Some New Fractional Integral Inequalities in the Sense of Conformable Fractional Derivative

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Abstract—**In the paper, basing on the definitions of the conformable fractional derivative and integral as well as the properties of fractional calculus, the authors present some new fractional integral inequalities, from which explicit bounds for concerned but unknown functions are derived. Basing on these inequalities, the authors also establish Volterra-Fredholm type fractional integral inequalities. These inequalities generalize some existing results in the literature and can be used in the research of certain qualitative properties such as boundedness and continuous dependence on the initial value of solutions of fractional differential equations. The authors also present some applications of the main results.**

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I. Introduction

Fractional differential equations(FDEs) are widely used in various domains including engineering, physics, biology, signal processing, systems identification, control theory, finance, fractional dynamics and so on [1-3]. In particular, the fractional derivative has proved to be very useful in describing the memory and hereditary properties of materials and processes. One of its most important applications is to model the process of subdiffusion and superdiffusion of particles in physics, where the fractional diffusion equation is usually used for modeling this movement. Recently, various aspects for fractional diffusion equations have been researched by many authors. In [4,5], the authors proposed two effective methods for finding numerical solutions of fractional differential equations, while in [6,7], the authors proposed certain methods for finding analytical solutions or numerical solutions of fractional differential equations. In [8-10], qualitative and quantitative properties of solutions of several fractional differential equations were investigated.

In the research of FDEs, seeking solutions of FDEs are a hot topic, and have been paid much attention by many authors. However, in most cases, it is difficult to obtain exact solutions for FDEs due to the complexity of fractional operators and fractional calculus. Thus, it becomes very important to research qualitative and quantitative properties of solutions of FDEs.

It is well known that inequalities especially the Gronwall-Bellman type inequalities play an important role in the research of qualitative and quantitative properties of solutions of differential and integral equations, difference equations and dynamic equations on time scales. During the past decades, a lot of various differential and integral inequalities (see [11-18] for example), difference inequalities [19-20], dynamic inequalities on time scales [21-26] have been established. These inequalities play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations, difference equations as well as dynamic equations on time scales. In the investigation of various inequalities, some fractional differential and integral inequalities have been established (see [27-31] for example), which have proved to be very useful in the research of solutions of certain fractional differential and integral equations. However, for some fractional differential and integral equations as well as some Volterra-Fredholm type fractional equations with more complicate forms, for example,

$$D_t^\alpha u(t) = f_1(t)[\omega_1(u(t)) + I^\alpha(f_2(t)\omega_2(u(t)))]^p, \ t \geq 0,$$

the existing inequalities in the literature is inadequate for the research of qualitative properties of solutions of them, and it is necessary to establish new fractional differential and integral inequalities so as to obtain the desired results.

In this paper, based on some basic properties of the conformable fractional derivative and fractional calculus, we establish some new fractional integral inequalities, and based on them, present some new Volterra-Fredholm type fractional integral inequalities. The present inequalities can be used to research qualitative properties of solutions of some fractional differential and integral equations with certain forms.

Definition 1 [32, Definition 2.1]. The conformable fractional derivative of order α is defined by
$$D_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

Definition 2 [32, Definition 3.1]. The conformable fractional integral of order α on the interval $[0, t]$ is defined by

$$I^\alpha f(t) = \int_0^t s^{\alpha-1} f(s) ds.$$

The following properties can be easily proved due to the definition of the conformable fractional derivative and the conformable fractional integral:

(i) $D^\alpha[af(t) + bg(t)] = aD^\alpha f(t) + bD^\alpha g(t).$

(ii) $D^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}.$

(iii) $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t).$

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(iv) $D^\alpha C = 0$, where C is a constant.

(v) $D_t^\alpha f[g(t)] = f_g'[g(t)] D_t^\alpha g(t)$.

(vi) $D_t^\alpha (\frac{f}{g})(t) = \frac{g(t) D^\alpha f(t) - f(t) D^\alpha g(t)}{g^2(t)}$.

(vii) $D_t^\alpha f(t) = t^{1-\alpha} f'(t)$.

$(viii)$ $D_t^\alpha (I^\alpha f(t)) = f(t)$

(ix) $I_t^\alpha (D_t^\alpha f(t)) = f(t) - f(0)$ on the interval $[0, t]$.

Many authors have investigated various applications of the conformable fractional derivative (see [33-36] for example).

The paper is organized as follows. In Section 2, we present the main inequalities, and derive explicit bounds for unknown functions by use of these inequalities. In Section 3, for illustrating the validity of the established results, we apply them to research boundedness and continuous dependence on initial value for the solutions of certain fractional differential equations. Some conclusions are given at the end of the paper.

II. Main Results

First we consider the following fractional integral inequality

$$u(t) \le a(t) + b(t) \int_0^t s^{\alpha-1} f(s) [\omega_1(u(s)) + \int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi]^p ds, \ t \ge 0, \tag{1}$$

that is,

$$u(t) \le a(t) + b(t) I^\alpha \{ f(t) [\omega_1(u(t)) + I^\alpha (g(t) \omega_2(u(t)))]^p \}, \ t \ge 0, \tag{2}$$

where $0 < \alpha < 1$, the functions $u, \ f, \ g, \ a, \ b, \ \omega_1, \ \omega_2$ are nonnegative continuous functions defined on $t \ge 0$ with $a, \ b, \ \omega_1, \ \omega_2$ nondecreasing, $p > 0$ is a constant.

Theorem 1. Define $J(v) = \int_1^v \frac{1}{[\omega_1(r) + \omega_2(r)]^p} dr$, and assume $J(v) < \infty$ for $v < \infty$. If the inequality (1) satisfies, then one has the following explicit estimate for $u(t)$:

$$u(t) \le J^{-1} \{ J(a(t)) + b(t) \int_0^t s^{\alpha-1} f(s) [1 + \int_0^s \xi^{\alpha-1} g(\xi) d\xi]^p ds \}, \ t \ge 0. \tag{3}$$

Proof. Fix $T \ge 0$, and let $t \in [0, T]$. Set

$$v(t) = a(T) + b(T) \int_0^t s^{\alpha-1} f(s) [\omega_1(u(s)) + \int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi]^p ds.$$
$$= a(T) + b(T) I^\alpha \{ f(t) [\omega_1(u(t)) + I^\alpha (g(t) \omega_2(u(t)))]^p \}.$$

Then

$$u(t) \le v(t), \ t \in [0, T]. \tag{4}$$

Since $u, \ f, \ g, \ \omega_1, \ \omega_2$ are continuous, then there exists a positive number A_1 such that $|g(\xi) \omega_2(u(\xi))| \le A_1$ for $\xi \in [0, s], \ s \in [0, t]$ and $t \in [0, \varepsilon]$, where $\varepsilon > 0$. So for $t \in [0, \varepsilon]$, one has $| \int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi| \le A_1 \int_0^s \xi^{\alpha-1} d\xi = \frac{A_1 s^\alpha}{\alpha}$, which implies the integral $\int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi$ is convergent. So there exists a positive number A_2 such that $|f(s)[\omega_1(u(s)) + \int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi]^p| \le A_2$. Then one has

$$a(T) \le v(t) = a(T) + b(T) \int_0^t s^{\alpha-1} f(s) [\omega_1(u(s)) + \int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi]^p ds$$
$$\le a(T) + A_2 b(T) \int_0^t s^{\alpha-1} ds$$
$$= a(T) + \frac{A_2 b(T) t^\alpha}{\alpha}.$$

From above one can see $v(0) = a(T)$. On the other hand, there also exists a nonnegative number A_3 such that $|f(s)[\omega_1(u(s)) + \int_0^s \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi]^p| \ge A_3$ for $s \in [0, t]$. So combining with the uniformly convergence of the integral denoted by $v(t)$, one can obtain $v(t) \ge b(T) A_3 \int_0^t s^{\alpha-1} ds = \frac{b(T) A_3 t^\alpha}{\alpha}$. Furthermore, $v'(t) \ge b(T) A_3 t^{\alpha-1} \ge 0$ for $t > 0$, which implies $v(t)$ is nondecreasing for $t \ge 0$. So under the condition that $\omega_1, \ \omega_2$ are nondecreasing, by use of the properties (iv) and $(viii)$, one can obtain

$$D_t^\alpha v(t) = b(T) f(t) [\omega_1(u(t)) + \int_0^t \xi^{\alpha-1} g(\xi) \omega_2(u(\xi)) d\xi]^p$$
$$\le b(T) f(t) [\omega_1(v(t)) + \int_0^t \xi^{\alpha-1} g(\xi) \omega_2(v(\xi)) d\xi]^p$$
$$\le b(T) f(t) [\omega_1(v(t)) + \omega_2(v(t)) \int_0^t \xi^{\alpha-1} g(\xi) d\xi]^p$$
$$\le b(T) f(t) [1 + \int_0^t \xi^{\alpha-1} g(\xi) d\xi]^p [\omega_1(v(t)) + \omega_2(v(t))]^p,$$

which implies

$$\frac{D_t^\alpha v(t)}{[\omega_1(v(t)) + \omega_2(v(t))]^p} \le b(T) f(t) [1 + \int_0^t \xi^{\alpha-1} g(\xi) d\xi]^p.$$

By use of the property (v), one can obtain

$$D_t^\alpha J(v(t)) = \frac{D_t^\alpha v(t)}{[\omega_1(v(t)) + \omega_2(v(t))]^p} \le b(T) f(t) [1 + \int_0^t \xi^{\alpha-1} g(\xi) d\xi]^p, \tag{5}$$

Substituting t with s, fulfilling fractional integral of order α for (5) with respect to s from 0 to t, together with the use of the property (ix), one can deduce

$$J(v(t)) - J(v(0)) \le b(T) \int_0^t s^{\alpha-1} f(s) [1 + \int_0^s \xi^{\alpha-1} g(\xi) d\xi]^p ds,$$

which implies

$$v(t) \le J^{-1} \{ J(a(T)) + b(T) \int_0^t s^{\alpha-1} f(s) [1 + \int_0^s \xi^{\alpha-1} g(\xi) d\xi]^p ds \}. \tag{6}$$

Furthermore,

$$u(t) \le J^{-1}\{J(a(T)) + b(T)\int_0^t s^{\alpha-1} f(s)[1+$$

$$\int_0^s \xi^{\alpha-1} g(\xi) d\xi]^p ds\},\ t \in [0, T]. \tag{7}$$

Letting $t = T$ in (7), one can obtain

$$u(T) \le J^{-1}\{J(a(T))+$$

$$b(T)\int_0^T s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi) d\xi]^p ds\}. \tag{8}$$

Since T is selected arbitrarily, substituting T with t in (8), one can deduce the desired result.

Lemma 2. Suppose $a, b, x, \gamma > 0$. If $0 < \gamma \le 1$, then $(a + x)^\gamma \le a^\gamma + x^\gamma$.

Proof. Set $f(x) = (a + x)^\gamma - [a^\gamma + x^\gamma]$. Then $f(0) = 0$, and $f'(x) = \gamma(a + x)^{\gamma-1} - \gamma x^{\gamma-1} \le 0$. So $f(x)$ is decreasing for $x > 0$, and the proof is complete.

Theorem 3. Under the conditions of Theorem 1, furthermore, assume $0 < p \le 1$, a, b are not necessarily nondecreasing, ω_1, ω_2 are subadditive and submultiplicative, that is, for $\forall\alpha \ge 0,\ \beta \ge 0$, $\omega_i(\alpha + \beta) \le \omega(\alpha) + \omega(\beta)$ and $\omega_i(\alpha\beta) \le \omega(\alpha)\omega(\beta),\ i = 1, 2$ always hold. If the inequality (1) satisfies, then for $t \ge 0$, one has the following explicit estimate for $u(t)$:

$$u(t) \le a(t) + b(t)J^{-1}\{J(H_1(t))+$$

$$\int_0^t s^{\alpha-1} f(s)[\omega_1(b(s)) + \int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))d\xi]^p ds\}, \tag{9}$$

where

$$H_1(t) = \int_0^t s^{\alpha-1} f(s)[\omega_1(a(s))+$$

$$\int_0^s \xi^{\alpha-1} g(\xi)\omega_2(a(\xi))d\xi]^p ds. \tag{10}$$

Proof. Denote $v(t)$ by $\int_0^t s^{\alpha-1} f(s)[\omega_1(u(s)) + \int_0^s \xi^{\alpha-1} g(\xi)\omega_2(u(\xi))d\xi]^p ds$. Then $u(t) \le a(t) + b(t)v(t)$. Similar to the analysis in Theorem 1, one can obtain $v(t)$ is nondecreasing, and $v(0) = 0$. Furthermore,

$$v(t) \le \int_0^t s^{\alpha-1} f(s)[\omega_1(a(s) + b(s)v(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)\omega_2(a(\xi) + b(\xi)v(\xi))d\xi]^p ds$$

Under the assumptions $0 < p \le 1$, and that ω_1, ω_2 are subadditive and submultiplicative, together with the use of Lemma 2, one can deduce

$$v(t) \le \int_0^t s^{\alpha-1} f(s)[\omega_1(a(s)) + \omega_1(b(s)v(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)[\omega_2(a(\xi)) + \omega_2(b(\xi)v(\xi))]d\xi]^p ds$$

$$\le \int_0^t s^{\alpha-1} f(s)[\omega_1(a(s)) + \omega_1(b(s))\omega_1(v(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)[\omega_2(a(\xi)) + \omega_2(b(\xi))\omega_2(v(\xi))]d\xi]^p ds$$

$$\le \int_0^t s^{\alpha-1} f(s)[\omega_1(a(s)) + \int_0^s \xi^{\alpha-1} g(\xi)\omega_2(a(\xi))d\xi]^p ds$$

$$+ \int_0^t s^{\alpha-1} f(s)[\omega_1(b(s))\omega_1(v(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)[\omega_2(b(\xi))\omega_2(v(\xi))]d\xi]^p ds$$

$$= H_1(t) + \int_0^t s^{\alpha-1} f(s)[\omega_1(b(s))\omega_1(v(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))\omega_2(v(\xi))d\xi]^p ds, \tag{11}$$

where $H_1(t)$ is defined in (10).

By the similar analysis as in Theorem 1, one can see $H_1(t)$ is nondecreasing. If we fix $T \ge 0$ and let $t \in [0, T]$, then

$$v(t) \le H_1(T) + \int_0^t s^{\alpha-1} f(s)[\omega_1(b(s))\omega_1(v(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))\omega_2(v(\xi))d\xi]^p ds,\ t \in [0, T]. \tag{12}$$

Denote the right hand side of (12) by $z(t)$. Then $v(t) \le z(t),\ t \in [0, T]$. By use of the properties (iv) and $(viii)$ one has

$$D_t^\alpha z(t) = f(t)[\omega_1(b(t))\omega_1(v(t))$$

$$+ \int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))\omega_2(v(\xi))d\xi]^p$$

$$\le f(t)[\omega_1(b(t))\omega_1(z(t))$$

$$+\omega_2(z(t)) \int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))d\xi]^p$$

$$\le f(t)[\omega_1(b(t))$$

$$+ \int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))d\xi]^p[\omega_1(z(t)) + \omega_2(z(t))]^p,$$

which implies

$$\frac{D_t^\alpha z(t)}{[\omega_1(z(t)) + \omega_2(z(t))]^p} \le$$

$$f(t)[\omega_1(b(t)) + \int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))d\xi]^p.$$

Then fulfilling a similar process to (5)-(8), one can deduce

$$z(t) \le J^{-1}\{J(H_1(T))+$$

$$\int_0^t s^{\alpha-1} f(s)[\omega_1(b(s))+\int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))d\xi]^p ds\}. \tag{13}$$

Furthermore,

$$u(t) \le a(t) + b(t)J^{-1}\{J(H_1(T)) + \int_0^t s^{\alpha-1} f(s)$$

$$[\omega_1(b(s))+\int_0^t \xi^{\alpha-1} g(\xi)\omega_2(b(\xi))d\xi]^p ds\},\ t \in [0, T]. \tag{14}$$

Setting $t = T$ in (14), and after substituting T with t, one can obtain the desired estimate (9).

Next we present the following Volterra-Fredholm type inequality based on the inequality (1).

$$u(t) \le C + \int_0^t s^{\alpha-1} f(s)[\omega_1(u(s))$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)\omega_2(u(\xi))d\xi]^p ds + \int_0^T s^{\alpha-1} f(s)[\omega_1(u(s))$$

$$+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds,\ t\in[0,T], \quad (15)$$

where $C,\ T>0$ are constants, $\alpha,\ u,\ f,\ g,\ \omega_1,\ \omega_2,\ p$ are defined as in the inequality (1).

Theorem 4. Suppose J is defined as in Theorem 1, and define G by $G(x)=J(2x-C)-J(x),\ x\geq C$. If the inequality (15) holds, and G is strictly increasing. Then one has the following estimate for $u(t)$:

$$u(t)\leq J^{-1}\{J\{G^{-1}\{\int_0^T s^{\alpha-1}f(s)[1+\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\}\}$$

$$+\int_0^t s^{\alpha-1}f(s)[1+\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\},\ t\in[0,T]. \quad (16)$$

Proof. Denote the right hand side of (15) by $v(t)$. Then one has

$$u(t)\leq v(t),\ t\in[0,T], \quad (17)$$

Similar to Theorem 1, one can obtain $v(t)$ is nondecreasing, and

$$v(0)=C+\int_0^T s^{\alpha-1}f(s)[\omega_1(u(s))$$

$$+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds.$$

So

$$v(t)=v(0)+$$

$$\int_0^t s^{\alpha-1}f(s)[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds$$

$$\leq v(0)+\int_0^t s^{\alpha-1}f(s)[\omega_1(v(s))$$

$$+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(v(\xi))d\xi]^p ds,\ t\in[0,T]. \quad (18)$$

Then a suitable application of Theorem 1 to (18) yields

$$v(t)\leq J^{-1}\{J(v(0))+\int_0^t s^{\alpha-1}f(s)[1+$$

$$\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\},\ t\in[0,T], \quad (19)$$

that is,

$$J(v(t))-J(v(0))\leq \int_0^t s^{\alpha-1}f(s)[1+$$

$$\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds,\ t\in[0,T], \quad (20)$$

Since $2v(0)-C=v(T)$, from (20) one can obtain

$$J(v(T))-J(v(0))=J(2v(0)-C)-J(v(0))$$

$$\leq \int_0^T s^{\alpha-1}f(s)[1+\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds,$$

which implies

$$G(v(0))\leq \int_0^T s^{\alpha-1}f(s)[1+\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds,$$

Using G is strictly increasing, it follows that

$$v(0)\leq G^{-1}\{\int_0^T s^{\alpha-1}f(s)[1+\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\}, \quad (21)$$

Combining (17), (19) and (21), one can get the desired result.

Now we consider the following fractional integral inequality

$$u(t)\leq a(t)+\int_0^t s^{\alpha-1}h(s)u(s)ds+b(t)\int_0^t s^{\alpha-1}f(s)$$

$$[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds,\ t\geq 0, \quad (22)$$

where h is a nonnegative continuous function defined on $t\geq 0$, and $\alpha,\ u,\ a,\ b,\ f,\ g,\ \omega_1,\ \omega_2,\ p$ are defined as in the inequality (1).

Lemma 5. Let $0<\alpha<1,\ a,\ b,\ u$ be continuous functions defined on $t\geq 0$. Then for $t\geq 0$,

$$D_t^\alpha u(t)\leq a(t)+b(t)u(t)$$

implies

$$u(t)\leq u(0)\exp[\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]$$

$$+\int_0^t \tau^{\alpha-1}a(\tau)\exp[\int_{\frac{\tau^\alpha}{\alpha}}^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]d\tau.$$

Proof. By the properties (ii), (iii), (v), one has the following observation

$$D_t^\alpha\{u(t)\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]\}$$

$$=\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]D_t^\alpha u(t)$$

$$+u(t)D_t^\alpha\{\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]\}$$

$$=\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]D_t^\alpha u(t)$$

$$-b(t)u(t)\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]D_t^\alpha(\frac{t^\alpha}{\alpha})$$

$$=\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds][D_t^\alpha u(t)-b(t)u(t)]$$

$$\leq a(t)\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds].$$

Substituting t with τ, fulfilling fractional integral of order α with respect to τ from 0 to t, one can deduce

$$u(t)\exp[-\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]$$

$$\leq u(0)+\int_0^t \tau^{\alpha-1}a(\tau)\exp[-\int_0^{\frac{\tau^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]d\tau,$$

which implies

$$u(t)\leq \exp[\int_0^{\frac{t^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]\{u(0)+$$

$$\int_0^t \tau^{\alpha-1}a(\tau)\exp[-\int_0^{\frac{\tau^\alpha}{\alpha}} b((s\alpha)^{\frac{1}{\alpha}})ds]d\tau\},$$

The desired result can be obtained subsequently.

Theorem 6. Assume $\omega_1,\ \omega_2$ are submultiplicative, that is, for $\forall\alpha\geq 0,\ \beta\geq 0,\ \omega_i(\alpha\beta)\leq\omega(\alpha)\omega(\beta),\ i=1,2$ always holds. If the inequality (22) satisfies, then one has the following estimate for $u(t)$:

$$u(t)\leq \widetilde{J}^{-1}\{\widetilde{J}(a(t))+b(t)\int_0^t s^{\alpha-1}f(s)[1+$$

$$\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\}\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds],\ t\geq 0, \quad (23)$$

where $\widetilde{J}$ is defined by

$$\widetilde{J}(v)=\int_1^v \frac{1}{[\widetilde{\omega}_1(r)+\widetilde{\omega}_2(r)]^p}dr, \quad (24)$$

with $\widetilde{J}(v)<\infty$ for $v<\infty$, and

$$\widetilde{\omega}_1(v(t))=\omega_1(v(t))\omega_1(\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((\tau\alpha)^{\frac{1}{\alpha}})d\tau]),$$

$$\widetilde{\omega}_2(v(t))=\omega_2(v(t))\omega_2(\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((\tau\alpha)^{\frac{1}{\alpha}})d\tau]), \quad (25)$$

Proof. Let $v(t)=a(t)+b(t)\int_0^t s^{\alpha-1}f(s)[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds$.

If we fix $T\geq 0$ and let $t\in[0,T]$, then one has

$$u(t)\leq v(t)+\int_0^t s^{\alpha-1}h(s)u(s)ds$$

$$\leq v(T)+\int_0^t s^{\alpha-1}h(s)u(s)ds,\ t\in[0,T].$$

Setting $z(t)=v(T)+\int_0^t s^{\alpha-1}h(s)u(s)ds$, one has

$$D_t^\alpha z(t)=h(t)u(t)\leq h(t)z(t).$$

By use of Lemma 5, one can obtain

$$z(t)\leq z(0)\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds]$$

$$=v(T)\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds].$$

So it follows that

$$u(t)\leq v(T)\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds],\ t\in[0,T].$$

Setting $t=T$ one has

$$u(T)\leq v(T)\exp[\int_0^{\frac{T^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds].$$

After substituting T with t, one can obtain

$$u(t)\leq v(t)\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds],\ t\geq 0. \quad (26)$$

So

$$v(t)\leq a(t)+b(t)\int_0^t s^{\alpha-1}f(s)[\omega_1(v(s)\exp[\int_0^{\frac{s^\alpha}{\alpha}} h((\tau\alpha)^{\frac{1}{\alpha}})d\tau]))$$
$$+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(v(\xi)\exp[\int_0^{\frac{\xi^\alpha}{\alpha}} h((\tau\alpha)^{\frac{1}{\alpha}})d\tau])d\xi]^p ds.$$

Under the assumption that $\omega_1,\ \omega_2$ are submultiplicative, one can obtain

$$v(t)\leq a(t)+$$

$$b(t)\int_0^t s^{\alpha-1}f(s)[\omega_1(v(s))\omega_1(\exp[\int_0^{\frac{s^\alpha}{\alpha}} h((\tau\alpha)^{\frac{1}{\alpha}})d\tau])$$

$$+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(\exp[\int_0^{\frac{\xi^\alpha}{\alpha}} h((\tau\alpha)^{\frac{1}{\alpha}})d\tau])\omega_2(v(\xi))d\xi]^p ds$$

$$=a(t)+b(t)\int_0^t s^{\alpha-1}f(s)[\widetilde{\omega}_1(v(s))$$

$$+\int_0^s \xi^{\alpha-1}g(\xi)\widetilde{\omega}_2(v(\xi))d\xi]^p ds, \quad (27)$$

where $\widetilde{\omega}_1,\ \widetilde{\omega}_2$ are defined in (25).

Applying Theorem 1 to (27), one can deduce

$$v(t)\leq \widetilde{J}^{-1}\{\widetilde{J}(a(t))+$$

$$b(t)\int_0^t s^{\alpha-1}f(s)[1+\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\},\ t\geq 0. \quad (28)$$

where $\widetilde{J}$ is defined in (24).

Combining (26) and (28), one can deduce the desired result.

Finally, based on the inequality (22), we consider the following Volterra-Fredholm type fractional integral inequality

$$u(t)\leq C+\int_0^t s^{\alpha-1}h(s)u(s)ds+\int_0^t s^{\alpha-1}f(s)$$

$$[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds$$

$$+\int_0^T s^{\alpha-1}h(s)u(s)ds+\int_0^T s^{\alpha-1}f(s)$$

$$[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds,\ t\in[0,T], \quad (29)$$

where $C,\ T>0$ are constants, $\alpha,\ u,\ f,\ g,\ h,\ \omega_1,\ \omega_2,\ p$ are defined as in the inequality (22).

Theorem 7. Suppose $\widetilde{J}$ is defined as in Theorem 6, and define $\widetilde{G}$ by $\widetilde{G}(x)=\widetilde{J}(2K_Tx-K_TC)-\widetilde{J}(x),\ x\geq C$, where $K_T=\exp[-\int_0^{\frac{T^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds]$. If the inequality (29) holds, and G is strictly increasing, then one has the following estimate for $u(t)$:

$$u(t)\leq \widetilde{J}^{-1}\{\widetilde{J}\{\widetilde{G}^{-1}\{\int_0^T s^{\alpha-1}f(s)[1+$$

$$\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\}\}+\int_0^t s^{\alpha-1}f(s)[1+$$

$$\int_0^s \xi^{\alpha-1}g(\xi)d\xi]^p ds\}\exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds],\ t\in[0,T]. \quad (30)$$

Proof. Denote the right hand side of (29) by $v(t)$. Then

$$u(t)\leq v(t),\ t\in[0,T], \quad (31)$$

Furthermore, one can see $v(t)$ is nondecreasing, and

$$v(0)=C+\int_0^T s^{\alpha-1}h(s)u(s)ds+$$

$$\int_0^T s^{\alpha-1}f(s)[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds.$$

So

$$v(t)=v(0)+\int_0^t s^{\alpha-1}h(s)u(s)ds+$$

$$\int_0^t s^{\alpha-1}f(s)[\omega_1(u(s))+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(u(\xi))d\xi]^p ds$$

$$\leq v(0)+\int_0^t s^{\alpha-1}h(s)v(s)ds+\int_0^t s^{\alpha-1}f(s)[\omega_1(v(s))$$

$$+\int_0^s \xi^{\alpha-1}g(\xi)\omega_2(v(\xi))d\xi]^p ds,\ t\in[0,T]. \quad (32)$$

Applying Theorem 6 to (32), one can obtain

$$v(t) \le \widetilde{J}^{-1}\{\widetilde{J}(v(0)) + \int_0^t s^{\alpha-1} f(s)[1+$$

$$\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^p ds\} \exp[\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds],\ t \in [0,T],\ (33)$$

Furthermore,

$$\widetilde{J}\{v(t)\exp[-\int_0^{\frac{t^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds]\} - \widetilde{J}(v(0))$$

$$\le \int_0^t s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^p ds,\ t \in [0,T],\quad (34)$$

Considering $2v(0) - C = v(T)$, from (34) one has

$$\widetilde{J}\{v(T)\exp[-\int_0^{\frac{T^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds]\}$$

$$= \widetilde{J}\{(2v(0)-C)\exp[-\int_0^{\frac{T^\alpha}{\alpha}} h((s\alpha)^{\frac{1}{\alpha}})ds]\}$$

$$\le \widetilde{J}(v(0)) + \int_0^T s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^p ds,$$

which can be rewritten as

$$\widetilde{G}(v(0)) \le \int_0^T s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^p ds. \quad (35)$$

Under the assumption that G is strictly increasing, one can obtain

$$v(0) \le \widetilde{G}^{-1}\{\int_0^T s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^p ds\}, (36)$$

Combining (31), (33) and (36), one can deduce the desired result.

Remark. If we set $\alpha = 1,\ a(t) = b(t) \equiv 1,\ \omega_1(u) = u^{2-p},\ \omega_2(u) = u^q$, where $0 < p \le 2,\ 0 \le q < 1$, then the inequality (1) reduces to the inequality (2.28) in [7, Theorem 2.5]. If we set $\alpha = 1,\ b(t) \equiv 1,\ \omega_1(u) = \omega_2(u) = u$, where $0 < p < 1$, then the inequality (1) reduces to the inequality (2.73) in [17, Theorem 2.9].

III. Applications

In this section, we present some applications for the results established above. The boundedness and continuous dependence on the initial value of solutions of certain fractional differential and integral equations will be investigated.

Example 1. Consider the following IVP of fractional differential-integral equation:

$$\begin{cases} D_t^\alpha u(t) = F(t, u(t), I^\alpha M(t, u(t))),\ t \ge 0, \\ u(0) = u_0, \end{cases} \quad (37)$$

where $0 < \alpha < 1,\ u \in \mathbf{C}([0,\infty), R),\ M \in \mathbf{C}(R \times R, R),\ F \in \mathbf{C}([0,\infty) \times R^2, R)$.

Theorem 8. Suppose $u(t)$ is a solution of the IVP (37). If $|F(t,x,y)| \le f(t)(\sqrt{|x|} + |y|)^2$, and $|M(t,z)| \le g(t)\sqrt{|z|}$, where $f,\ g$ are nonnegative continuous functions on $[0,\infty)$, then one has the following estimate for $u(t)$:

$$|u(t)| \le |u_0| \exp\{4\int_0^t s^{\alpha-1} f(s)[1+$$

$$\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^2 ds\}\ ,\ t \ge 0, \quad (38)$$

Proof. Using the property (ix) and the definition of the conformable fractional integral, one can deduce

$$u(t) = u_0 + \int_0^t s^{\alpha-1} F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)ds. (39)$$

So

$$|u(t)| \le |u_0| + \int_0^t s^{\alpha-1} |F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)| ds$$

$$\le |u_0| + \int_0^t s^{\alpha-1} f(s)[\sqrt{|u(s)|}$$

$$+ |\int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi|]^2 ds$$

$$\le |u_0| + \int_0^t s^{\alpha-1} f(s)[\sqrt{|u(s)|} + \int_0^s \xi^{\alpha-1} |M(\xi, u(\xi))| d\xi]^2$$

$$\le |u_0| + \int_0^t s^{\alpha-1} f(s)[\sqrt{|u(s)|}$$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)\sqrt{|u(\xi)|} d\xi]^2 ds. \quad (40)$$

After applying Theorem 1 to (40) (with $\omega_1(x) = \omega_2(x) = \sqrt{x},\ p = 2$), one can obtain

$$|u(t)| \le J^{-1}\{J(|u_0|)+$$

$$\int_0^t s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^2 ds\},\ t \ge 0,$$

where $J(v) = \int_1^v \frac{1}{[\omega_1(r)+\omega_2(r)]^p} dr = \int_1^v \frac{1}{4r} dr = \frac{1}{4}\ln|v|$. After some basic computations, one can get the desired estimate (38).

Theorem 9. Suppose $|F(t,x_1,y_1) - F(t,x_2,y_2)| \le f(t)(\sqrt{|x_1-x_2|} + |y_1-y_2|)^2$, $|M(t,x_1) - M(t,x_2)| \le g(t)\sqrt{|x_1-x_2|}$. Then the solution of (37) depends continuously on the initial value u_0.

Proof. Suppose $\widetilde{u}(t)$ is the solution of the following IVP

$$\begin{cases} D_t^\alpha \widetilde{u}(t) = F(t, \widetilde{u}(t), I^\alpha M(t, \widetilde{u}(t))),\ t \ge 0, \\ \widetilde{u}(0) = \widetilde{u}_0, \end{cases} \quad (41)$$

Then one has

$$\widetilde{u}(t) = \widetilde{u}_0 + \int_0^t s^{\alpha-1} F(s, \widetilde{u}(s), \int_0^s \xi^{\alpha-1} M(\xi, \widetilde{u}(\xi))d\xi)ds, \quad (42)$$

By a combination of (39) and (42), one can deduce

$$|u(t) - \widetilde{u}(t)| \le |u_0 - \widetilde{u}_0|$$

$$+ \int_0^t s^{\alpha-1} |F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)$$

$$-F(s, \widetilde{u}(s), \int_0^s \xi^{\alpha-1} M(\xi, \widetilde{u}(\xi))d\xi)| ds$$

$$\le |u_0 - \widetilde{u}_0| + \int_0^t s^{\alpha-1} f(s)\{\sqrt{|u(s) - \widetilde{u}_2(s)|}$$

$$+ |\int_0^s \xi^{\alpha-1} [M(\xi, u(\xi)) - M(\xi, \widetilde{u}_2(\xi))] d\xi|\}^2 ds$$

$$\le |u_0 - \widetilde{u}_0| + \int_0^t s^{\alpha-1} f(s)\{\sqrt{|u(s) - \widetilde{u}_2(s)|}$$

$$+ \int_0^s \xi^{\alpha-1} |M(\xi, u(\xi)) - M(\xi, \widetilde{u}_2(\xi))| d\xi\}^2 ds$$

$\leq |u_0 - \widetilde{u}_0| + \int_0^t s^{\alpha-1} f(s)\{\sqrt{|u(s) - \widetilde{u}_2(s)|}$

$$+ \int_0^s \xi^{\alpha-1} g(\xi)\sqrt{|u(\xi) - \widetilde{u}_2(\xi)|}d\xi\}^2 ds, \quad (43)$$

Applying Theorem 1 to (43) (with $\omega_1(x) = \omega_2(x) = \sqrt{x},\ p = 2$ in Theorem 1), one can deduce $|u(t) - \widetilde{u}_2(t)| \leq |u_0-\widetilde{u}_0| \exp\{4\int_0^t s^{\alpha-1} f(s)[1+\int_0^s \xi^{\alpha-1} g(\xi)d\xi]^2 ds\}$, $t \geq 0$, which implies a little vibration of the initial value u_0 leads to little vibration for the solution $u(t)$. So the proof is complete.

Example 2. Consider the following Volterra-Fredholm type fractional integral equation:

$$u(t) = C + I^\alpha F(t, u(t), I^\alpha M(t, u(t))$$

$$+[I^\alpha F(t, u(t), I^\alpha M(t, u(t))]_{t=T},\ t \in [0, T], \quad (44)$$

where $0 < \alpha < 1,\ T,\ C > 0,\ u \in \mathbf{C}([0, \infty), R),\ M \in \mathbf{C}(R \times R, R),\ F \in \mathbf{C}([0, \infty) \times R^2, R)$.

Theorem 10. Suppose $u(t)$ is a solution of (44). If $|F(t, x, y)| \leq f(t)(\sqrt{|x|} + |y|)$, and $|M(t, z)| \leq g(t)\sqrt{|z|}$, where $f,\ g$ are nonnegative continuous functions on $[0, \infty)$, then for $t \in [0, T]$, one has the following estimate for $u(t)$:

$$|u(t)| \leq \{\sqrt{G^{-1}\{\int_0^T s^{\alpha-1} f(s)[1 + \int_0^s \xi^{\alpha-1} g(\xi)d\xi]ds\}}$$

$$+ \int_0^t s^{\alpha-1} f(s)[1 + \frac{1}{\Gamma(\alpha)}\int_0^s \xi^{\alpha-1} g(\xi)d\xi]ds\}^2, \quad (45)$$

where G is defined by $G(x) = \sqrt{2x - C} - \sqrt{x},\ x \geq C$.

Proof. Similar to Theorem 8, by (44) one has

$$u(t) = C + \int_0^t s^{\alpha-1} F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)ds$$

$$+ \int_0^T s^{\alpha-1} F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)ds. \quad (46)$$

So

$$|u(t)| \leq |C| + \int_0^t s^{\alpha-1}|F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)|ds$$

$$+ \int_0^T s^{\alpha-1}|F(s, u(s), \int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi)|ds$$

$$\leq |C| + \int_0^s s^{\alpha-1} f(s)[\sqrt{|u(s)|}$$

$$+|\int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi|]ds$$

$$+ \int_0^T s^{\alpha-1} f(s)[\sqrt{|u(s)|} + |\int_0^s \xi^{\alpha-1} M(\xi, u(\xi))d\xi|]ds$$

$$\leq |C| + \int_0^t s^{\alpha-1} f(s)[\sqrt{|u(s)|} + \int_0^s \xi^{\alpha-1}|M(\xi, u(\xi))|d\xi]$$

$$+ \int_0^T s^{\alpha-1} f(s)[\sqrt{|u(s)|} + \int_0^s \xi^{\alpha-1}|M(\xi, u(\xi))|d\xi]$$

$$\leq |C|+\int_0^t s^{\alpha-1} f(s)[\sqrt{|u(s)|}+\int_0^s \xi^{\alpha-1} g(\xi)\sqrt{|u(\xi)|}d\xi]ds$$

$$+ \int_0^T s^{\alpha-1} f(s)[\sqrt{|u(s)|}+\int_0^s \xi^{\alpha-1} g(\xi)\sqrt{|u(\xi)|}d\xi]ds. \quad (47)$$

To apply Theorem 4, one can see that $\omega_1(x) = \omega_2(x) = \sqrt{x},\ p = 1,\ J(v) = \int_1^v \frac{1}{[\omega_1(r) + \omega_2(r)]^p}dr = \int_1^v \frac{1}{2\sqrt{r}}dr = \sqrt{v}-1, G(x) = J(2x-C)-J(x) = \sqrt{2x - C}-\sqrt{x},\ x \geq C$. So $G'(x) = \frac{1}{\sqrt{2x - C}} - \frac{1}{2\sqrt{x}} > 0$ for $x \geq C$, which means G is strictly increasing. Then a suitable application of Theorem 4 to (47) one can deduce

$$|u(t)| \leq J^{-1}\{J\{G^{-1}\{\int_0^T s^{\alpha-1} f(s)[1 + \int_0^s \xi^{\alpha-1} g(\xi)d\xi]ds\}\}$$

$$+ \int_0^t s^{\alpha-1} f(s)[1 + \int_0^s \xi^{\alpha-1} g(\xi)d\xi]ds\},\ t \in [0, T]. \quad (48)$$

After basic computations one can get the desired estimate (45).

IV. Conclusions

We have presented some new fractional integral inequalities, and by use of them derived explicit bounds for unknown functions concerned. Based on these inequalities, some Volterra-Fredholm type fractional integral inequalities are also established. For illustrating the validity of the established results, we presented some examples, and researched boundedness, continuous dependence on initial value of solutions of two fractional differential and integral equations.

References

[1] A. Kilbas, H. Srivastava and J. Trujillo, "Theory and Applications of Fractional Differential Equations," *Elsevier*, Boston, 2006.
[2] A. M. Bijura, " Systems of Singularly Perturbed Fractional Integral Equations II," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 4, pp. 198-203, 2012.
[3] A. Bouhassoun, " Multistage Telescoping Decomposition Method for Solving Fractional Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 1, pp. 10-16, 2013.
[4] Q.H. Feng, " Jacobi Elliptic Function Solutions For Fractional Partial Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 121-129, 2016.
[5] H.M. Jaradat, S. Al-Shar'a, Q. J.A. Khan, M. Alquran and K. Al-Khaled, " Analytical Solution of Time-Fractional Drinfeld-Sokolov-Wilson System Using Residual Power Series Method," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 64-70, 2016.
[6] H. Song, M. Yi, J. Huang and Y. Pan, " Bernstein Polynomials Method for a Class of Generalized Variable Order Fractional Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 437-444, 2016.
[7] S.X. Zhou, F.W. Meng, Q.H. Feng and L. Dong, " A Spatial Sixth Order Finite Difference Scheme for Time Fractional Sub-diffusion Equation with Variable Coefficient," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 2, pp. 175-181, 2017.
[8] Z. Yan and F. Lu, " Existence of A New Class of Impulsive Riemann-Liouville Fractional Partial Neutral Functional Differential Equations with Infinite Delay," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 300-312, 2015.
[9] B. Rebiai and K. Haouam, " Nonexistence of Global Solutions to a Nonlinear Fractional Reaction-Diffusion System," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 259-262, 2015.
[10] M. A. Abdellaoui, Z. Dahmani, and N. Bedjaoui, " Applications of Fixed Point Theorems for Coupled Systems of Fractional Integro-Differential Equations Involving Convergent Series," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 273-278, 2015.
[11] Pachpatte BG: Inequalities for Differential and Integral Equations," *Academic Press*, New York, 1998.
[12] A. Abdeldaim and M. Yakout " On some new integral inequalities of Gronwall-Bellman-Pachpatte type," *Appl. Math. Comput.*, vol. 217, pp. 7887-7899, 2011.
[13] R. Xu and X. Ma, "Some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two variables and their applications," *J. Ineq. Appl.*, vol. 2017:187, pp. 1-25, 2017.
[14] Z. Zheng, X. Gao and J. Shao, " Some new generalized retarded inequalities for discontinuous functions and their applications," *J. Ineq. Appl.*, vol. 2016:7, pp. 1-14, 2016.
[15] Q. Feng and F. Meng, " Some generalized Ostrowski-Grüss type integral inequalities," *Comput. Math. Appl.*, vol. 63, pp. 652-659, 2012.

[16] W.S. Wang, " A class of retarded nonlinear integral inequalities and its application in nonlinear differential-integral equation," *J. Inequal. Appl.*, vol. 2012:154, pp. 1-10, 2012.
[17] L. Li, F. Meng and P. Ju, " Some new integral inequalities and their applications in studying the stability of nonlinear integro differential equations with time delay," *J. Math. Anal. Appl.*, vol. 377, pp. 853-862, 2011.
[18] H. El-Owaidy, A.A. Ragab, W. Abuelela and A.A. El-Deeb, " On some new nonlinear integral inequalities of Gronwall-Bellman type," *KKYUNGPOOK Math. J.*, vol. 54, pp. 555-575, 2014.
[19] Q. Feng, F. Meng and B. Fu, " Some new generalized Volterra-Fredholm type finite difference inequalities involving four iterated sums," *Appl. Math. Comput.*, vol. 219, pp. 8247-8258, 2013.
[20] Q.H. Ma, " Estimates on some power nonlinear Volterra-Fredholm type discrete inequalities and their applications," *J. Comupt. Appl. Math.*, vol. 233, pp. 2170-2180, 2010.
[21] J. Wang, F. Meng and J. Gu, " Estimates on some power nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales," *Adv. Diff. Equ.*, vol. 2017:257, pp. 1-16, 2017.
[22] H. Liu, " A class of retarded Volterra-Fredholm type integral inequalities on time scales and their applications," *J. Ineq. Appl.*, vol. 2017:293, pp. 1-15, 2017.
[23] R. Agarwal, M. Bohner and A. Peterson, "Inequalities on time scales: a survey," *Math. Inequal. Appl.*, vol. 4(4), pp. 535-557, 2001.
[24] S.H. Saker, "Some nonlinear dynamic inequalities on time scales and applications," *J. Math. Ineq.*, vol. 4, pp. 561-579, 2010.
[25] J. Gu and F. Meng, "Some new nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales," *Appl. Math. Comput.*, vol. 245, pp. 235-242, 2014.
[26] F. Meng and J. Shao, "Some new Volterra-Fredholm type dynamic integral inequalities on time scales," *Appl. Math. Comput.*, vol. 223, pp. 444-451, 2013.
[27] R. Xu, "Some New Nonlinear Weakly Singular Integral Inequalities and Their Applications," *J. Math. Ineq.*, vol. 11(4), pp. 1007-1018, 2017.
[28] H. Liu and F. Meng, "Some new generalized Volterra-Fredholm type discrete fractional sum inequalities and their applications," *J. Ineq. Appl.*, vol. 2016:213, pp. 1-16, 2016.
[29] H.P. Ye, J.M. Gao and Y.S. Ding, "A generalized Gronwall inequality and ins application to a fractional differential equation," *J. Math. Anal. Appl.*, vol. 328, pp. 1075-1081, 2007.
[30] Q.H. Feng, " Some New Gronwall-Bellmann Type Discrete Fractional Inequalities Arising in the Theory of Discrete Fractional Calculus," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 527-534, 2016.
[31] B. Zheng, "Some New Gronwall-Bellman Type And Volterra-Fredholm Type Fractional Integral Inequalities And Their Applications in Fractional Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 3, pp. 288-296, 2018.
[32] R. Khalil, M. Al-Horani, A. Yousef and M. Sababheh, "A new definition of fractional derivative," *J. Comput. Appl. Math.*, vol. 264, pp. 65-70, 2014.
[33] K. Hosseini, A. Bekir and R. Ansari, "New exact solutions of the conformable time-fractional Cahn-Allen and Cahn-Hilliard equations using the modified Kudryashov method," *Optik*, vol. 132, pp. 203-209, 2017.
[34] H.W. Zhou, S. Yang and S.Q. Zhang, "Conformable derivative approach to anomalous diffusion," *Physica A*, vol. 491, pp. 1001-1013, 2018.
[35] S. Yang, L. Wang and S. Zhang, "Conformable derivative: Application to non-Darcian flow in low-permeability porous media," *Appl. Math. Lett.*, vol. 79, pp. 105-110, 2018.
[36] O. Tasbozan, Y. Çenesiz, A. Kurt and D. Baleanu, "New analytical solutions for conformable fractional PDEs arising in mathematical physics by exp-function method," *Open Phys.*, vol. 15, pp. 647-651, 2017.