\gamma)[\frac{\tau(\Gamma(2-\gamma)+\tau^{1-\gamma})}{2\tau\Gamma(2-\gamma)(1+\tau)}\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\frac{2\tau\Gamma(2-\gamma)(1+\tau)}{4\tau(\Gamma(2-\gamma)+\tau^{1-\gamma})}\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}],$$

and

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$\leq\sum_{k=1}^{n-1}\tau^{1-\gamma}(1+\tau)(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})\|\widehat{u}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\tau^{1-\gamma}(1+\tau)a^{(\gamma)}_{n-1}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}+(1+\tau)\Gamma(2-\gamma)\|\widehat{u}^{n-1}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\frac{[\Gamma(2-\gamma)]^2}{(\Gamma(2-\gamma)+\tau^{1-\gamma})}\tau(1+\tau)^2\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}],$$
$$\leq\sum_{k=1}^{n-1}\tau^{1-\gamma}(1+\tau)(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})\|\widehat{u}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)\Gamma(2-\gamma)\|\widehat{u}^{n-1}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\tau^{1-\gamma}(1+\tau)a^{(\gamma)}_{n-1}$$
$$\{\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\frac{\Gamma(2-\gamma)}{a^{(\gamma)}_{n-1}}\tau^{\gamma}(1+\tau)\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}]\}.$$

Since $(1-\gamma)n^{-\gamma}<a^{(\gamma)}_{n-1}$ according to Lemma 2, then furthermore we have

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$\leq\sum_{k=1}^{n-1}\tau^{1-\gamma}(1+\tau)(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})\|\widehat{u}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)\Gamma(2-\gamma)\|\widehat{u}^{n-1}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\tau^{1-\gamma}(1+\tau)a^{(\gamma)}_{n-1}$$
$$\{\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\Gamma(1-\gamma)t^{\gamma}_{n}(1+\tau)\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}]\}. \quad (24)$$

Now we prove the following inequality by use of the mathematical induction method

$$\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}\leq(1+\tau)^{n}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)^{n+1}\Gamma(1-\gamma)t^{\gamma}_{n}\max_{1\leq k\leq n}\|\widehat{f}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2},\ n\geq 1. \quad (25)$$

If $n=1$, then from (24) one can derive that

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^{1}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$\leq(1+\tau)\Gamma(2-\gamma)\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}+\tau^{1-\gamma}(1+\tau)\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\tau^{1-\gamma}\Gamma(1-\gamma)t^{\gamma}_{1}(1+\tau)^2\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2},$$

which implies

$$\|\widehat{u}^{1}\|^2_{\mathcal{A}_1\mathcal{A}_2}\leq(1+\tau)\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)^2\frac{\tau^{1-\gamma}\Gamma(1-\gamma)}{(\Gamma(2-\gamma)+\tau^{1-\gamma})}t^{\gamma}_{1}\|\widehat{f}^{1}\|^2_{\mathcal{A}_1\mathcal{A}_2},$$
$$\leq(1+\tau)\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)^2\Gamma(1-\gamma)t^{\gamma}_{1}\|\widehat{f}^{1}\|^2_{\mathcal{A}_1\mathcal{A}_2},$$

and then (25) holds for $n=1$.

Suppose (25) holds for the levels $1,2,...,n-1$, then for the level n, by (24) one can obtain that

$$(\Gamma(2-\gamma)+\tau^{1-\gamma})\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$\leq\tau^{1-\gamma}(1+\tau)\sum_{k=1}^{n-1}(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})\{(1+\tau)^{k}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)^{k+1}\Gamma(1-\gamma)t^{\gamma}_{k}\|\widehat{f}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}\}$$
$$+(1+\tau)^{n-1}\Gamma(2-\gamma)\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(1+\tau)^{n}\Gamma(2-\gamma)\Gamma(1-\gamma)t^{\gamma}_{n-1}\|\widehat{f}^{n-1}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\tau^{1-\gamma}(1+\tau)a^{(\gamma)}_{n-1}\{\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\Gamma(1-\gamma)(1+\tau)t^{\gamma}_{n}\|\widehat{f}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}\}$$
$$\leq(1+\tau)^{n}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}\{\tau^{1-\gamma}[\sum_{k=1}^{n-1}(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})$$
$$+a^{(\gamma)}_{n-1}]+\Gamma(2-\gamma)\}+(1+\tau)^{n+1}\Gamma(1-\gamma)t^{\gamma}_{n}\max_{1\leq k\leq n}$$
$$\|\widehat{f}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}\{\tau^{1-\gamma}[\sum_{k=1}^{n-1}(a^{(\gamma)}_{n-k-1}-a^{(\gamma)}_{n-k})+a^{(\gamma)}_{n-1}]+\Gamma(2-\gamma)\}$$
$$=(\Gamma(2-\gamma)+\tau^{1-\gamma})(1+\tau)^{n}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+(\Gamma(2-\gamma)+\tau^{1-\gamma})(1+\tau)^{n+1}\Gamma(1-\gamma)t^{\gamma}_{n}\max_{1\leq k\leq n}\|\widehat{f}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2},$$

which implies (25) holds. So (25) always holds according to the the mathematical induction method.

Moreover, from (25) one can deduce that

$$\|\widehat{u}^{n}\|^2_{\mathcal{A}_1\mathcal{A}_2}\leq\exp^{n\tau}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+\exp^{(n+1)\tau}\Gamma(1-\gamma)t^{\gamma}_{n}\max_{1\leq k\leq n}\|\widehat{f}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$\leq\exp^{T}\|\widehat{u}^{0}\|^2_{\mathcal{A}_1\mathcal{A}_2}$$
$$+T^{\gamma}\exp^{2T}\Gamma(1-\gamma)\max_{1\leq k\leq n}\|\widehat{f}^{k}\|^2_{\mathcal{A}_1\mathcal{A}_2}. \quad (26)$$

From (26) one can see that the solution $\widehat{u}^{n}$ of Eq. (20) depends continuously on the initial value $\widehat{u}^{0}$ and the right term $\widehat{f}$. So the difference scheme (20) is unconditionally stable, and furthermore, the difference scheme (13) is also unconditionally stable on the initial value and the right term f. The proof is complete.

Similarly we have the following theorem:

Theorem 4. The difference scheme denoted by (19) is unconditionally stable on the initial value and the right term f.

C. Convergence

Theorem 5. The difference scheme denoted by (13) is convergent.

Proof. Let $\varepsilon^n = \widehat{u}^n - U^n,\ n = 0, 1, ..., N$ denotes the errors between the numerical solutions and the exact solutions. Then from (12), (13) and (20) we have

$$\begin{cases} \mathcal{A}_1\mathcal{A}_2[(\Gamma(2-\gamma)+\tau^{1-\gamma}a_0^{(\gamma)})\varepsilon_i^n - \Gamma(2-\gamma)\varepsilon_i^{n-1} \\ -\sum\limits_{k=1}^{n-1}\tau^{1-\gamma}(a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)})\varepsilon_i^k - \tau^{1-\gamma}a_{n-1}^{(\gamma)}\varepsilon_i^0] = \\ \mu\sum\limits_{k=-\infty}^{\infty}\lambda_{i-k}^{(\alpha)}\widehat{\kappa}_k\varepsilon_k^n + \tau^\gamma\Gamma(2-\gamma)\mathcal{A}_1\mathcal{A}_2R(\tau,h), \\ \quad 1 \le n \le N,\ i = 0, \pm1, \pm2, ..., \\ \varepsilon_i^0 = 0,\ i = 0, \pm1, \pm2, ..., \end{cases} \tag{27}$$

where $\mathcal{A}_1\mathcal{A}_2R(\tau,h) = O(\tau + \tau^{2-\gamma} + h^2)$.

Similar to the proof of Theorem 3 one has that

$$\|\varepsilon^n\|_{\mathcal{A}_1\mathcal{A}_2}^2 \le \exp^T\|\varepsilon^0\|_{\mathcal{A}_1\mathcal{A}_2}^2$$

$$+T^\gamma\exp^{2T}\Gamma(1-\gamma)\|R(\tau,h)\|_{\mathcal{A}_1\mathcal{A}_2}^2$$

$$= T^\gamma\exp^{2T}\Gamma(1-\gamma)\|R(\tau,h)\|_{\mathcal{A}_1\mathcal{A}_2}^2,$$

which implies that

$$\|\varepsilon^n\|_{\mathcal{A}_1\mathcal{A}_2} \le T^{\frac{\gamma}{2}}\exp^T\sqrt{\Gamma(1-\gamma)}\|R(\tau,h)\|_{\mathcal{A}_1\mathcal{A}_2}.$$

Furthermore, according to Lemma 4 and Remark 4, there exist three positive constants $C_1,\ C_2,\ C_3$ such that

$$\|\varepsilon^n\| \le C_1\tau + C_2\tau^{2-\gamma} + C_3h^3.$$

So $\lim\limits_{\tau,h\to0}\|\varepsilon^n\| = 0$. The proof is complete.

Similarly we have the following theorem:

Theorem 6. The difference scheme denoted by (19) is also convergent.

IV. NUMERICAL EXPERIMENTS

In this section, we propose one numerical example for the present difference schemes (13) and (19).

Consider the problem (1) with an exact analytical solution

$$u(x,t) = \begin{cases} (t+1)x^2(1-x)^2,\ x \in (0,1), \\ 0,\ x \in (-\infty,0]\bigcup[1,\infty), \end{cases}$$

and satisfies

$$\begin{cases} \kappa(x) = x^3(1-x)^3, \\ f(x,t) = \begin{cases} x^2(1-x)^2[1+\dfrac{t^{1-\beta}}{\Gamma(2-\beta)}] - \\ \sum\limits_{m=2}^{4}[\dfrac{c_m m! x^{-\alpha+m}}{\Gamma(1-\alpha+m)} - \dfrac{c_m m!(1-x)^{-\alpha+m}}{\Gamma(1-\alpha+m)}] \\ (t+1)x^3(1-x)^3, \qquad \alpha \in (2,3), \\ x^2(1-x)^2[1+\dfrac{t^{1-\beta}}{\Gamma(2-\beta)}] - \\ \sum\limits_{m=2}^{4}[\dfrac{c_m m! x^{-\alpha+m}}{\Gamma(1-\alpha+m)} - \dfrac{c_m m!(1-x)^{-\alpha+m}}{\Gamma(1-\alpha+m)}] \\ (t+1)x^3(1-x)^3, \qquad \alpha \in (3,4), \end{cases} \\ u(x,0) = h(x) = x^2(1-x)^2, \end{cases}$$

where $x^2(1-x)^2 = \sum\limits_{m=2}^{4}c_m x^m$.

Let $\|e_1\| = \sqrt{\sum\limits_{i=1}^{M-1}h|U_i^n - u_i^n|^2}$ and $\|e_2\| = \sqrt{\sum\limits_{i=1}^{M-1}h|\dfrac{U_i^n - u_i^n}{U_i^n}\times100|^2}$ denote the absolute error and the relative error in L_2 norm respectively.

In Figs. 1-2 and Tables 1-2, the errors between the numerical solutions and the exact solutions are shown under certain conditions, while in Figs. 3-4, comparison between the exact solutions and the numerical solutions is demonstrated under certain selected parameters..

Table 1: The absolute errors and relative errors for the difference scheme (13) at $\beta = 0.5,\ \tau = 10^{-3},\ t = 0.05$

	$\alpha = 2.3$		$\alpha = 2.5$	
h	$\|e_1\|$	$\|e_2\|$	$\|e_1\|$	$\|e_2\|$
$\frac{1}{6}$	5.2081×10^{-4}	1.1454	7.9389×10^{-4}	1.8180
$\frac{1}{8}$	3.9798×10^{-4}	0.9359	5.0502×10^{-4}	1.2962
$\frac{1}{10}$	2.7637×10^{-4}	0.8926	2.7637×10^{-4}	0.8926
$\frac{1}{12}$	1.7577×10^{-4}	0.5491	1.7236×10^{-4}	0.6562
$\frac{1}{14}$	1.1364×10^{-4}	0.4228	1.7069×10^{-4}	0.5615

Table 2: The absolute errors and relative errors for (19) at $\beta = 0.8,\ h = \frac{1}{6}$ after 50 time steps

	$\alpha = 3.3$		$\alpha = 3.5$	
τ	$\|e_1\|$	$\|e_2\|$	$\|e_1\|$	$\|e_2\|$
1×10^{-5}	7.7863×10^{-5}	0.1549	8.4678×10^{-5}	0.2775
2×10^{-5}	1.5316×10^{-4}	0.3037	1.6571×10^{-4}	0.5403
3×10^{-5}	2.2797×10^{-4}	0.4505	2.4576×10^{-4}	0.7952
4×10^{-5}	3.0282×10^{-4}	0.5964	3.2567×10^{-4}	1.0431
5×10^{-5}	3.7798×10^{-4}	0.7420	4.0595×10^{-4}	1.2843

From Figs. 1-2 one can see that the absolute errors and relative errors can be bounded to a low level, and do not increase sharply with the time steps increase, which illustrate the stability of the present difference schemes. The results of Tables 1-2 show that the absolute and relative errors can be restricted to a accepted level even with large spatial time step size. Figs. 3-4 show that the numerical solutions can approximate the exact solutions satisfactorily.

V. CONCLUSIONS

In this paper, we have proposed two unconditionally stable compact finite difference schemes by use of a combination of the order reduction method and the weighted shifted

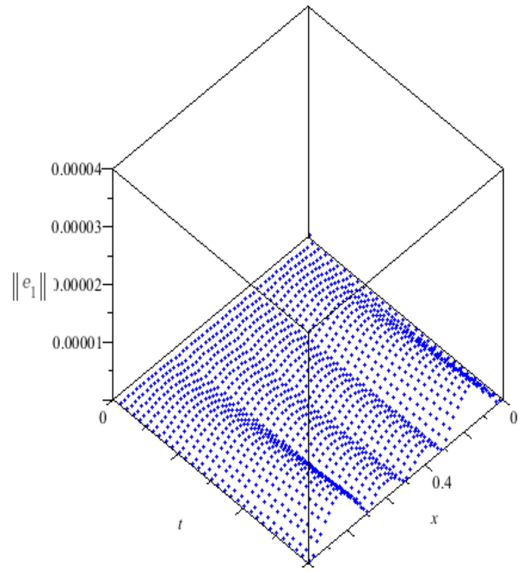

Fig 1. The absolute errors with h=1/51;τ=0.001;α=2.1;β=0.5

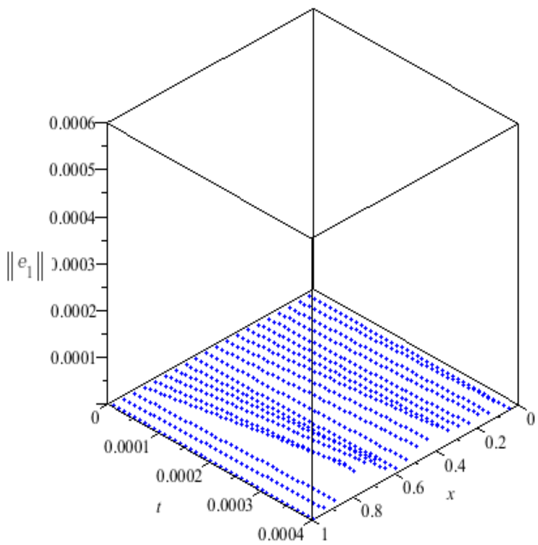

Fig 2. The absolute errors with h=1/21;τ=0.00001;α=3.2;β=0.7

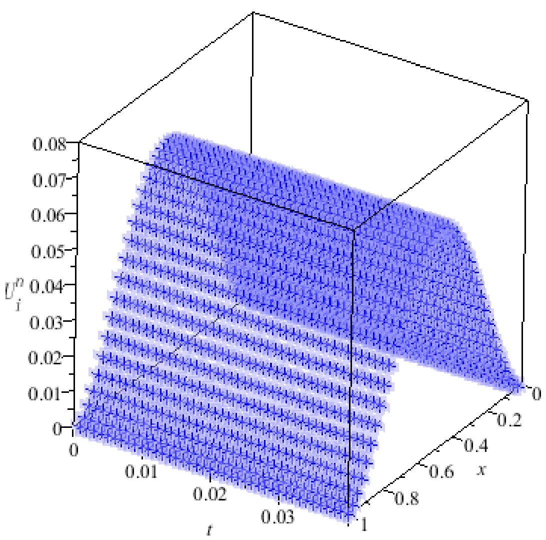

Fig 3. The exact solutions with h=1/21;τ=0.001;α=2.7;β=0.7

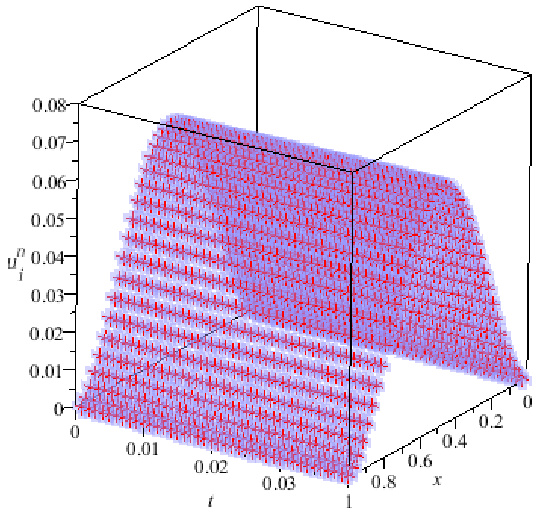

Fig 4. The numerical solutions with h=1/21;τ=0.001;α=2.7;β=0.7

Grünwald-Letnikov derivative approximation formulas for a class f space-time fractional differential equations with the order of the spatial fractional derivative more than two. Analysis of unique solvability, stability and convergence in L_2 norm for the two difference schemes are fulfilled. For testing the validity of the present difference schemes, numerical experiments are carried out, and the numerical results show their coincidence with the theoretical analysis.

Finally, further research can be done based on the proposed method in this paper.

(1) How to improve the accuracy of the difference schemes in both time and spatial directions.

(2) How to derive stable difference schemes with high accuracy for other types of fractional differential equations including multi-term time fractional differential equations, space-time fractional diffusion equations with time distributed-order derivative and so on.

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