The Influence of Density Dependent Birth Rate to a Commensal Symbiosis Model with Holling Type Functional Response

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***Abstract*—A two species commensal symbiosis model with Holling type functional response and density dependent birth rate takes the form**

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a_{11}}{a_{12}+a_{13}x}-a_{14}-b_1x+\frac{c_1y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y\big(a_2-b_2y\big)\end{aligned}$$

is proposed and studied in this paper. For autonomous case, i.e., $a_{ij}, b_i, i=1,2, j=1,2,3,4$ p and c_1 are all positive constants, $p\geq 1$, sufficient conditions which ensure the local and global stability of the boundary equilibrium and positive equilibrium are obtained, respectively; For non-autonomous case, i.e., a_{ij}, b_i and c_1 are all continuous functions bounded above and below by positive constans, $p\geq 1$ is a positive constant, sufficient conditions which ensure the permanence and partial survival of the system are obtained, respectively. Sufficient conditions -p which ensure the existence of at least one positive T-periodic solution is obtained. Our study shows that the birth rate of species and the commensal intensity between the species play important role on the dynamic behaviors of the system.

***Index Terms*—commensalism system, density dependent birth rate, global asymptotic stability.**

I. INTRODUCTION

MUTUALISM, which describes that two different species exist in a relationship in which each individual fitness benefits from the activity of the other, has recently been studied by many shcolars([1]-[17]). Xie, Chen, Yang et al[1], Yang, Xie and Chen[2], Xie, Chen et al [3], Lei[4] and Chen, Wu and Xie[5] focus on the stability property of the mutualism model, by using the iterative method, they obtained some sufficient conditions to ensure the global attractivity of the positive equilibrium of the system they considered. Some scholars([6]-[10]) argued that the system maybe disturbed by some unpredictable factors, and one should introduce the feedback control variables to describe such kind of phenomenon, Chen and Xie[7] showed that feedback control variables have no influence to the persistent property of the system, Yang and Miao[8] showed that a system with single feedback control variables may have complex dynamic behaviors, Han et al[10] investigated the stability property of the May cooperation system with feedback controls; Some scholars([11]-[12]) considered the influence of the stage structure to the mutualism model, for example, Chen, Xie et al [11] showed that the stage structure of the species plays important role on the persistent or extinct property of the system. Some scholars[13]-[14] considered the multispecies system, they also obtained the conditions which ensure the permanence of the system.

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Commensalism is a long-term biological interaction (symbiosis) in which members of one species gain benefits while those of the other species neither benefit nor are harmed. As was pointed out by Georgescu, D. Maxin and H. Zhang[15], commensalism can be thought as mutualism in which one of the two interspecies interaction terms is zero, so at a glance everything should be simpler. However, this is not actually the case. Recently, many scholars ([15]-[29]) paid their attention to the dynamic behaviors of the commensalism model, and some essential progress had made on this direction.

Han and Chen[21] incorporated the feedback control variables to the commensal symbiosis model, and proposed the following model:

$$\begin{aligned}\dot{x} &= x\big(b_1-a_{11}x+a_{12}y-\alpha_1u_1\big),\\ \dot{y} &= y\big(b_2-a_{22}y-\alpha_2u_2\big),\\ \dot{u}_1 &= -\eta_1u_1+a_1x,\\ \dot{u}_2 &= -\eta_2u_2+a_2y.\end{aligned}\tag{1.1}$$

They showed that system (1.1) admits a unique globally stable positive equilibrium.

Xie et al. [22] proposed the following discrete commensal symbiosis model

$$\begin{aligned}x_1(k+1) &= x_1(k)\exp\big\{a_1(k)-b_1(k)x_1(k)\\ &\qquad +c_1(k)x_2(k)\big\},\\ x_2(k+1) &= x_2(k)\exp\big\{a_2(k)-b_2(k)x_2(k)\big\}.\end{aligned}\tag{1.2}$$

Sufficient conditions which ensure the existence of at least one positive ω-periodic solution of the system (1.2) is obtained.

Xue et al[23] further incorporated the delay to system (1.2), and they proposed the following discrete commensalism system

$$\begin{aligned}x(n+1) &= x(n)\exp\Big[r_1(n)\Big(1-\frac{x(n-\tau_1)}{K_1(n)}\\ &\qquad +\alpha(n)\frac{y(n-\tau_2)}{K_1(n)}\Big)\Big],\\ y(n+1) &= y(n)\exp\Big[r_2(n)\Big(1-\frac{y(n-\tau_3)}{K_2(n)}\Big)\Big].\end{aligned}\tag{1.3}$$

They investigated the almost periodic solution of the system (1.3).

Wu et al[17] assumed that the the functional response

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between two species is of Holling type, and they established the following two species commensal symbiosis model

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1x + \frac{c_1y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y(a_2 - b_2y),\end{aligned}\tag{1.4}$$

where $a_i, b_i, i = 1, 2$ p and c_1 are all positive constants, $p \geq 1$. Their study indicates that the unique positive equilibrium of the system is globally stable.

Wu and Lin[18] proposed the following commensalism model with ratio-dependent functional response and one party can not survive independently:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(-a_1 - b_1x + \frac{c_1y}{x+y}\Big),\\ \frac{dy}{dt} &= y(a_2 - b_2y).\end{aligned}\tag{1.5}$$

For the autonomous case, they obtained sufficient conditions which ensure the existence, local and global stability property of the equilibria.

Recently, several scholars ([19], [25], [29]) also investigated the influence of Allee effect to the commensalism model, for example, based on the work of Wu and Lin[18], Chen[29] investigated the dynamic behaviors of the following two species commensal symbiosis model involving Allee effect and one party can not survive independently:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(-a_1 - b_1x + \frac{c_1y}{x+y}\Big),\\ \frac{dy}{dt} &= y\big(a_2 - b_2y\big)\frac{y}{u+y},\end{aligned}\tag{1.6}$$

where $a_1, b_1, c_1, a_2, b_2, p \geq 1$ and u are all positive constants. They showed that the unique positive equilibrium is globally stable if $a_1 < c_1$ holds; and the boundary equilibrium $(0, \frac{a_2}{b_2})$ is globally stable if $a_1 > c_1$ holds. Numeric simulations showed that with the increasing of Allee effect, it takes much more time for the system to reach its stable steady-state solution.

It bring to our attention that all of the modelings of (1.1)-(1.6) are based on the traditional Logistic model, for example, in system (1.5), if we did not consider the relationship of the two species, then the first species takes the form

$$\frac{dx}{dt} = x\Big(a_1 - b_1x\Big),\tag{1.7}$$

where a_1 is the intrinsic growth rate, and b_1 is the density dependent coefficients. System (1.7) could be revised as

$$\frac{dx}{dt} = x\Big(a_{11} - a_{14} - b_1x\Big),\tag{1.8}$$

where a_{11} is the birth rate of the species and a_{14} is the death rate of the species. Already, Brauer and Castillo-Chavez[30], Tang and Chen[31], Berezansky, Braverman and Idels[32] had showed that in some cases, the density dependent birth rate of the species is more suitable. If we take the famous Beverton-Holt function ([32]) as the birth rate, then the system (1.8) should be revised to

$$\frac{dx}{dt} = x\Big(\frac{a_{11}}{a_{12} + a_{13}x} - a_{14} - b_1x\Big).\tag{1.9}$$

Now, combine with (1.4) and (1.9), we could obtain the following two species commensal symbiosis model with Holling type functional response and density dependent birth rate

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a_{11}}{a_{12} + a_{13}x} - a_{14} - b_1x + \frac{c_1y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y\big(a_2 - b_2y\big),\end{aligned}\tag{1.10}$$

where $a_{ij}, b_i, i = 1, 2, j = 1, 2, 3, 4$ p and c_1 are all positive constants, $p \geq 1$.

Since the environment is vary with seasonal, it is naturally to consider the non-autonomous case of system (1.10), i.e,

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)x} - a_{14}(t)\\ &\qquad -b_1(t)x + \frac{c_1(t)y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y\big(a_2(t) - b_2(t)y\big),\end{aligned}\tag{1.11}$$

where $a_{ij}(t), b_i(t), i = 1, 2, j = 1, 2, 3, 4$ and $c_1(t)$ are all continuous functions bounded above and below by positive constants, $p \geq 1$ is a positive constant.

In the study of the dynamic behaviors of the ecosystem, the stability, persistent and extinction are the most important topics, see [32]-[42] and the references cited therein. As far as system (1.10) and (1.11) are concerned, it is naturally to investigate the extinction and stability property of the system, the existence of the positive periodic solution and to find out the influence of the density dependent birth rate and the influence of the commensalism.

The rest of the paper is arranged as follows. In section 2, we investigate the dynamic behaviors of the system (1.10), specially focus on the local and global stability property of the equilibrium; In section 3, we investigate the dynamic behaviors of the system (1.11), specially focus on the extinction and persistent property of the system; Section 4 presents some numerical simulations to show the feasibility of the main results. We end this paper by a briefly discussion.

II. Autonomous case

A. The existence and local stability of the equilibria

The equilibria of system (1.10) is determined by the system

$$\begin{aligned}&x\Big(\frac{a_{11}}{a_{12} + a_{13}x} - a_{14} - b_1x + \frac{c_1y^p}{1+y^p}\Big) = 0,\\ &y(a_2 - b_2y) = 0.\end{aligned}\tag{2.1}$$

Hence, system (1.10) admits four possible equilibria, $A_0(0,0)$, $A_1\big(x_1, 0\big)$, $A_2\big(0, \frac{a_2}{b_2}\big)$ and $A_3\big(x^*, y^*\big)$, where

$$x_1 = \frac{-C_1 + \sqrt{C_1^2 - C_2}}{2b_1a_{13}}.\tag{2.2}$$

$$x^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1},\ \ y^* = \frac{a_2}{b_2},\tag{2.3}$$

here

$$\begin{aligned} C_1 &= b_1 a_{12} + a_{14} a_{13}, \\ C_2 &= 4 b_1 a_{13} (a_{14} a_{12} - a_{11}) \\ B_1 &= a_{13}\, b_1\, (y^*)^p + a_{13}\, b_1, \\ B_2 &= \big(a_{12}\, b_1 + a_{13} a_{14} - a_{13} c_1\big)\,(y^*)^p \\ &\quad + a_{12}\, b_1 + a_{13}\, a_{14}, \\ B_3 &= -\,(y^*)^p\,(-a_{12}\, a_{14} + a_{12}\, c_1 + a_{11}) \\ &\quad - a_{11} + a_{12} a_{14}. \end{aligned} \tag{2.4}$$

Concerned with the local stability property of the above four equilibria, we have

Theorem 2.1. *$A_0(0,0)$ and $A_1(\frac{a_1}{b_1},0)$ are unstable; Assume that*

$$\frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1 (y^*)^p}{1+(y^*)^p} < 0, \tag{2.5}$$

then $A_2\big(0,\frac{a_2}{b_2}\big)$ is locally stable; Assume that

$$\frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1 (y^*)^p}{1+(y^*)^p} > 0, \tag{2.6}$$

then $A_3\big(x^*,y^*\big)$ is locally stable.

Proof. The Jacobian matrix of the system (1.10) is calculated as

$$J(x,y) = \begin{pmatrix} K_1 & \dfrac{c_1 p x y^{p-1}}{(1+y^p)^2} \\ 0 & -2b_2 y + a_2 \end{pmatrix}, \tag{2.7}$$

where

$$\begin{aligned} K_1 &= -\frac{a_{11}\, a_{13}\, x}{\left(a_{13}\, x + a_{12}\right)^2} + \frac{a_{11}}{a_{13}\, x + a_{12}} \\ &\quad - a_{14} - 2\, b_1\, x + \frac{c_1\, y^p}{1+y^p}. \end{aligned} \tag{2.8}$$

Then the Jacobian matrix of the system (1.10) about the equilibrium $A_0(0,0)$ is given by

$$\begin{pmatrix} \dfrac{a_{11}}{a_{12}} - a_{14} & 0 \\ 0 & a_2 \end{pmatrix}. \tag{2.9}$$

The eigenvalues of the above matrix are $\lambda_1 = \frac{a_{11}}{a_{12}} - a_{14}, \lambda_2 = a_2 > 0$. Hence, $A_0(0,0)$ is a unstable.

The Jacobian matrix of the system (1.10) about the equilibrium $A_1(x_1,0)$ is given by

$$\begin{pmatrix} -\dfrac{a_{11}\, a_{13}\, x_1}{\left(a_{13}\, x_1 + a_{12}\right)^2} - b_1 x_1 & 0 \\ 0 & a_2 \end{pmatrix}. \tag{2.10}$$

The eigenvalues of the above matrix are $\lambda_1 = -\frac{a_{11}\, a_{13}\, x_1}{\left(a_{13}\, x_1 + a_{12}\right)^2} - b_1 x_1 < 0, \lambda_2 = a_2 > 0$. Hence, $A_1(x_1,0)$ is unstable.

For $A_2(0,\frac{a_2}{b_2})$, its Jacobian matrix is given by

$$\begin{pmatrix} \dfrac{a_{11}}{a_{12}} - a_{14} + \dfrac{c_1 (y^*)^p}{1+(y^*)^p} & 0 \\ 0 & -a_2 \end{pmatrix}. \tag{2.11}$$

Hence, under the assumption (2.5) holds, the eigenvalues of the above matrix are $\lambda_1 = \frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1 (y^*)^p}{1+(y^*)^p} < 0, \lambda_2 = -a_2 < 0$. Therefore, $A_2(0,\frac{a_2}{b_2})$ is locally stable.

Under the assumption (2.6) holds, A_3 is the positive equilibrium. The Jacobian matrix about the equilibrium A_3 is given by

$$\begin{pmatrix} -\dfrac{a_{11}\, a_{13}\, x^*}{\left(a_{13}\, x^* + a_{12}\right)^2} - b_1 x^* & F_{12} \\ 0 & -a_2 \end{pmatrix}, \tag{2.12}$$

where

$$F_{12} = \frac{c_1 p x^* (y^*)^{p-1}}{(1+(y^*)^p)^2}. \tag{2.13}$$

The eigenvalues of the above matrix are $\lambda_1 = -\frac{a_{11}\, a_{13}\, x_1}{\left(a_{13}\, x^* + a_{12}\right)^2} - b_1 x^* < 0, \lambda_2 = -a_2 < 0$. Hence, $A_3(x^*,y^*)$ is locally stable.

This ends the proof of Theorem 2.1.

B. Global stability of the equilibria

Theorem 2.1 shows that depending on the sign of the term $\frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1 (y^*)^p}{1+(y^*)^p}$, the boundary equilibrium $A_2(0,\frac{a_2}{b_2})$ and the positive equilibrium $A_3(x^*,y^*)$ are all possible locally stable. Following we will further investigate the global stability property of those two equilibria.

Lemma 2.1.[35] System

$$\frac{dy}{dt} = y(a - by) \tag{2.14}$$

has a unique globally attractive positive equilibrium $y^* = \frac{a}{b}$.

Theorem 2.2. *Assume that (2.6) holds, then $A_3\big(x^*,y^*\big)$ is globally asymptotically stable; Assume that (2.5) holds, then $A_2(0,\frac{a_2}{b_2})$ is globally asymptotically stable.*

Proof. (1) We assume that (2.6) holds. Firstly we prove that every solution of system (1.10) that starts in R^2_+ is uniformly bounded. From the second equation of (1.10) one has

$$\frac{dy}{dt} = y(a_2 - b_2 y). \tag{2.15}$$

It follows from Lemma 3.1 that system (2.15) has a unique globally attractive positive equilibrium $y^* = \frac{a_2}{b_2}$.

Also, from the first equation of system (1.10), one has

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{a_{11}}{a_{12} + a_{13} x} - a_{14} - b_1 x + \frac{c_1 y^p}{1+y^p}\Big) \\ &\le x\Big(\frac{a_{11}}{a_{12}} - a_{14} - b_1 x + c_1\Big). \end{aligned} \tag{2.16}$$

By using the differential inequality theory and applying Lemma 2.1, we obtain

$$\limsup_{t\to+\infty} x(t) \le \frac{\Upsilon}{b_1}. \tag{2.17}$$

where

$$\Upsilon = \frac{a_{11}}{a_{12}} - a_{14} + c_1.$$

Hence, there exists a $\varepsilon > 0$ such that for all $t > T$

$$x(t) < \frac{\Upsilon}{b_1} + \varepsilon, \;\; y(t) < \frac{a_2}{b_2} + \varepsilon. \tag{2.18}$$

Let

$$D=\Big\{(x,y)\in R_+^2|x<\frac{\Upsilon}{b_1}+\varepsilon,\ \ y<\frac{a_2}{b_2}+\varepsilon\Big\}.$$

Then every solution of system (1.10) starts in R_+^2 is uniformly bounded on D. Also, from Theorem 2.1, under the assumption (2.6), system (1.10) admits an unique local stable positive equilibrium $A_3(x^*,y^*)$. To ensure $A_3(x^*,y^*)$ is globally stable in above area, we consider the Dulac function $u(x,y)=x^{-1}y^{-1}$, then

$$\begin{aligned}
&\frac{\partial(uP)}{\partial x}+\frac{\partial(uQ)}{\partial y}\\
=\ &\frac{1}{xy}\left(K_1-a_{14}-2\,b_1\,x+\frac{c_1\,y^p}{1+y^p}\right)\\
&-\frac{1}{x^2y}\left(K_2x-b_1\,x^2+\frac{c_1\,xy^p}{1+y^p}\right)\\
&+\frac{-2\,b_2\,y+a_2}{xy}-\frac{-b_2\,y^2+a_2\,y}{xy^2}\\
=\ &-\frac{\Delta(x,y)}{xy\,(a_{13}\,x+a_{12})^2}\\
<\ &0,
\end{aligned}\tag{2.19}$$

where

$$\begin{aligned}
K_1&=-\frac{a_{11}\,a_{13}\,x}{(a_{13}\,x+a_{12})^2}+\frac{a_{11}}{a_{13}\,x+a_{12}},\\
K_2&=\frac{a_{11}}{a_{13}\,x+a_{12}}-a_{14},\\
P(x,y)&=x\Big(\frac{a_{11}}{a_{12}+x}-a_{13}-b_1x+\frac{c_1y^p}{1+y^p}\Big),\\
Q(x,y)&=y\Big(a_2-b_2y\Big)\\
\Delta(x,y)&=a_{13}{}^2b_1\,x^3+a_{13}{}^2b_2\,x^2y\\
&\quad+2\,a_{12}\,a_{13}\,b_1\,x^2+2\,a_{12}\,a_{13}\,b_2\,xy\\
&\quad+a_{12}{}^2b_1\,x+a_{12}{}^2b_2\,y+a_{11}\,a_{13}\,x.
\end{aligned}$$

By Dulac Theorem[36], there is no closed orbit in area D. So $A_3(x^*,y^*)$ is globally asymptotically stable.

(2) Now let's assume that (2.5) holds, from the continuous of the function $\dfrac{y^p}{1+y^p}$, one could see that for $\varepsilon>0$ enough small,

$$\frac{a_{11}}{a_{12}}-a_{14}+\frac{c_1(y^*+\varepsilon)^p}{1+(y^*+\varepsilon)^p}<0.\tag{2.20}$$

From the second equation of (1.10) one has

$$\frac{dy}{dt}=y(a_2-b_2y).$$

It follows from Lemma 2.1 that above system has a unique globally attractive positive equilibrium $y^*=\frac{a_2}{b_2}$. Hence, for above $\varepsilon>0$ enough small, there exists a $T_1>0$ such that for all $t>T_1$,

$$y(t)<y^*+\varepsilon.\tag{2.21}$$

Hence, for $t>T_1$, from the first equation of system (1.10), one has

$$\begin{aligned}
\frac{dx}{dt}&=x\Big(\frac{a_{11}}{a_{12}+a_{13}x}-a_{14}-b_1x+\frac{c_1y^p}{1+y^p}\Big)\\
&\le x\Big(\frac{a_{11}}{a_{12}}-a_{14}-b_1x+\frac{c_1(y^*+\varepsilon)^p}{1+(y^*+\varepsilon)^p}\Big)\\
&\le \Delta x,
\end{aligned}\tag{2.22}$$

where

$$\Delta=\frac{a_{11}}{a_{12}}-a_{14}+\frac{c_1(y^*+\varepsilon)^p}{1+(y^*+\varepsilon)^p}<0,$$

and so, by using (2.20), one has

$$x(t)\le x(T_1)\exp\Big\{\Delta(t-T_1)\Big\}\to 0\ \text{ as }\ t\to+\infty.\tag{2.23}$$

That is,

$$\lim_{t\to+\infty}x(t)=0.\tag{2.24}$$

This completes the proof of Theorem 2.2.

III. Non-autonomous case

Given a function g defined on R, let g^L and g^M be defined as

$$g_L=\inf_{t\in R}g(t),\ \ g_M=\sup_{t\in R}g(t).$$

From Lemma 2.1 of Liu, Xie and Lin[26], we have

Lemma 3.1. *If $a>0,b>0$ and $\dot{x}\ge x(b-ax)$, when $t\ge0$ and $x(0)>0$, we have*

$$\liminf_{t\to+\infty}x(t)\ge\frac{b}{a}.$$

If $a>0,b>0$ and $\dot{x}\le x(b-ax)$, when $t\ge0$ and $x(0)>0$, we have

$$\limsup_{t\to+\infty}x(t)\le\frac{b}{a}.$$

Theorem 3.1 *Assume that the inequality*

$$\frac{a_{11}^M}{a_{12}^L}-a_{14}^L+\frac{c_1^M\left(\frac{a_2^M}{b_2^L}\right)^p}{1+\left(\frac{a_2^M}{b_2^L}\right)^p}<0\tag{3.1}$$

holds, then

$$\lim_{t\to+\infty}x(t)=0,$$

$$\frac{a_2^L}{b_2^M}\le\liminf_{t\to+\infty}y(t)\le\limsup_{t\to+\infty}y(t)\le\frac{a_2^M}{b_2^L}.$$

i.e., the first species will be driven to extinction, while the second species is permanent.

Proof. Condition (3.1) implies that for enough small positive constant $\varepsilon>0$, the following inequality holds

$$\Gamma_\varepsilon\overset{\text{def}}{=}\frac{a_{11}^M}{a_{12}^L}-a_{14}^L+\frac{c_1^M\left(\frac{a_2^M}{b_2^L}+\varepsilon\right)^p}{1+\left(\frac{a_2^M}{b_2^L}+\varepsilon\right)^p}<0\tag{3.2}$$

From the second equation of system (1.11), one has

$$\frac{dy}{dt}\le y\big(a_2^M-b_2^Ly\big).\tag{3.3}$$

It follows from Lemma 3.1 and (3.1) that

$$\limsup_{t\to+\infty} y(t) \le \frac{a_2^M}{b_2^L}. \tag{3.4}$$

Hence, there exists a $T_1 > 0$ such that

$$y(t) < \frac{a_2^M}{b_2^L} + \varepsilon \ \text{ for all } \ t \ge T_2. \tag{3.5}$$

For $t > T_2$, from the first equation of (1.11) and (3.5), we have

$$\begin{aligned}\frac{dx}{dt} &\le x\left(\frac{a_{11}^M}{a_{12}^L} - a_{14}^M + \frac{c_1^M\left(\frac{a_2^M}{b_2^L}+\varepsilon\right)^p}{1+\left(\frac{a_2^M}{b_2^L}+\varepsilon\right)^p}\right) \\ &= \Gamma_\varepsilon x.\end{aligned} \tag{3.6}$$

Hence

$$x(t) \le x(T_2)\exp\Big\{\Gamma_\varepsilon(t-T_2)\Big\} \to 0 \ \text{ as } \ t\to+\infty. \tag{3.7}$$

From the second equation of system (1.11), we also have

$$\frac{dy}{dt} \ge y\big(a_2^L - b_2^M y\big). \tag{3.8}$$

It follows from Lemma 3.1 and (3.8) that

$$\liminf_{t\to+\infty} y(t) \ge \frac{a_2^L}{b_2^M}. \tag{3.9}$$

It follows from (3.5), (3.7) and (3.9) that the conclusions of Theorem 3.1 holds. This ends the proof of Theorem 3.1.

Theorem 3.2 *Assume that the inequality*

$$\frac{a_{11}^L}{a_{12}^M + a_{13}^M M_1} - a_{14}^M + \frac{c_1^L\left(\frac{a_2^L}{b_2^M}\right)^p}{1+\left(\frac{a_2^L}{b_2^M}\right)^p} > 0 \tag{3.10}$$

holds, then

$$m_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_1,$$

$$\frac{a_2^L}{b_2^M} \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le \frac{a_2^M}{b_2^L}.$$

i.e., the system (1.11) is permanent, where M_1, m_1 will be defined in (3.15) and (3.23), respectively.

Proof. From (3.3)-(3.5), one could easily see that

$$\limsup_{t\to+\infty} y(t) \le \frac{a_2^M}{b_2^L} \tag{3.11}$$

holds. Hence, for $\varepsilon_1 > 0$ small enough, there exists a $T_3 > 0$ such that

$$y(t) < \frac{a_2^M}{b_2^L} + \varepsilon_1 \ \text{ for all } \ t \ge T_3. \tag{3.12}$$

For $t > T_3$, from the second equation of (1.11) and (3.12), we have

$$\frac{dx}{dt} \le x\left(\frac{a_{11}^M}{a_{12}^L} - a_{14}^M - b_1^L x + \frac{c_1^M\left(\frac{a_2^M}{b_2^L}+\varepsilon_1\right)^p}{1+\left(\frac{a_2^M}{b_2^L}+\varepsilon_1\right)^p}\right). \tag{3.13}$$

Applying Lemma 3.1 to (3.13) leads to

$$\limsup_{t\to+\infty} x(t) \le \frac{\frac{a_{11}^M}{a_{12}^L} - a_{14}^M + \frac{c_1^M\left(\frac{a_2^M}{b_2^L}+\varepsilon_1\right)^p}{1+\left(\frac{a_2^M}{b_2^L}+\varepsilon_1\right)^p}}{b_1^L}. \tag{3.14}$$

Since ε_1 is any positive constants small enough, letting $\varepsilon_1 \to 0$ in (3.14) leads to

$$\limsup_{t\to+\infty} x(t) \le \frac{\frac{a_{11}^M}{a_{12}^L} - a_{14}^M + \frac{c_1^M\left(\frac{a_2^M}{b_2^L}\right)^p}{1+\left(\frac{a_2^M}{b_2^L}\right)^p}}{b_1^L} \overset{\text{def}}{=} M_1. \tag{3.15}$$

Condition (3.10) implies that for enough small positive constants ε_2, one has

$$\frac{a_{11}^L}{a_{12}^M + a_{13}^M(M_1+\varepsilon_2)} - a_{14}^M + \frac{c_1^L\left(\frac{a_2^L}{b_2^M}-\varepsilon_2\right)^p}{1+\left(\frac{a_2^L}{b_2^M}-\varepsilon_2\right)^p} > 0 \tag{3.16}$$

From the second equation of system (1.1), similarly to the analysis of (3.8)-(3.9), one could easily see that

$$\liminf_{t\to+\infty} y(t) \ge \frac{a_2^L}{b_2^M}. \tag{3.17}$$

Hence, for above $\varepsilon_2 > 0$, from (3.16) and (3.17), there exists a $T_4 > 0$ such that

$$y(t) > \frac{a_2^L}{b_2^M} - \varepsilon_2 \ \text{ for all } \ t \ge T_4. \tag{3.18}$$

$$x(t) < M_1 + \varepsilon_2 \ \text{ for all } \ t \ge T_4. \tag{3.19}$$

For $t > T_4$, (3.18), (3.19) together with the first equation of system (1.11) lead to

$$\frac{dx}{dt} \ge x\Big(\Delta_{\varepsilon_2} - b_1^M x\Big). \tag{3.20}$$

where

$$\begin{aligned}\Delta_{\varepsilon_2} \overset{\text{def}}{=}\ & \frac{a_{11}^L}{a_{12}^M + a_{13}^M(M_1+\varepsilon_2)} - a_{14}^M \\ &+ \frac{c_1^L\left(\frac{a_2^L}{b_2^M}-\varepsilon_2\right)^p}{1+\left(\frac{a_2^L}{b_2^M}-\varepsilon_2\right)^p}.\end{aligned} \tag{3.21}$$

Applying Lemma 3.1 to (3.21) leads to

$$\liminf_{t\to+\infty} x(t) \ge \frac{\Delta_{\varepsilon_2}}{b_1^M}. \tag{3.22}$$

Since ε_2 is any positive constant small enough, setting $\varepsilon_2 \to 0$ in (3.22) leads to

$$\liminf_{t\to+\infty} x(t) \ge \frac{\Delta}{b_1^M} \overset{\text{def}}{=} m_1. \tag{3.23}$$

where

$$\Delta \overset{\text{def}}{=} \frac{a_{11}^L}{a_{12}^M + a_{13}^M M_1} - a_{14}^M + \frac{c_1^L\left(\frac{a_2^L}{b_2^M}\right)^p}{1+\left(\frac{a_2^L}{b_2^M}\right)^p}. \tag{3.24}$$

(3.11), (3.15), (3.17) and (3.23) shows the conclusion of Theorem 3.2 holds, this ends the proof of Theorem 3.2.

Now let's further consider the periodic case of the system (1.11), As a direct corollary of Theorem 2 in [37], from Theorem 3.2, we have

Corollary 2.1. Under the assumption (3.10) holds, assume further that all the coefficients of the system (1.11) are the continuous positive T-periodic function, then system (1.3) admits at least one positive T-periodic solution.

IV. Numeric simulations

Now let's consider the following examples.

Example 4.1

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{2}{3+2x} - 1 - \frac{1}{2}x + \frac{3}{2}\frac{y}{1+y}\Big), \\ \frac{dy}{dt} &= y\big(1-y\big). \end{aligned} \tag{4.1}$$

In this system, corresponding to system (1.10), we take $a_{11} = 2, a_{12} = 3, a_{13} = 2, a_{14} = 1, b_1 = \frac{1}{2}, c_1 = \frac{3}{2}, p = 1, a_2 = 1, b_2 = 1$. From the second equation of system (4.1), we have $y^* = 1$, and

$$\frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1 y^*}{1+y^*} = \frac{5}{12} > 0,$$

it follows from Theorem 3.1 that the positive equilibrium $A_3(x^*, y^*) = (0.5, 1)$ is globally asymptotically stable. Fig. 1 supports this assertion.

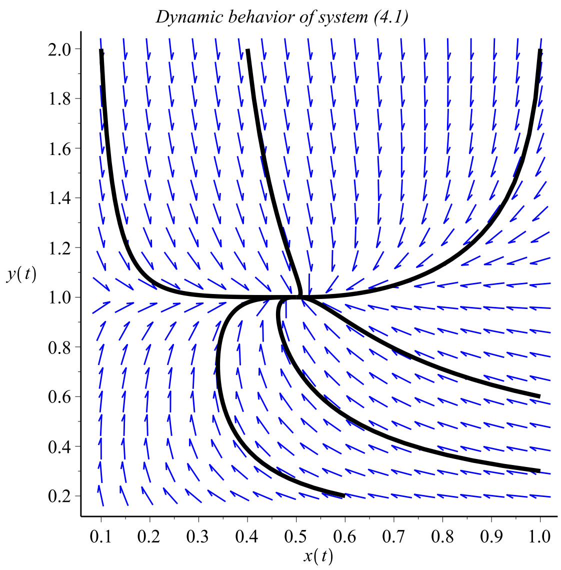

Fig. 1. Dynamic behaviors of the system (4.1), the initial condition $(x(0), y(0)) = (1, 0.3), (0.4, 2), (1, 2), (0.1, 2), (0.6, 0.2)$ and $(1, 0.6)$, respectively.

Example 4.2

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{2}{3+2x} - 1 - \frac{1}{2}x + \frac{1}{6}\frac{y}{1+y}\Big), \\ \frac{dy}{dt} &= y\Big(1-y\Big). \end{aligned} \tag{4.2}$$

In this system, all the coefficients are the same as that of system (4.1), only with c_1 changed to $\frac{1}{6}$. From the second equation of system (4.2), we also have $y^* = 1$, and so,

$$\frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1 y^*}{1+y^*} < -\frac{1}{6} < 0,$$

it follows from Theorem 3.1 that the boundary equilibrium $A_2(0, \frac{a_2}{b_2}) = (0, 1)$ is globally asymptotically stable. Fig. 2 supports this assertion.

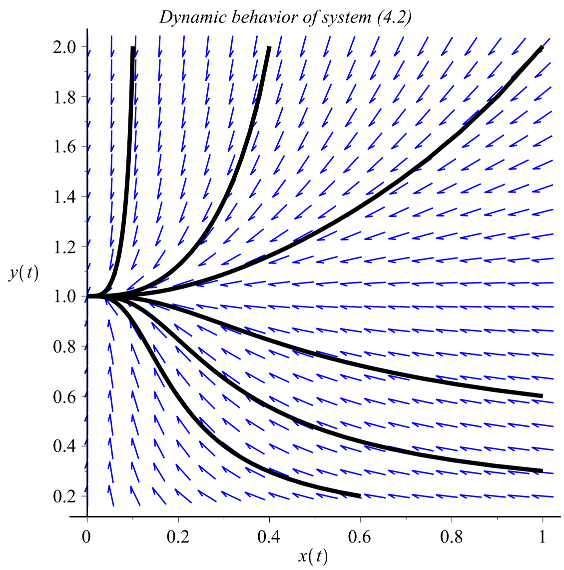

Fig. 2. Dynamic behaviors of the system (4.2), the initial condition $(x(0), y(0)) = (1, 0.3), (0.4, 2), (1, 2), (0.1, 2), (0.6, 0.2)$ and $(1, 0.6)$, respectively.

Example 4.3

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{1.5+0.5\sin(t)}{1.5-0.5\sin(t)+x(t)} - 3 - x + \frac{1}{2}\frac{y}{1+y}\Big), \\ \frac{dy}{dt} &= y\Big(1 - 0.5\cos(t) - (1+0.5\sin(t))y\Big). \end{aligned} \tag{4.3}$$

One could easily check that the coefficients of system (4.3) satisfies the inequality (3.1), hence it follows from Theorem 3.1 that the first species will be driven to extinction, while the second species is permanent. Fig. 3 and 4 support this assertion.

Example 4.4

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{1.5+0.5\sin(t)}{1.5-0.5\sin(t)+x(t)} - 3 - x + \frac{1}{2}\frac{y}{1+y}\Big), \\ \frac{dy}{dt} &= y\Big(1 - 0.5\cos(t) - (1+0.5\sin(t))y\Big). \end{aligned} \tag{4.4}$$

One could easily check that the coefficients of system (4.4) satisfies the inequality (3.10), hence it follows from Theorem 3.2 and corollary 3.1 that the system is permanent and admits at least one 2π-periodic solution. Fig. 5 and 6 support this assertion.

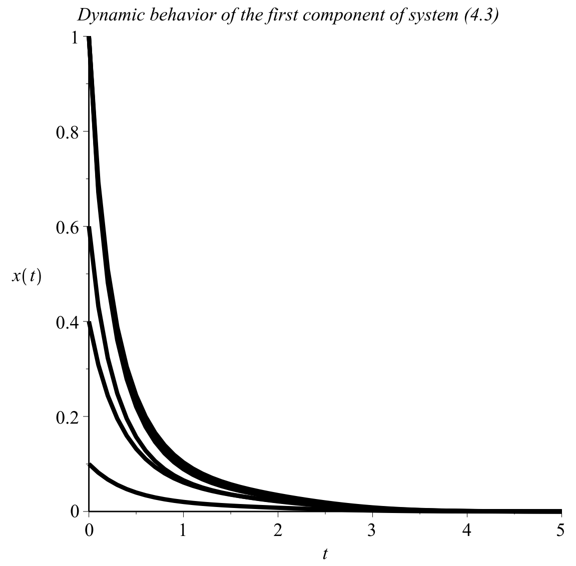

Fig. 3. Dynamic behaviors of the first component of system (4.3), the initial condition $(x(0), y(0)) = (0.5, 0.01), (1.5, 1), (0.7, 0.3)$, $(1, 0.7)$ and $(0.3, 1)$, respectively.

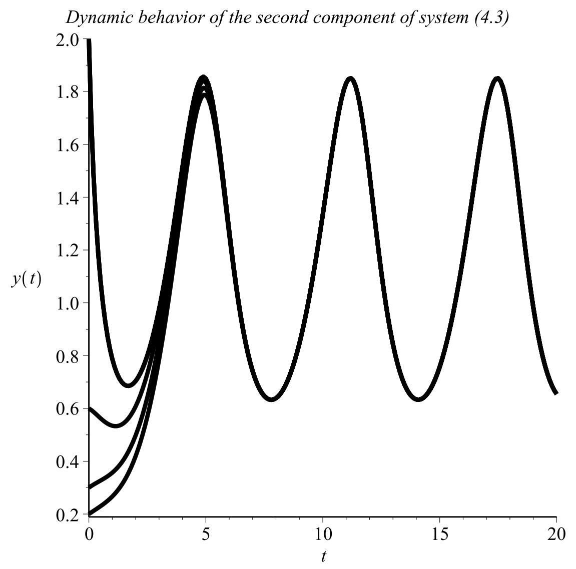

Fig. 4. Dynamic behaviors of the second component of system (4.3), the initial condition $(x(0), y(0)) = (0.5, 0.01), (1.5, 1), (0.7, 0.3)$, $(1, 0.7)$ and $(0.3, 1)$, respectively.

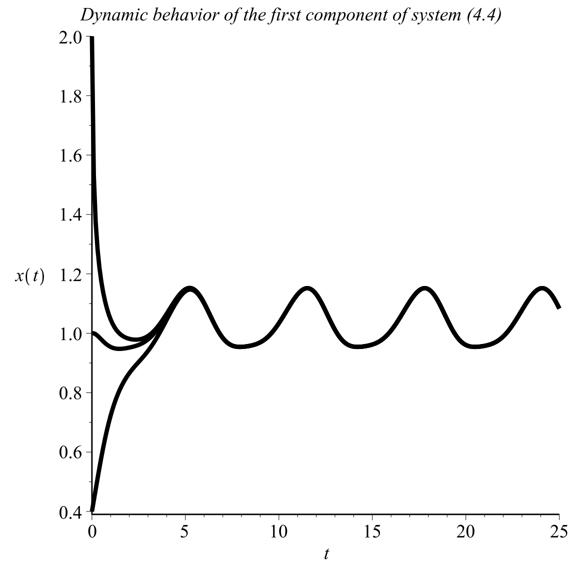

Fig. 5. Dynamic behaviors of the first component of system (4.4), the initial condition $(x(0), y(0)) = (1, 1), (0.4, 0.4)$ and $(2, 2)$, respectively.

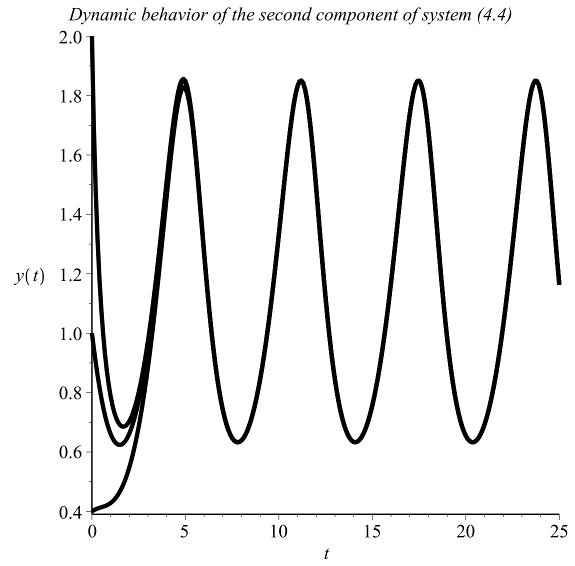

Fig. 6. Dynamic behaviors of the second component of system (4.4), the initial condition $(x(0), y(0)) = (1, 1), (0.4, 0.4)$ and $(2, 2)$, respectively.

V. Discussion

Recently, many scholars studied the dynamic behaviors of the commensalism symbiosis model ([17]-[28]). All of the works of [17]-[28] are based on the traditional Logistic model. Specially, Wu et al[17] had showed that the system (1.4) admits a unique positive equilibrium which is globally asymptotically stable, which means that the two species could be coexist in a stable state.

In this paper, we argued that it is more suitalbe to consider the density dependent birth rate of the species, consequently, we propose the system (1.10). Theorem 2.2 shows that depending on the sign of the term $\frac{a_{11}}{a_{12}} - a_{14} + \frac{c_1(y^*)^p}{1+(y^*)^p}$, the positive equilibrium A_3 or the boundary equilibrium A_2 maybe globally asymptotically stable, respectively. Noting that the stability of A_2 means that the extinction of the first species. That is, by introducing the density dependent birth rate, the dynamic behaviors of the system becomes complicated. Also, from (2.5) and (2.6) we can see that the cooperate intensity between the species (represented by c_1) plays important role on the persistent or extinct of the species, this is confirmed by the numeric simulation.

Next, we consider the non-autonomous case of system

(1.10), i.e, system (1.11), by using the differential inequality theory, we obtain sufficient conditions which ensure the permanence and partial survival of the system, respectively. Also, for the periodic case, sufficient conditions which ensure the existence of at least one positive T-periodic solution is obtained.

We mentioned here that we only consider the density dependent birth rate of the first species, one could expect that if both species have the density dependent birth rate, the dynamic behaviors of the system maybe become more complicated.

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