Analytical Solutions of a Continuous Arithmetic Asian Model for Option Pricing using Projected Differential Transform Method

Sunday O. EDEKI, *Member, IAENG*, Olabisi O. UGBEBOR, and Paul O. OGUNDILE

Abstract—**In financial mathematics, the procedure for finding the solutions of the Black-Scholes model defined in terms of Arithmetic Asian Option (AAO) is one of the most tasking issues when considering its corresponding analytical solutions. In this paper, such analytical solution of a continuous arithmetic Asian option is obtained through the application of a proposed semi-analytical approach known as Projected Differential transform Method (PDTM). The Asian option model in continuous arithmetic form witnesses the application of PDTM for the first time in literature (to the authors' best of knowledge). The PDTM entails less computational work, even without neglecting the high level of accuracy. The obtained solution is in agreement with those in literature via other solution methods. In terms of recommendation, the proposed solution method will be of great interest for related versions or forms of Asian option pricing models (geometric) likewise other financial nonlinear differential models.**

Index Terms— **Option pricing, Asian option, Black-Scholes model, Adomian polynomials, Analytical solution**

I. INTRODUCTION

OPTION pricing is a basic aspect of financial fields such as financial mathematics, financial engineering, and financial physics. Asian option is a special form of option contracts. The payoff functions associated with Asian options depend significantly on the average underlying price over some specified period of time usually referred to as the option life time [2, 3]. They are thus, classified as path dependent options since the payoff function depends on the sequence average of the prices of the underlying asset over some specified period of time. This indeed is a different case compared to the usual American and European style of options, where the option payoff function depends on the price of the underlying stock at exercise [4-7]. Asian options are therefore referred to as one of the main forms of exotic options. Asian options are advantageous because they minimize the risk of market influence of the underlying stock at maturity, and also encompass relative cost compared to American or European options [8].

Averages in option are of two forms namely: geometric and arithmetic. Both of these can be structured as either calls or puts. It is noted that the geometric Asian option has a closed form solution, therefore the ease of pricing [9]. The arithmetic type happens to be the most frequently used but it seems challenging for pricing in terms of closed form solution since the average of a class of lognormal random variables (LNRVs) is not log-normally distributed [10]. Providing a simple analytical expression for the valuation of Asian (arithmetic) options has posed a lot of problems in financial mathematics [1].

In a bid to handle this, some researchers have developed and adopted varieties of solution techniques: German and Yor [10], consider Asian option price via Laplace transform but for some cases based on the conferment region of the transform. Rogers and Shi [1], consider the problem of valuing the Asian option in two ways by reducing the problem to a two-variable parabolic partial differential equation (PDE) using scaling property, and thereafter provide a lower bound formulae on the basis of some zero-mean Gaussian variable. In [11],Vecer builds his approach on traded account by providing a PDE for Asian option in one-dimensional form. Zhang [12], considers Asian option pricing via a continuously-sampled theory in line with a perturbation technique.

Chen and Lyuu [13], extended the work of Rogers and Shi [1] by including general maturity terms instead of one year with regard to the lower bound pricing formulae.

Elshegmani, Ahmad and Zakaria [14] using Fourier Transform, present a simple solution for the Asian option PDE proposed by Rogers and Shi [1]. Kumar, Waikos and Chakrabarty [15], obtain a standard partial differential equation for the price of arithmetic average strike Asian call option via the derivation of a Crank-Nicolson implicit approach and a higher order compact finite different scheme for the considered pricing problem at various values of risk-free and volatility rates. Zhang, Yu, and Wang [16], study the numerical solution for the Delta of Asian arithmetic option and provide a simple, fast and reliable solution on the basis of Monte Carlo Simulation (MCS). Fadugba [17], applies the Mellin transform method for analytical solution of geometric Asian option and obtain a closed form solution for the continuous geometric version of the Asian option. Elshegmani, and Ahmad [18], obtain a closed form solution for a continuous arithmetic Asian (CAA) option via PDE by Mellin transform. Recently, Elshegmani and Ahmad [2],

Manuscript received March 17, 2018; revised October 15, 2018.

S. O. Edeki (soedeki@yahoo.com) is with the Department of Mathematics, Covenant University, Ota, Nigeria.

O. O. Ugbebor is with the Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

P. O. Ogundile is with the Department of Mathematics, Covenant University, Ota, Nigeria.

apply the PDE approach coupled with Laplace transform as an efficient method for pricing an arithmetic Asian option of continuous type. They [2] do this by transforming the PDE of the Asian option from three dimensional-form to two dimensional ordinary differential equation (ODE).

In this paper, PDTM which is an improved form of the standard differential transform method (DTM) is basically applied to a CAA option pricing model for analytical solutions.

In terms of structure, in the rest sections of the paper are organised as follows: we have a concise note as regards Asian option pricing model in section 2; section 3 is on PDTM- the proposed method of solution; applications and examples are considered in section 4 while section 5 is on concluding remarks.

II. Asian Option Pricing Model via the Two-Dimensional Black-Scholes Equation

Suppose the price of the stock, $S(t)$ follows a geometric (exponential) Brownian Motion (BM) governed by the stochastic differential (dynamical) equation of the form:

$$dS(t) = S(t)\left(rdt + \sigma dW(t)\right),\ t \in \mathbb{R}^{+} \tag{2.1}$$

where r is a drift term (average rate of growth), σ is a volatility coefficient, $0 \le t \le T$, and $W(t)$ a BM. Then the payoff function for an Asian option associated with an arithmetic average strike (AAS) [19-21] is defined and denoted by:

$$\Lambda(T) = \left(-\frac{1}{T}\int_0^T S(\varsigma)\,d\varsigma + S(T),\ 0\right)^{+}. \tag{2.2}$$

The price of the option at $0 \le t \le T$ is an equivalent martingale pricing formula (risk-neutral form) defined and denoted by [19] as:

$$\Lambda(t) = \mathrm{E}\left(e^{-r(T-t)}\Lambda(T)\middle|F_t\right) \tag{2.3}$$

where $E(\cdot)$ and F_t represent expected-value operator and filtration respectively.

Now let us consider $\Lambda(S,I,t)$ as the price function and $I = I(t)$ the path dependent variable be defined as:

$$I(t) = \int_0^t h\left(S(\varsigma),\varsigma\right)d\varsigma \tag{2.4}$$

where $h(\cdot)$ is a specific function for any possible path-dependent option under consideration. It is obvious that $\Lambda = \Lambda(S,I,t)$ is a function of three variables since $I = I(t)$ does not depend on S (the current asset price). Hence,

$$\frac{dI}{dt} = h\left(S(t),t\right). \tag{2.5}$$

As such, applying the multidimensional Ito's lemma on $\Lambda(S,I,t)$ gives:

$$d\Lambda = \begin{cases} \left(\begin{matrix} \frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2} + rS\frac{\partial\Lambda}{\partial S} + \frac{\partial\Lambda}{\partial t} \\ +h(S,t)\frac{\partial\Lambda}{\partial I}\end{matrix}\right)dt \\ +\sigma S\frac{\partial\Lambda}{\partial S}dW \end{cases}. \tag{2.6}$$

Suppose we construct a delta-hedge portfolio, Π, by longing a contingent claim value, and shorting a delta unit of the asset price, then we have:

$$\begin{cases} \Pi = \Lambda - \Delta S \\ d\Pi = d\Lambda - \Delta dS. \end{cases} \tag{2.7}$$

So, (2.1) and (2.6) in (2.7) gives:

$$\begin{aligned} d\Pi &= d\Lambda - \Delta dS \\ &= \begin{cases} \left\{\begin{matrix}\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2} + rS\frac{\partial\Lambda}{\partial S} + \frac{\partial\Lambda}{\partial t} + h(S,t)\frac{\partial\Lambda}{\partial I}\right)dt \\ +\sigma S\frac{\partial\Lambda}{\partial S}dW\end{matrix}\right\} \\ -\Delta\left\{rS(t)dt + \sigma S(t)dW(t)\right\}. \end{cases} \end{aligned} \tag{2.8}$$

We set $\Delta = \frac{\partial\Lambda}{\partial S}$ in order to eliminate the stochastic term from the portfolio, hence,

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2} + \frac{\partial\Lambda}{\partial t} + h(S,t)\frac{\partial\Lambda}{\partial I}\right)dt. \tag{2.9}$$

For adopting the notion of no arbitrage opportunities, (it implies that change in the portfolio needs to coincide with the change of the corresponding monetary value as deposit in the bank account for a risk-free interest rate). Thus,

$$\begin{aligned} d\Pi &= r\Pi dt \\ &= r\left(\Lambda - \Delta S\right)dt \\ &= r\left(\Lambda - \frac{\partial\Lambda}{\partial S}S\right)dt. \end{aligned} \tag{2.10}$$

So, combining (2.9) and (2.10) gives:

$$\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2} + \frac{\partial\Lambda}{\partial t} + h(S,t)\frac{\partial\Lambda}{\partial I}\right)dt = r\left(\Lambda - \frac{\partial\Lambda}{\partial S}S\right)dt. \tag{2.11}$$

That implies that:

$$\left(\frac{\partial\Lambda}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2} + rS\frac{\partial\Lambda}{\partial S} + h(S,t)\frac{\partial\Lambda}{\partial I}\right) - r\Lambda = 0. \tag{2.12}$$

In order to consider the Black-Scholes model for Asian option, the payoff $\Lambda(T)$ is based the trajectory past knowledge (path dependent). Thus, a stochastic process in (2.13) is introduced [19]:

$$I(t)=\int_0^t S(\omega)\,d\omega \tag{2.13}$$

whose first derivative w.r.t. time, t, gives $h(\cdot)$, and $I(t)$ denotes the strike price running total. Hence,

$$\frac{dI}{dt}=S(t)=h(S,t). \tag{2.14}$$

So, using (2.14) in (2.12) gives the associated option price of the Asian call described by the following model:

$$\left(\frac{\partial\Lambda}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2\Lambda}{\partial S^2}+rS\frac{\partial\Lambda}{\partial S}+S\frac{\partial\Lambda}{\partial I}\right)-r\Lambda=0 \tag{2.15}$$

solved by $\Lambda(S,I,t)$ for $t\in[0,\infty)$ and $S\in(0,\infty)$. It is obvious that (2.15) is comparable to the classical form of the time-fractional Black-Scholes (BS) option pricing model at $\alpha=1$ but for the averaging term denoted by $\left(S\frac{\partial\Lambda}{\partial I}\right)$ [22-23]. In most cases, modification may be required leading to the adoption of numerical-approximate or semi-analytical methods [24-29]. Though, the associated volatility in (2.5) can be further viewed from the aspect of transaction cost inclusion, and stochastic point while adopting a simple and a class control variate methods [30-32]. Rehurek [33] considered (2.15) by discussing some numerical techniques as well as their corresponding variations. Recently, analytical solutions and the associated Greek (sensitivities) parameters of a CAA option pricing model were considered [34, 35]. It is obvious that (2.15) will finally yield a problematic issue in terms of computational since it is of three (3) dimensional. Therefore, the notion of reduction to a lower level in dimension is needed with aid of transformation variables [2, 36]:

$$\begin{cases}\Lambda(S,I,t)=Sg(t,\psi),\\ \psi S=k-\dfrac{I}{T}.\end{cases} \tag{2.16}$$

Hence, (2.5) becomes:

$$\begin{cases}\dfrac{\partial g}{\partial t}+\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 g}{\partial\psi^2}-\left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial g}{\partial\psi}=0,\\ g(T,\psi)=\varphi(\psi).\end{cases} \tag{2.17}$$

It can thus be remarked that (2.17) is now in two dimensional form. The solution of this (2.17) will yield the price of the Asian option via the relation in (2.16).

III. The DTM and Projected DTM

A. Analysis of a Two-Dimensional DTM

Here, we make remarks on the preliminaries of two dimensional version of the DTM as follows.

Suppose the two-variable function: $\upsilon(x,y)$ is analytic function at (x_*,y_*) w.r.t. the Domain, D_* so, the differential transform of $\upsilon(x,y)$ is presented as follows:

$$\Upsilon(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}\upsilon(x,y)}{\partial x^k\partial y^h}\right]_{(x,y)=(x_*,y_*)}$$

while the differential inverse transform of $\Upsilon(k,h)$ is:

$$\upsilon(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\Upsilon(k,h)(x-x_*)^k(y-y_*)^h,$$

where the capital and the small letters denote the transformed and the original functions respectively.

B. Projected Differential Transform Method

We present a brief but concise introduction of the proposed method (PDTM) with emphasis on the basic features and properties [37, 38].

Let $z(\omega,t)$ be considered analytic w.r.t (ω_*,t_*) in a given domain D, then by the Taylor series expansion of $z(\omega,t)$, with a preference to some variables, say: $s_v=t$ rather than all the concerned variables as presumed in the standard DTM. Therefore, the projected DTM of $z(\omega,t)$ with respect to t at t_* is defined and denoted by:

$$Z(\omega,\varepsilon)=\frac{1}{\varepsilon!}\left[\frac{\partial^{\varepsilon}z(\omega,t)}{\partial t^{\varepsilon}}\right]_{t=t_*} \tag{3.1}$$

and consequently:

$$z(\omega,\varepsilon)=\sum_{\varepsilon=0}^{\infty}Z(\omega,\varepsilon)(-t_*+t)^{\varepsilon}. \tag{3.2}$$

Equation (3.2) is the inverse PDTM of $Z(\omega,\varepsilon)$ in (3.1).

Note, at $t=t_*=0$, (3.1) helps in realizing $Z(x,0)$ from the given initial condition(s) $z(x,0)$, where the capital and the small letters denote the transformed and the original functions respectively.

C. Properties of the proposed method: PDTM

Let $z(\omega,t)$ be as defined above, then the following hold:

a: $$\begin{cases}z(\omega,t)=\alpha z_a(\omega,t)+\beta z_b(\omega,t)\\ \Rightarrow\ Z(\omega,\varepsilon)=\alpha Z_a(\omega,\varepsilon)+\beta Z_b(\omega,\varepsilon)\end{cases},$$

b: $$\begin{cases}z(\omega,t)=\beta\dfrac{\partial^n z_*(\omega,\mathrm{t})}{\partial t^n}\\ \Rightarrow Z(\omega,\varepsilon)=\beta\dfrac{(\varepsilon+n)!}{\varepsilon!}Z_*(\omega,\varepsilon+n)\end{cases},$$

c: $\begin{cases} z(\omega,t)=\alpha\dfrac{\partial z_*(\omega,t)}{\partial t} \\ \Rightarrow Z(\omega,\varepsilon)=\alpha\dfrac{(\varepsilon+1)!Z_*(\omega,\varepsilon+1)}{\varepsilon!} \\ \qquad =\alpha(\varepsilon+1)Z_*(\omega,\varepsilon+1) \end{cases}$,

d: $\begin{cases} z(\omega,t)=p(\omega)\dfrac{\partial^n z_*(\omega,\mathrm{t})}{\partial\omega^n} \\ \Rightarrow Z(\omega,\varepsilon)=p(\omega)\dfrac{\partial^n Z_*(\omega,\varepsilon)}{\partial\omega^n} \end{cases}$,

e: $\begin{cases} z(\omega,t)=p(\omega)z_*^2(\omega,t) \\ \Rightarrow Z(\omega,h)=p(\omega)\sum_{r=0}^{h} Z_*(\omega,r)Z_*(\omega,h-r), \end{cases}$

f: $\begin{cases} l(\omega,y)=\omega^n y^{n^*} \\ \Rightarrow L(k,h)=\delta(k-n,h-n^*) \\ \qquad =\delta(k-n)\delta(h-n^*), \end{cases}$

where

$$\delta(k-n)=\begin{cases}1\ ,\ \text{if } k=n \\ 0\ ,\ \text{if } k\neq n\end{cases},$$

$$\delta(k-n^*)=\begin{cases}1\ ,\ \text{if } k=n^* \\ 0\ ,\ \text{if } k\neq n^*.\end{cases}$$

IV. APPLICATIONS / ILLUSTRATIVE EXAMPLES

In this section, analytical solutions the Asian option pricing model are considered based on the PDTM with regard to three cases: I, II and III as follows:

A. Case I

Consider (2.15) via (2.16-2.17) in the following form:

$$\begin{cases} \dfrac{\partial g}{\partial t}=\left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial g}{\partial\psi}-\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 g}{\partial\psi^2} \\ g(0,\psi)=\varphi(\psi). \end{cases} \tag{4.1}$$

Taking the PDTM of (4.1) gives:

$$G(\psi,h+1)=\frac{1}{h+1}\begin{Bmatrix} \left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,h)}{\partial\psi} \\ -\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 G(\psi,h)}{\partial\psi^2} \end{Bmatrix}, \tag{4.2}$$

$$G(\psi,0)=\frac{1}{rT}\left(1-e^{-rT}\right)-\psi e^{-rT}. \tag{4.3}$$

Hence, for $h\geq 0$, we have:

$$G(\psi,1)=\begin{Bmatrix} \left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,0)}{\partial\psi} \\ -\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 G(\psi,0)}{\partial\psi^2} \end{Bmatrix},$$

$$G(\psi,2)=\frac{1}{2}\begin{Bmatrix} \left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,1)}{\partial\psi} \\ -\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 G(\psi,1)}{\partial\psi^2} \end{Bmatrix},$$

$$G(\psi,3)=\frac{1}{3}\begin{Bmatrix} \left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,2)}{\partial\psi} \\ -\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 G(\psi,2)}{\partial\psi^2} \end{Bmatrix},$$

$$G(\psi,4)=\frac{1}{4}\begin{Bmatrix} \left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,3)}{\partial\psi} \\ -\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 G(\psi,3)}{\partial\psi^2} \end{Bmatrix},$$

$\vdots$

$$G(\psi,p)=\frac{1}{p}\begin{Bmatrix} \left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,p-1)}{\partial\psi} \\ -\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 G(\psi,p-1)}{\partial\psi^2} \end{Bmatrix}_{p\geq 1.} \tag{4.4}$$

Thus, by subjecting (4.1) to:

$$g(\psi,0)=\frac{1}{rT}\left(1-e^{-rT}\right)-\psi e^{-rT}, \tag{4.5}$$

The following are obtained:

$$G(\psi,1)=-\left(\frac{1}{T}+r\psi\right)e^{-rT},$$

$$G(\psi,2)=-\frac{1}{2!}\left(\frac{1}{T}+r\psi\right)re^{-rT},$$

$$G(\psi,3)=-\frac{1}{3!}\left(\frac{1}{T}+r\psi\right)r^2e^{-rT},$$

$$G(\psi,4)=-\frac{1}{4!}\left(\frac{1}{T}+r\psi\right)r^3e^{-rT},$$

$$G(\psi,5)=-\frac{1}{5!}\left(\frac{1}{T}+r\psi\right)r^4e^{-rT},$$

$$G(\psi,6)=-\frac{1}{6!}\left(\frac{1}{T}+r\psi\right)r^5e^{-rT},$$

$$G(\psi,7)=-\frac{1}{7!}\left(\frac{1}{T}+r\psi\right)r^6e^{-rT},$$

$$G(\psi,8)=-\frac{1}{8!}\left(\frac{1}{T}+r\psi\right)r^7e^{-rT},$$

$$G(\psi,9)=-\frac{1}{9!}\left(\frac{1}{T}+r\psi\right)r^8e^{-rT},.$$

$\vdots$

$$G(\psi,p)=-\frac{1}{p!}\left(\frac{1}{T}+r\psi\right)r^{p-1}e^{-rT},\ p\geq 1$$

Hence,

$$g(\psi,t)=\sum_{m=0}^{\infty}G(\psi,m)t^{m}$$

$$=\begin{cases}\left(\frac{1}{rT}\left(1-e^{-rT}\right)-\psi e^{-rT}\right)\\-\left(t+\frac{t^2r}{2!}+\frac{t^3r^2}{3!}+\frac{t^4r^3}{4!}+\cdots\right)\left(\frac{1}{T}+r\psi\right)e^{-rT}\end{cases}$$

$$=\begin{cases}\left(\frac{1}{rT}\left(-e^{-rT}+1\right)-\psi e^{-rT}\right)\\-\frac{e^{-Tr}}{r}\left(rt+\frac{(rt)^2}{2!}+\frac{(rt)^3}{3!}+\frac{(rt)^4}{4!}+\cdots\right)\left(\frac{1}{T}+r\psi\right)\end{cases}$$

$$=\left(-\psi e^{-rT}+\frac{1}{Tr}\left(-e^{-Tr}+1\right)\right)-\frac{1}{r}\sum_{i=1}^{\infty}\frac{(rt)^i}{i!}\left(\frac{1}{T}+r\psi\right)e^{-rT}$$

$$=\begin{cases}\left(\frac{1}{rT}\left(-e^{-rT}+1\right)-\psi e^{-rT}\right)\\-\frac{e^{-rT}}{r}\left(-1+\sum_{i=0}^{\infty}\frac{(rt)^i}{i!}\right)\left(\frac{1}{T}+r\psi\right)\end{cases}$$

$$=\begin{cases}\left(\frac{1}{rT}\left(-e^{-rT}+1\right)-\psi e^{-rT}\right)\\-\frac{e^{-rT}}{r}\left(-1+e^{rt}\right)\left(\frac{1}{T}+r\psi\right)\end{cases},$$

$$=\left(\frac{1}{rT}\left(-e^{-r(T-t)}+1\right)-\psi e^{-r(T-t)}\right),\ \psi\le 0. \tag{4.6}$$

But from (2.16), $\begin{cases}\Lambda(S,I,t)=Sg(t,\psi),\\ \psi S=-\frac{I}{T}-k.\end{cases}$

Therefore,

$$\Lambda(S,I,t)=\left(\frac{S}{rT}\left(-e^{-r(T-t)}+1\right)-\left(k-\frac{I}{T}\right)e^{-(T-t)r}\right). \tag{4.7}$$

Equation (4.7) is the analytical solution of (2.15) corresponding to the CAA option pricing model. This can be validated as follows. From (4.5), we have for $\Lambda(S,I,t)=\Lambda$:

$$\frac{\partial\Lambda}{\partial t}=\left[\frac{-Se^{-r(T-t)}}{T}-r\left(k-\frac{I}{T}\right)e^{-r(T-t)}\right], \tag{4.8}$$

$$\begin{cases}\frac{\partial\Lambda}{\partial S}=\left(\frac{1}{rT}\left(-e^{-r(T-t)}+1\right)\right)\\ \frac{\partial^2\Lambda}{\partial S^2}=0\end{cases}, \tag{4.9}$$

$$T\partial\Lambda=e^{-r(T-t)}\partial I. \tag{4.10}$$

Thus, using (4.5) through (4.8) in (2.15) gives:

$$\frac{\partial\Lambda}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2\Lambda}{\partial S^2}+rS\frac{\partial\Lambda}{\partial S}+S\frac{\partial\Lambda}{\partial I}-r\Lambda=H(S,I,t).$$

Thus, for $H(S,I,t)=H$, we have:

$$H=\left(\begin{array}{c}\left\{\frac{-Se^{-r(T-t)}}{T}-r\left(k-\frac{I}{T}\right)e^{-r(T-t)}\right\}\\ +rS\left(\frac{1}{rT}\left(1-e^{-r(T-t)}\right)\right)+\frac{Se^{-r(T-t)}}{T}\\ -r\left(\frac{S}{rT}\left(1-e^{-r(T-t)}\right)-\left(k-\frac{I}{T}\right)e^{-r(T-t)}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\frac{-Se^{-r(T-t)}}{T}-r\left(k-\frac{I}{T}\right)e^{-r(T-t)}+\left(\frac{S}{T}\left(1-e^{-r(T-t)}\right)\right)\\ +\frac{Se^{-r(T-t)}}{T}-\frac{S}{T}\left(1-e^{-r(T-t)}\right)+r\left(k-\frac{I}{T}\right)e^{-r(T-t)}\end{array}\right)$$

$$=0=H(S,I,t).$$

B. Case II

Consider (2.15) via (2.16-2.17) subject to the initial condition:

$$g(\psi,0)=\left(\psi r-\frac{1}{rT}\right)Se^{-rT}. \tag{4.11}$$

This gives:

$$\begin{cases}\frac{\partial g}{\partial t}=\left(\frac{1}{T}+r\psi\right)\frac{\partial g}{\partial\psi}-\frac{1}{2}\sigma^2\psi^2\frac{\partial^2 g}{\partial\psi^2}\\ g(\psi,0)=\left(\psi r-\frac{1}{rT}\right)Se^{-rT}\end{cases} \tag{4.12}$$

Taking the PDTM of (4.12) yields:

$$\begin{cases}G(\psi,h+1)=\frac{1}{h+1}\begin{Bmatrix}\left(\frac{1}{T}+r\psi\right)\frac{\partial G(\psi,h)}{\partial\psi}\\ -\frac{1}{2}\sigma^2\psi^2\frac{\partial^2G(\psi,h)}{\partial\psi^2}\end{Bmatrix},\\ G(\psi,0)=\left(\psi r-\frac{1}{rT}\right)Se^{-rT},\ h\in\mathbb{N}\cup\{0\}.\end{cases} \tag{4.13}$$

Thus, following the approach in *Case I*, the following are obtained:

$$G(\psi,1)=\left(\frac{1}{T}+r\psi\right)rSe^{-rT},$$

$$G(\psi,2)=\frac{1}{2!}\left(\frac{1}{T}+r\psi\right)r^2Se^{-rT},$$

$$G(\psi,3)=\frac{1}{3!}\left(\frac{1}{T}+r\psi\right)r^3Se^{-rT},$$

$$G(\psi,4)=\frac{1}{4!}\left(\frac{1}{T}+r\psi\right)r^4Se^{-rT},$$

$$G(\psi,5)=\frac{1}{5!}\left(\frac{1}{T}+r\psi\right)r^5Se^{-rT},$$

$$G(\psi,6)=\frac{1}{6!}\left(\frac{1}{T}+r\psi\right)r^6Se^{-rT},$$

$$G(\psi,7)=\frac{1}{7!}\left(\frac{1}{T}+r\psi\right)r^7Se^{-rT},$$

$$G(\psi,8)=\frac{1}{8!}\left(\frac{1}{T}+r\psi\right)r^8Se^{-rT},$$

$$G(\psi,9)=\frac{1}{9!}\left(\frac{1}{T}+r\psi\right)r^9e^{-rT},$$

$$\cdots G(\psi,p)=\frac{1}{p!}\left(\frac{1}{T}+r\psi\right)r^pSe^{-rT},\ p\in\mathbb{N}.$$

So, we have:

$$g(\psi,t)=\sum_{m=0}^{\infty}G(\psi,m)t^m$$

$$=\begin{cases}\left(\psi r-\dfrac{1}{rT}\right)Se^{-rT}\\+\left(rt+\dfrac{(rt)^2}{2!}+\dfrac{(rt)^3}{3!}+\dfrac{(rt)^4}{4!}+\cdots\right)\left(\dfrac{1}{T}+r\psi\right)Se^{-rT}\end{cases}$$

$$=\left(\psi r-\frac{1}{rT}\right)Se^{-rT}+\left(-1+\sum_{n=0}^{\infty}\frac{(rt)^n}{n!}\right)\left(\frac{1}{T}+r\psi\right)Se^{-rT}$$

$$=\left\{\left(\psi r-\frac{1}{rT}\right)+\left(-1+e^{rt}\right)\left(\frac{1}{T}+r\psi\right)\right\}Se^{-rT}.$$

Thus, we have:

$$\begin{cases}\Lambda(S,I,t)=Sg(t,\psi),\\ \psi S=k-\dfrac{I}{T}.\end{cases}$$

$$\therefore\ \begin{cases}\Lambda=\left\{\left(\psi r-\dfrac{1}{rT}\right)+\left(-1+e^{rt}\right)\left(\dfrac{1}{T}+r\psi\right)\right\}S^2e^{-rT}\\ \Lambda(S,I,t)=\Lambda.\end{cases} \tag{4.14}$$

Equation (4.14) represents the analytical solution of (2.15) subject to (4.11).

C. *Case III*

Suppose the initial condition associated with (2.15) via (2.16-2.17) is

$$g(\psi,0)=\left(1+\frac{1}{rT}\right)\psi S^2e^{-rT} \tag{4.15}$$

then we have:

$$\begin{cases}\dfrac{\partial g}{\partial t}=\left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial g}{\partial\psi}-\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2 g}{\partial\psi^2}\\ g(\psi,0)=\left(1+\dfrac{1}{rT}\right)\psi S^2e^{-rT}.\end{cases} \tag{4.16}$$

Taking the PDTM of (4.16) gives:

$$\begin{cases}G(\psi,h+1)=\dfrac{1}{h+1}\begin{Bmatrix}\left(\dfrac{1}{T}+r\psi\right)\dfrac{\partial G(\psi,h)}{\partial\psi}\\-\dfrac{1}{2}\sigma^2\psi^2\dfrac{\partial^2G(\psi,h)}{\partial\psi^2}\end{Bmatrix}\\ G(\psi,0)=\left(1+\dfrac{1}{rT}\right)\psi S^2e^{-rT}.\end{cases} \tag{4.17}$$

Hence, for $h\geq 0$, we have:

$$G(\psi,0)=\left(1+\frac{1}{rT}\right)\psi S^2e^{-rT}$$

$$G(\psi,1)=\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)S^2e^{-rT},$$

$$G(\psi,2)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)rS^2e^{-rT}}{2!},$$

$$G(\psi,3)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^2S^2e^{-rT}}{3!},$$

$$G(\psi,4)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^3S^2e^{-rT}}{4!},$$

$$G(\psi,5)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^4S^2e^{-rT}}{5!},$$

$$G(\psi,6)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^5S^2e^{-rT}}{6!},$$

$$G(\psi,7)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^6S^2e^{-rT}}{7!},$$

$$G(\psi,8)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^7S^2e^{-rT}}{8!},$$

$$G(\psi,9)=\frac{\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^8S^2e^{-rT}}{9!},$$

$$\cdots G(\psi,p)=-\frac{1}{p!}\left(\frac{1}{T}+r\psi\right)r^{p-1}e^{-rT},\ p\geq 1.$$

Hence,

$$g(\psi,t)=\sum_{m=0}^{\infty}G(\psi,m)t^{m}$$

$$=\begin{Bmatrix}\left(1+\frac{1}{rT}\right)\psi S^{2}e^{-rT}\\+\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)tS^{2}e^{-rT}\end{Bmatrix}$$

$$=\begin{Bmatrix}\frac{1}{2!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)rS^{2}e^{-rT}t^{2}\\+\frac{1}{3!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{2}S^{2}e^{-rT}t^{3}\end{Bmatrix}$$

$$=\begin{Bmatrix}\frac{1}{4!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{3}S^{2}e^{-rT}t^{4}\\+\frac{1}{5!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{4}S^{2}e^{-rT}t^{5}\end{Bmatrix}$$

$$=\begin{Bmatrix}\frac{1}{6!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{5}S^{2}e^{-rT}t^{6}\\+\frac{1}{7!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{6}S^{2}e^{-rT}t^{7}\end{Bmatrix}$$

$$=\begin{Bmatrix}\frac{1}{8!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{7}S^{2}e^{-rT}t^{8}\\+\frac{1}{9!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{8}S^{2}e^{-rT}t^{9}+\cdots\end{Bmatrix}$$

$$g(\psi,t)=\begin{Bmatrix}\left(1+\frac{1}{rT}\right)\psi S^{2}e^{-rT}\\+\sum_{k=1}^{n}\frac{1}{k!}\left(\frac{1}{T}+r\psi\right)\left(1+\frac{1}{rT}\right)r^{k-1}S^{2}e^{-rT}t^{k}\end{Bmatrix},$$

$$=\left(1+\frac{1}{rT}\right)S^{2}e^{-rT}\left\{\psi+\left(\frac{1}{T}+r\psi\right)\frac{1}{r}\sum_{k=1}^{\infty}\frac{(rt)^{k}}{k!}\right\}. \tag{4.18}$$

Hence, by (2.6), we obtain the following:

$$\Lambda=\begin{cases}\left(1+\frac{1}{rT}\right)S^{3}e^{-rT}\left\{\psi+\left(\frac{1}{T}+r\psi\right)\frac{1}{r}\left(-1+e^{rt}\right)\right\},\\ \psi S=k-\frac{I}{T},\ \Lambda=\Lambda(S,I,t).\end{cases} \tag{4.19}$$

Equation (4.19) represents the analytical solution of (2.15) subject to (4.15).

V. Conclusion and Remarks

In this work, we have successfully obtained analytical solutions of the CAA option pricing model via the application of a proposed PDTM. The method involved less computational work without compromising the level of accuracy. Three cases were considered as regards illustration whose results ascertained the method's efficiency and effectiveness. The works of Rogers & Shi [1], and Elshegmani & Ahmad [2] serve as benchmarks to this current work. The proposed semi-analytical approach is therefore endorsed for obtaining analytical and/or approximate solutions of similar forms of Asian option pricing models likewise other financial pricing models. In addition, other financial PDEs resulting from stochastic dynamics can be considered via this approach.

Conflict of Interests

Conflict of interest is not declared by the authors as regards the publication of this paper..

Acknowledgment

The authors acknowledged the Covenant University for support, and encouragement. Sincere regards to the anonymous referees for meaningful comments.

References

[1] L. C. G. Rogers and Z. Shi, "The value of an Asian option", *Journal of Applied Probability*, vol. 32, pp. 1077-1088, 1995.

[2] Z. A. Elshegmania, and R. R. Ahmad, "Solving an Asian option PDE via the Laplace transform", *ScienceAsia*, vol. 39S, pp. 67–69, 2013..

[3] W. Falloon and D. Turner, "The evolution of a market in Managing Energy Price Risk", *Risk Books*, London, UK, 1999.

[4] O. González-Gaxiola, S. O. Edeki, O.O. Ugbebor, and J.R. de Chávez, "Solving the Ivancevic pricing model using the He's frequency amplitude formulation", *European Journal of Pure and Applied Mathematics*, vol. 10, no. 4, pp. 631-637, 2017.

[5] S. O. Edeki, O.O. Ugbebor, and E. A. Owoloko, "On a dividend-paying stock options pricing model (SOPM) using constant elasticity of variance stochastic dynamics" *International Journal of Pure and Applied Mathematics,* vol. 106 (4), pp. 1029-1036, 2016.

[6] S.O. Edeki, E.A. Owoloko, and O.O. Ugbebor, "The modified Black-Scholes model via constant elasticity of variance for stock options valuation", *2015 Progress in Applied Mathematics in Science and Engineering (PIAMSE), Conference proceedings*, September 29-October 1, 2015,*AIP Conference Proceedings* 1705.

[7] T. Allahviranloo, Sh. S. Behzadi, "The use of iterative methods for solving Black-Scholes equations", *Int. J. Industrial Mathematics, vol.* 5, no. 1, pp. 1-11, 2013.

[8] A.G.Z. Kemna and A.C.F. Vorst, "A pricing method for options based on average asset values", *Journal of Banking and Finance,* vol. 14, pp. 113-129, 1990.

[9] E. Barucci, S. Polidoro and V. Vespri, "Some results on partial differential equations and Asian options", *Math. Models Methods Appl. Sci.* vol. 11, no. 3, pp. 475-497, 2001.

[10] H. Geman and M. Yor. Bessel process, "Asian options and perpetuities", *Mathematical Finance,* vol. 3, no. 4, pp. 349-75, 1993.

[11] J. Vecer, "A new PDE approach for pricing arithmetic average Asian option", *Journal of Computational Finance*, vol. 4, pp. 105-113, 2001.

[12] J. Zhang, "Theory of continuously-sampled Asian option pricing", (working paper) City University of Hong Kong, 2000.

[13] K. Chen and Y. Lyuu, "Accurate pricing formula for Asian options", *Journal of applied Mathematics and Computation*, vol. 188, pp.1711-1724, 2007.

[14] Z.A Elshegmani, R. R. Ahmad and R.H. Zakaria, "New pricing formula for arithmetic Asian options using PDE Approach", *Applied Mathematical Sciences*, vol. 5, no. 77, pp. 3801–3809, 2011.

[15] A. Kumar, A. Waikos and S. P. Chakrabarty, "Pricing of average strike Asian call option using numerical PDE methods, arXiv:1106.1999v1 [q-fin.CP] 10 Jun 2011.

[16] B. Zhang, Y. Yu, and W. Wang, "Numerical algorithm for delta of Asian option", *The Scientific World Journal*, Vol. 2015, Article ID 692847, 6 pages, http://dx.doi.org/10.1155/2015/692847.

[17] S.E. Fadugba , "The Mellin transform method as an alternative analytic solution for the valuation of geometric Asian option", *Applied and Computational Mathematics*, vol. 3, no. (6-1), pp. 1-7, 2014.
[18] Z.A Elshegmani, R. R. Ahmad, "Analytical solution for an arithmetic Asian option using Mellin transforms", *Int. Journal of Math. Analysis*, vol. 5, no. 26, 1259-1265, 2011.
[19] S. Shreve, "Stochastic calculus for finance, Volume II: Continuous-Time Models*"*, Springer-Verlag, 2004.
[20] Wilmott, S. Howison and J. Dewynne, "The Mathematics of Financial Derivatives", Cambridge University Press, 1995.
[21] D. J. Higham, "An Introduction to financial option valuation: mathematics, stochastics and computation*",* Cambridge University Press, 2004.
[22] S. O. Edeki, O.O. Ugbebor, and E. A. Owoloko, "Analytical solution of the time-fractional order Black-Scholes model for stock option valuation on no dividend yield basis", *IAENG International Journal of Applied Mathematics,* vol. 47, no. 4, pp. 407-416, 2017.
[23] S.O. Edeki, O.O. Ugbebor, and E.A. Owoloko, "He's polynomials for analytical solutions of the Black-Scholes pricing model for stock option valuation", *Proceedings of the World Congress on Engineering*, London, UK, 2016.
[24] G.O. Akinlabi, R. B. Adeniyi, E.A. Owoloko, "The solution of boundary value problems with mixed boundary conditions via boundary value methods", *International Journal of Circuits, Systems and Signal Processing*, vol. 12, pp. 1-6, 2018.
[25] G. O. Akinlabi, R. B. Adeniyi, E.A. Owoloko, "Hybrid boundary value methods for the solution of first order stiff systems", *International Journal of Circuits, Systems and Signal Processing*, vol. 11, pp. 332-337, 2017.
[26] O. A. Agbolade and T. A. Anake, "Solutions of first-order volterra type linear integrodifferential equations by collocation method", *Journal of Applied Mathematics*, 2017, 2017, Article ID: 1510267.
[27] S.O. Edeki, G.O. Akinlabi, "Zhou Method for the Solutions of System of Proportional Delay Differential Equations", *MATEC Web of Conferences* **125**, 02001, 2017.
[28] O. S. Adesina, O. O. Agboola, O. K, Famurewa, O. A. Odetunmibi and R. J. Dare, "The Mackey-Glass Type Delay Differential Equation with Uniformly Generated Constants", *International Journal of Mechanical Engineering and Technology*, vol. **9** (9), 2018, pp. 467–477.
[29] J. Biazar and F. Goldoust, "The Adomian decomposition method for the Black-Scholes Equation", *3rd International Conference on Applied Mathematics and Pharmaceutical Sciences (ICAMP'2013), April 29-30, pp. 321-323, 2013,* Singapore.
[30] K. Du, G. Liu and G. Gu, "Accelerating Monte Carlo method for pricing multi-asset options under stochastic volatility models", *IAENG International Journal of Applied Mathematics*, vol. 44, no. 2, pp. 62-70, 2014.
[31] G. Liu, Q. Zhao, and G. Gu, "A simple control variate method for options pricing with stochastic volatility models", *IAENG International Journal of Applied Mathematics*, vol. 45, no. 1, pp. 64-70, 2015.
[32] K. Du, G. Liu and G. Gu, "A class of control variates for pricing Asian options under stochastic volatility models," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 2, pp. 45-53, 2013.
[33] A. . Rehurek, "Stable Numerical Methods for PDE Models of Asian Options", *Technical report*, IDE1119 , September 27, (2011).
[34] S.O. Edeki, G.O. Akinlabi, O. González-Gaxiola, "Adomian decomposition method for analytical solution of a continuous arithmetic Asian option pricing model", *TELKOMNIKA*, 17 (2), 2019.
[35] S.O. Edeki, T. Motsepa, C.M. Khalique, G.O. Akinlabi, "The Greek parameters of a continuous arithmetic Asian option pricing model via Laplace Adomian decomposition method", *Open Physics* **,** vol. 16, no. 1, pp. 780-785, 2018.
[36] Z. A. Elshegmani and R. R. Ahmed, "Analytical solution for an arithmetic Asian option using Mellin transforms, *Int. Journal of Math. Analysis"*, vol. 5, no. 26, 1259-1265, 2011.
[37] G.O. Akinlabi, S.O. Edeki, "On approximate and closed-form solution method for initial-value wave-like models", *International Journal of Pure and Applied Mathematics*, vol. 107, no. 2, pp. 449-456, 2016.
[38] B. K. Singh, and V. K. Srivastava, "Approximate series solution of multi-dimensional time fractional-order (heat-like) diffusion equations using FDRM", *Royal Society Open Science,* vol. 2 2015, 140511.