Dynamic Analysis of an Almost Periodic Predator-Prey Model with Water Level Fluctuations

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***Abstract*—Water level may influence local community dynamics. In this paper, an almost predator-prey model is investigated to study the influence of water level variations on the interaction between two species. By using several comparison theorems and some analytical technicals, we derive some sufficient conditions for permanence of the system. Moreover, by using the properties of almost periodic functions and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the existence of uniformly asymptotically stable almost periodic solution of the system are obtained. Finally, numerical simulations are carried out to illustrate the feasibility of the main results. The theoretical results confirm the assumption that the water exerts a strong influence on the interaction between the species.**

***Index Terms*—water level fluctuations; permanence; almost periodic solution; asymptotical stability.**

I. Introduction

It is well known that significant variations of water level can have a strong impact on the persistence of some species. In fact, the increase of water volume hinders the capture of the prey by the predator. The same reasoning is applied when there is a decrease in the volume of water, favoring the capture of the prey by the predator. During the last decade, the impact of water level fluctuations on the species communities has been widely studied in rivers, lakes and reservoirs; see, for example, [1-7].

Notice that ecosystems are often disturbed by outside continuous forces in the real world, such as seasonal effects and variations in weather conditions, food supplies, mating habits, etc., the assumption of almost periodicity of the parameters is a way of incorporating the almost periodicity of a temporally nonuniform environment with incommensurable periods (nonintegral multiples).

Motivated by the works [3] and [4], we propose an almost predator-prey model with water level fluctuations as follows: Let $G(t)$ and $B(t)$ be respectively the biomass of the prey and predator at time t. When a predator attacks a prey, it has access to a certain quantity of food depending on the water level. When water level is low the predator is more in contact with the prey. Let $r(t)$ be the accessibility function for the prey. It is assumed that the function $r(t)$ is almost periodic and continuous. The minimum value r^l is reached in spring and the maximum value r^u is attained during autumn, denoted respectively by $\gamma_G(t)$ and $\gamma_B(t)$ the consumption rate of the resource by the prey and predator. Let $e_B(t)$ be the conversion rate of the prey in biomass and $m_G(t)$, $m_B(t)$ be respectively the consumption rate of biomass by metabolism of the prey and predator. The predator needs a quantity $\gamma_B(t)B(t)$ for his food, but he has access to a quantity

$$r(t)\frac{G(t)B(t)}{B(t)+D(t)}\ (\le \gamma_B(t)B(t)),$$

where $D(t)$ measures the other causes of mortality outside the metabolism and predation. It gives the extent to which environment provides protection to the prey.

Accordingly, the predator-prey model can be expressed as

$$\begin{cases} \frac{dG(t)}{dt} &= G(t)(\gamma_G(t)-m_G(t)G(t)) \\ &\quad -r(t)\frac{G(t)B(t)}{B(t)+D(t)}, \\ \frac{dB(t)}{dt} &= e_B(t)r(t)\frac{G(t)B(t)}{B(t)+D(t)}-m_B(t)B(t). \end{cases} \tag{1}$$

The initial conditions for system (1) take the form of

$$\begin{aligned} &G(s)=\varphi_1(s)>0, B(s)=\varphi_2(s)>0, \\ &s\in(-\infty,0], \varphi_i(0)>0, \end{aligned} \tag{2}$$

where φ_i, $i=1,2$ are bounded and continuous functions on $(-\infty,0]$.

Let B_0, G_0 be respectively the initial density of the predator and prey with $B_0>0$ and $G_0>0$. By the theory of functional differential equations [8], it is clear that system (1) has a unique positive solution which satisfies the initial condition (2).

The following standard analysis shows that the model (1) is biologically sound.

Lemma 1. *Every solution of system (1) with initial conditions (2) exists in the interval* $[0,+\infty]$ *and remains positive for all* $t\ge 0$.

Proof: Firstly, we show that $G(t)>0$ for all $t\in[0,\alpha)$, where $0<\alpha\le+\infty$. Otherwise, there exists a $t_1\in[0,\alpha)$ such that $G(t_1)=0$, $\frac{dG}{dt}(t_1)<0$ and $G(t)>0$ for all $t\in[0,t_1)$.

Hence, there must have $B(t)>0$ for all $t\in[0,t_1)$. If this statement is not true, then there exists a $t_2\in[0,t_1)$ such that $B(t_2)=0$ and $B(t)>0$ on $[0,t_2)$. Furthermore,

$$\frac{dB(t)}{dt}\ge -m_B(t)B(t), \forall t\in[0,t_2],$$

then

$$B(t)\ge B_0\exp(-m_B(t)t), \forall t\in[0,t_2],$$

thus,

$$B(t_2)\ge B_0\exp(-m_B(t_2)t_2)>0.$$

It is a contradiction. Hence, $B(t)>0$ for all $t\in[0,t_1)$.

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On the other hand

$$\frac{dG(t)}{dt} \geq G(t)\Big(- r(t)\frac{B(t)}{B(t)+D(t)} + \gamma_G(t) \\ -m_G(t)G(t)\Big),$$

then

$$G(t) \geq G_0 \exp\bigg(\int_0^t \Big[- r(s)\frac{B(s)}{B(s)+D} + \gamma_G(s) \\ -m_G(s)G(s)\Big]\bigg) ds,$$

thus

$$G(t_1) \geq G_0 \exp\bigg(\int_0^{t_1} \Big[- r(s)\frac{B(s)}{B(s)+D} + \gamma_G(s) \\ -m_G(s)G(s)\Big]\bigg) ds > 0.$$

It is a contradiction with $G(t_1) = 0$, so $G(t) > 0$ for all $t \geq 0$. This completes the proof.

For convenience, we introduce the notation

$$f^u = \sup_{t\in R} f(t),\ f^l = \inf_{t\in R} f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that the coefficients of the almost periodic system (1) satisfy

$$\min\{\gamma_G^l, m_G^l, r^l, e_B^l, D^l, m_B^l\} > 0, \\ \max\{\gamma_G^u, m_G^u, r^u, e_B^u, D^u, m_B^u\} < +\infty.$$

In the following sections, we shall study the existence and stability of almost periodic solutions of system (1).

II. Basic results

In this section, we shall develop some preliminary results, which will be used to prove the main result.

Lemma 2. (see [9]) *If $a > 0, b > 0$, and $x' \geq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, then*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a > 0, b > 0$, and $x' \leq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, then

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}.$$

Let D denotes R or an open subset of R. The relevant definitions and the properties of almost periodic functions, see [10].

Definition 1. (see [10]) *$f \in C(R, R)$ is an almost periodic function if and only if for any sequence $\alpha_n' \subset \mathbb{T}$, there exists a subsequence $\alpha_n \subset \alpha_n'$ such that $f(t + \alpha_n)$ converges uniformly on R as $n \to +\infty$. Furthermore, the limit function is also an almost periodic function.*

Lemma 3. (see [10]) *If $f(t), g(t)$ are almost periodic functions, then, for any $\varepsilon > 0$, $E\{\varepsilon, f\} \cap E\{\varepsilon, g\}$ is a nonempty relatively dense set in R; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau \in E\{\varepsilon, f\} \cap E\{\varepsilon, g\}$ such that*

$$|f(t+\tau) - f(t)| < \varepsilon, |g(t+\tau) - g(t)| < \varepsilon, \forall t \in R.$$

Consider the following almost periodic dynamic equation

$$x' = f(t, x) \tag{3}$$

and the associate product system of (3)

$$x' = f(t, x),\ y' = f(t, y). \tag{4}$$

Lemma 4. (see [10]) *Suppose that there exists a Lyapunov function $V(t, x, y) \in C(R^+ \times D \times D, R)$ satisfying the following conditions:*

1. *$a(\|x - y\|) \leq V(t, x, y) \leq b(\|x - y\|)$, where $a, b \in K, K = \{a \in C(R^+, R^+) : a(0) = 0$ and a is increasing$\}$;*
2. *$|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|)$, where $L > 0$ is a constant;*
3. *$D^+V'_{(4)}(t, x, y) \leq -\mu V(t, x, y)$, where $\mu > 0$ is a constant.*

Moreover, if there exists a solution $x(t)$ of (3) such that $x(t) \in S$, where $S \subset D$ is a compact set. Then there exists a unique uniformly asymptotically stable almost periodic solution $p(t)$ of (3) in S. Furthermore, if $f(t, x)$ is periodic with period ω in t, then $p(t)$ is a periodic solution of (3) with period ω.

III. Permanence

Assume that the coefficients of (1) satisfy

(H_1) $\gamma_G^l > r^u \geq r^l > \frac{m_B^u D^u}{e_B^l m_1}$.

Theorem 1. *Let $(G(t), B(t))$ be any positive solution of system (1) with initial condition (2). If (H_1) holds, then system (1) is permanent, that is, any positive solution $(G(t), B(t))$ of system (1) satisfies*

$$m_1 \leq \liminf_{t\to+\infty} G(t) \leq \limsup_{t\to+\infty} G(t) \leq M_1, \tag{5}$$

$$m_2 \leq \liminf_{t\to+\infty} B(t) \leq \limsup_{t\to+\infty} B(t) \leq M_2, \tag{6}$$

especially if $m_1 \leq G_0 \leq M_1$, $m_2 \leq B_0 \leq M_2$, then

$$m_1 \leq G(t) \leq M_1,\ m_2 \leq B(t) \leq M_2,\ t \in [t_0, +\infty),$$

where

$$M_1 = \frac{\gamma_G^u}{m_G^l} - \frac{r^l m_1}{m_G^l(m_1 + D^u)},\ M_2 = \frac{e_B^u r^u M_1}{m_B^l} - D^l, \\ m_1 = \frac{\gamma_G^l - r^u}{m_G^u},\ m_2 = \frac{e_B^l r^l m_1}{m_B^u} - D^u.$$

Proof: Assume that $(G(t), B(t))$ be any positive solution of system (1) with initial condition (2). It follows from the first equation of system (1) that

$$G'(t) = G(t)(\gamma_G(t) - m_G(t)G(t)) \\ -r(t)\frac{G(t)B(t)}{B(t)+D(t)} \\ \geq G(t)[(\gamma_G^l - r^u) - m_G^u G(t)]. \tag{7}$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} G(t) \geq \frac{\gamma_G^l - r^u}{m_G^u} := m_1.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$G(t) > m_1 - \varepsilon,\ \forall t \in [T_1, +\infty].$$

From the second equation of system (1), when $t \in [T_1, +\infty)$,

$$\begin{aligned} B'(t) &= e_B(t) r(t) \frac{G(t)B(t)}{B(t)+D(t)} - m_B(t)B(t) \\ &> e_B^l r^l \frac{(m_1-\varepsilon)B(t)}{B(t)+D(t)} - m_B^u B(t). \end{aligned}$$

Let $\varepsilon \to 0$, then

$$\begin{aligned} B'(t) &\geq e_B^l r^l \frac{m_1 B(t)}{B(t)+D(t)} - m_B^u B(t) \\ &\geq \frac{B(t)}{B(t)+D(t)}[e_B^l r^l m_1 - m_B^u D^u \\ &\quad - m_B^u B(t)]. \end{aligned} \quad (8)$$

By Lemma 2, we can get

$$\liminf_{t \to +\infty} B(t) = \frac{e_B^l r^l m_1}{m_B^u} - D^u := m_2.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$B(t) > m_2 - \varepsilon,\ \forall t \in [T_2, +\infty).$$

On the other hand, from the first equation of system (1), when $t \in [T_2, +\infty)$,

$$\begin{aligned} G'(t) &= G(t)(\gamma_G(t) - m_G(t)G(t)) - r(t)\frac{G(t)B(t)}{B(t)+D(t)} \\ &< G(t)\left(\gamma_G^u - \frac{r^l(m_1-\varepsilon)}{(m_1-\varepsilon)+D^u} - m_G^l G(t)\right). \end{aligned}$$

Let $\varepsilon \to 0$, then

$$G'(t) \leq G(t)\left(\gamma_G^u - \frac{r^l m_1}{m_1 + D^u} - m_G^l G(t)\right). \quad (9)$$

By Lemma 2, we can get

$$\limsup_{t \to +\infty} G(t) = \frac{\gamma_G^u}{m_G^l} - \frac{r^l m_1}{m_G^l(m_1 + D^u)} := M_1.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$G(t) < M_1 + \varepsilon,\ \forall t \in [T_3, +\infty].$$

From the second equation of system (1), when $t \in [T_3, +\infty)$,

$$\begin{aligned} B'(t) &= e_B(t) r(t) \frac{G(t)B(t)}{B(t)+D(t)} - m_B(t)B(t) \\ &< \frac{B(t)}{B(t)+D(t)}[e_B^u r^u (M_1+\varepsilon) - m_B^l D^l \\ &\quad - m_B^l B(t)]. \end{aligned}$$

Let $\varepsilon \to 0$, then

$$B'(t) \leq \frac{B(t)}{B(t)+D(t)}[e_B^u r^u M_1 - m_B^l D^l - m_B^l B(t)]. (10)$$

By Lemma 2, we can get

$$\limsup_{t \to +\infty} B(t) = \frac{e_B^u r^u M_1}{m_B^l} - D^l := M_2.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_4 > T_3$ such that

$$B(t) < M_2 - \varepsilon,\ \forall t \in [T_4, +\infty].$$

In special case, if $m_1 \leq G_0 \leq M_1$, $m_2 \leq B_0 \leq M_2$, by Lemma 2, it follows from (7)-(8), (9)-(10) that

$$m_1 \leq G(t) \leq M_1,\ m_2 \leq B(t) \leq M_2,\ t \in [t_0, +\infty),$$

This completes the proof.

IV. Almost periodic solution

In this section, by constructing a suitable Lyapunov functional, we shall study the existence of a unique almost periodic solution of (1), which is uniformly asymptotically stable.

Let S be the set of all solutions $(G(t), B(t))$ of system (1) satisfying $m_1 \leq G(t) \leq M_1,\ m_2 \leq B(t) \leq M_2$ for all $t \in R^+$.

Lemma 5. $S \neq \emptyset$.

Proof: By Theorem 1, we see that for any $t_0 \in R^+$ with $m_1 \leq G_0 \leq M_1$, $m_2 \leq B_0 \leq M_2$, system (1) has a solution $(G(t), B(t))$ satisfying $m_1 \leq G(t) \leq M_1,\ m_2 \leq B(t) \leq M_2, t \in [t_0, +\infty)$.

Since $r(t)$ is almost periodic, it follows from Definition 1 that there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $\gamma_G(t+t_n) \to \gamma_G(t)$, $m_G(t+t_n) \to m_G(t)$, $r(t+t_n) \to r(t)$, $D(t+t_n) \to D(t)$, $e_B(t+t_n) \to e_B(t)$, $m_B(t+t_n) \to m_B(t)$ as $n \to +\infty$ uniformly on R^+.

Now, we claim that $\{G(t+t_n)\}$ and $\{B(t+t_n)\}$ are uniformly bounded and equi-continuous on any bounded interval in R^+.

In fact, for any bounded interval $[\alpha, \beta] \subset R^+$, when n is large enough, $\alpha + t_n > t_0$, then $t + t_n > t_0, \forall t \in [\alpha, \beta]$. So, $m_1 \leq G(t+t_n) \leq M_1,\ m_2 \leq B(t+t_n) \leq M_2$ for any $t \in [\alpha, \beta]$, that is, $\{G(t+t_n)\}$ and $\{B(t+t_n)\}$ are uniformly bounded. On the other hand, $\forall t_1, t_2 \in [\alpha, \beta]$, from the mean value theorem of differential calculus, we have

$$\begin{aligned} &|G(t_1+t_n) - G(t_2+t_n)| \\ &\leq \left(M_1(\gamma_G^u + m_G^u M_1) + r^u \frac{M_1 M_2}{m_2 + D^l}\right)|t_1 - t_2| \quad (11) \\ &|B(t_1+t_n) - B(t_2+t_n)| \\ &\leq \left(e_B^u r^u \frac{M_1 M_2}{m_2 + D^l} + m_B^u M_2\right)|t_1 - t_2|. \quad (12) \end{aligned}$$

The inequalities (11) and (12) show that $\{G(t+t_n)\}$ and $\{B(t+t_n)\}$ are equi-continuous on $[\alpha, \beta]$. By the arbitrary of $[\alpha, \beta]$, the conclusion is valid.

By Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, we still denote it as $\{t_n\}$, such that

$$G(t+t_n) \to p(t), B(t+t_n) \to q(t),$$

as $n \to +\infty$ uniformly in t on any bounded interval in R^+.

Furthermore,

$$\begin{cases} \frac{dG(t+t_n)}{dt} &= G(t+t_n)(\gamma_G(t+t_n) \\ &\quad -m_G(t+t_n)G(t+t_n)) \\ &\quad -r(t+t_n)\frac{G(t+t_n)B(t+t_n)}{B(t+t_n)+D(t+t_n)}, \\ \frac{dB(t+t_n)}{dt} &= e_B(t+t_n)r(t+t_n)\frac{G(t+t_n)B(t+t_n)}{B(t+t_n)+D(t+t_n)} \\ &\quad -m_B(t+t_n)B(t+t_n). \end{cases}$$

Let $n \to +\infty$, then

$$\begin{cases} \frac{dp(t)}{dt} &= p(t)(\gamma_G(t) - m_G(t)p(t)) - r(t)\frac{p(t)q(t)}{q(t)+D(t)}, \\ \frac{dq(t)}{dt} &= e_B(t)r(t)\frac{p(t)q(t)}{q(t)+D(t)} - m_B(t)q(t). \end{cases}$$

It is clear that $(p(t), q(t))$ is a solution of system (1). Moreover,

$$m_1 \le p(t) \le M_1,\ m_2 \le q(t) \le M_2,\ \forall t \in R^+.$$

This completes the proof.

Remark 1. *From the proofs of Theorem 1 and Lemma 5, we know that if the conditions of Theorem 1 hold, S is a positive invariant set of system (1).*

Theorem 2. *Suppose the condition (H_1) holds, assume further that*

(H_2) $\lambda < 0$, *where*

$$\lambda = \max\Big\{ -m_G^l m_1 + \frac{e_B^u r^u (M_1 + M_2 + D^u M_1)}{(m_2 + D^l)^2}, \frac{D^u r^u M_2 - e_B^l r^l (m_1 + m_2)}{(m_2 + D^l)^2} \Big\},$$

then there exists a unique uniformly asymptotically stable almost periodic solution $(G(t), B(t))$ of system (1) which is bounded by S^* for all $t \in R^+$.

Proof: Let $x(t) = \ln(G(t))$, $y(t) = \ln(B(t))$, then system (1) can be transformed into

$$\begin{cases} x'(t) &= \gamma_G(t) - m_G(t)\exp(x(t)) \\ &\quad -r(t)\frac{\exp(y(t))}{\exp(y(t))+D(t)}, \\ y'(t) &= e_B(t)r(t)\frac{\exp(x(t))}{\exp(y(t))+D(t)} - m_B(t). \end{cases} \tag{13}$$

From Lemma 5, system (13) has a bounded solution $(x(t), y(t))$ satisfying

$$\ln m_1 < x(t) < \ln M_1,\ \ln m_2 < y(t) < \ln M_2,\ \forall t \in R^+,$$

then

$$|x(t)| < \tilde{M_1}, |y(t)| < \tilde{M_2},$$

where

$$\tilde{M_1} = \max\{|\ln m_1|, |\ln M_1|\},$$
$$\tilde{M_2} = \max\{|\ln m_2|, |\ln M_2|\}.$$

For $(x, y) \in R^2$, we define the norm $\|(x, y)\| = |x| + |y|$. Consider the product system of system (13)

$$\begin{cases} x'(t) &= \gamma_G(t) - m_G(t)\exp(x(t)) \\ &\quad -r(t)\frac{\exp(y(t))}{\exp(y(t))+D(t)}, \\ y'(t) &= e_B(t)r(t)\frac{\exp(x(t))}{\exp(y(t))+D(t)} - m_B(t), \\ u'(t) &= \gamma_G(t) - m_G(t)\exp(u(t)) \\ &\quad -r(t)\frac{\exp(v(t))}{\exp(v(t))+D(t)}, \\ v'(t) &= e_B(t)r(t)\frac{\exp(u(t))}{\exp(v(t))+D(t)} - m_B(t). \end{cases} \tag{14}$$

Suppose $X = (x(t), y(t))$, $Y = (u(t), v(t))$ be any two solutions of system (13), then $\|X\| \le A, \|Y\| \le A$, where $A = \tilde{M_1} + \tilde{M_2}$. Set

$$S^* = \{(x(t), y(t)) | \ln m_1 < x(t) < \ln M_1, \ln m_2 < y(t) < \ln M_2,\ \forall t \in R^+\}.$$

Consider a Lyapunov functional defined on $R^+ \times S^* \times S^*$ as follows

$$V(t, X, Y) = |x(t) - u(t)| + |y(t) - v(t)|. \tag{15}$$

Since $\|X - Y\| = |x(t) - u(t)| + |y(t) - v(t)|$, we have

$$\frac{1}{2}\|X - Y\| \le V(t, X, Y) \le 2\|X - Y\|.$$

Let $a, b \in C(R^+, R^+)$, $a(x) = \frac{1}{2}x, b(x) = 2x$, thus the condition (1) of Lemma 4 is satisfied.

In addition,

$$\begin{aligned} &|V(t, X, Y) - V(t, \tilde{X}, \tilde{Y})| \\ =& \big||x(t) - u(t)| + |y(t) - v(t)| \\ &-|\tilde{x}(t) - \tilde{u}(t)| - |\tilde{y}(t) - \tilde{v}(t)|\big| \\ \le& \big|(x(t) - u(t)) - (\tilde{x}(t) - \tilde{u}(t))\big| \\ &+\big|(y(t) - v(t)) - (\tilde{y}(t) - \tilde{v}(t))\big| \\ \le& |x(t) - \tilde{x}(t)| + |u(t) - \tilde{u}(t)| \\ &+|y(t) - \tilde{y}(t)| + |v(t) - \tilde{v}(t)| \\ =& \|X - \tilde{X}\| + \|Y - \tilde{Y}\|. \end{aligned}$$

Let $L = 1$, then the condition (2) of Lemma 4 is satisfied.

Finally, calculate the $V'(t, X, Y)$ along the solutions of (13), we can obtain

$$\begin{aligned} &V'(t, X, Y) \\ =& \operatorname{sgn}(x(t) - u(t))(x(t) - u(t))' \\ &+\operatorname{sgn}(y(t) - v(t))(y(t) - v(t))' \\ =& \operatorname{sgn}(x(t) - u(t))\Big[m_G(t)(\exp(u(t)) - \exp(x(t))) \\ &+r(t)\Big(\frac{\exp(v(t))}{\exp(v(t)) + D(t)} - \frac{\exp(y(t))}{\exp(y(t)) + D(t)}\Big)\Big] \\ &+\operatorname{sgn}(y(t) - v(t))e_B(t)r(t)\Big[\frac{\exp(x(t))}{\exp(y(t)) + D(t)} \\ &-\frac{\exp(u(t))}{\exp(v(t)) + D(t)}\Big] \\ =& \operatorname{sgn}(x(t) - u(t))\Big[m_G(t)(\exp(u(t)) - \exp(x(t))) \\ &+D(t)r(t)\frac{\exp(v(t)) - \exp(y(t))}{(\exp(v(t)) + D(t))(\exp(y(t)) + D(t))}\Big] \\ &+\operatorname{sgn}(y(t) - v(t))e_B(t)r(t) \\ &\cdot\frac{\exp(x(t) + v(t)) - \exp(y(t) + u(t))}{(\exp(y(t)) + D(t))(\exp(v(t)) + D(t))} \\ &+\operatorname{sgn}(y(t) - v(t))De_B(t)r(t) \\ &\cdot\frac{\exp(x(t)) - \exp(u(t))}{(\exp(y(t)) + D(t))(\exp(v(t)) + D(t))}. \end{aligned} \tag{16}$$

By using the mean value theorem, we have

$$\begin{aligned} &\exp\{x(t)\} - \exp\{u(t)\} = \xi_1(t)(x(t) - u(t)), \\ &\exp\{y(t)\} - \exp\{v(t)\} = \xi_2(t)(y(t) - v(t)), \\ &\exp\{x(t) + v(t)\} - \exp\{y(t) + u(t)\} \\ &= \xi_3(t)[(x(t) - u(t)) + (v(t) - y(t))], \end{aligned} \tag{17}$$

where $\xi_1(t)$ lies between $\exp\{x(t)\}$ and $\exp\{u(t)\}$; $\xi_2(t)$ lies between $\exp\{y(t)\}$ and $\exp\{v(t)\}$; $\xi_3(t)$ lies between $\exp\{x(t)+v(t)\}$ and $\exp\{y(t)+u(t)\}$.

From (16) and (17), we have

$$
\begin{aligned}
&V'(t,X,Y)\\
=\ &\operatorname{sgn}(x(t)-u(t))\Big[m_G(t)(\exp(u(t))-\exp(x(t)))\\
&+D(t)r(t)\frac{\exp(v(t))-\exp(y(t))}{(\exp(v(t))+D(t))(\exp(y(t))+D(t))}\Big]\\
&+\operatorname{sgn}(y(t)-v(t))e_B(t)r(t)\\
&\cdot\frac{\exp(x(t)+v(t))-\exp(y(t)+u(t))}{(\exp(y(t))+D(t))(\exp(v(t))+D(t))}\\
&+\operatorname{sgn}(y(t)-v(t))D(t)e_B(t)r(t)\\
&\cdot\frac{\exp(x(t))-\exp(u(t))}{(\exp(y(t))+D(t))(\exp(v(t))+D(t))}\\
\le\ &\Big[-m_G^l m_1+\frac{e_B^u r^u(M_1+M_2+D^u M_1)}{(m_2+D^l)^2}\Big]\\
&\cdot|x(t)-u(t)|\\
&+\frac{D^u r^u M_2-e_B^l r^l(m_1+m_2)}{(m_2+D^l)^2}|y(t)-v(t)|\\
\le\ &\lambda V(t,X,Y),
\end{aligned}
\tag{18}
$$

where $\lambda = \max\Big\{-m_G^l m_1+\frac{e_B^u r^u(M_1+M_2+D^u M_1)}{(m_2+D^l)^2}, \frac{D^u r^u M_2-e_B^l r^l(m_1+m_2)}{(m_2+D^l)^2}\Big\}$. From the condition (H_2), $\lambda<0$, the condition (3) of Lemma 4 is satisfied.

To sum up, from Lemma 4, there exists a unique uniformly asymptotically stable almost periodic solution $(x(t),y(t))$ of system (13) which is bounded by S^* for all $t\in R^+$, which means that there exists a uniqueness uniformly asymptotically stable almost periodic solution $(G(t),B(t))$ of (1) which is bounded by S for all $t\in R^+$. This completes the proof.

Corollary 1. *Assume that (H_1)-(H_2) hold. Suppose that the nonnegative coefficient $r(t)$ is periodic of period ω; then system (1) has a unique uniformly asymptotically stable periodic solution of period ω.*

Remark 2. *Assume that (H_1)-(H_2) hold, then system (1) has a unique uniformly asymptotically stable almost periodic solution.*

Remark 3. *Assume that (H_1)-(H_2) hold. Suppose that the nonnegative coefficient $r(t)$ is periodic of period ω; then system (1) has a unique uniformly asymptotically stable periodic solution of period ω.*

V. An Example and Simulations

Consider the following system

$$
\begin{cases}
\frac{dG(t)}{dt} = -(0.9+0.1\sin(\sqrt{3}t))\frac{G(t)}{B(t)+0.4}B(t)\\
\qquad\qquad +G(t)[2.5+0.1\sin(\sqrt{5}t)-0.1G(t)],\\
\frac{dB(t)}{dt} = 0.3(0.9+0.1\sin(\sqrt{3}t))\frac{G(t)}{B(t)+0.4}B(t)\\
\qquad\qquad -0.5B(t).
\end{cases}
\tag{19}
$$

that is $r(t)=0.9+0.1\sin(\sqrt{3}t), D=0.4, \gamma_G(t)=2.5+0.1\sin(\sqrt{5}t), m_G(t)=0.1, m_B(t)=0.5, e_B(t)=0.3$.

By a direct calculation, we can get

$$
\begin{aligned}
m_1 &= 16.0000, M_1 = 16.1951,\\
m_2 &= 7.2800, M_2 = 9.3171,
\end{aligned}
$$

and

(H_1) $\gamma_G^l = 2.4 > r^u \ge r^l > 0.7583 = \frac{m_B(t)D}{e_B(t)m_1}$;

(H_2) $\lambda = \max\Big\{-m_G^l m_1+\frac{e_B^u r^u(M_1+M_2+D^u M_1)}{(m_2+D^l)^2}, \frac{D^u r^u M_2-e_B^l r^l(m_1+m_2)}{(m_2+D^l)^2}\Big\} = -0.0315 < 0.$

Therefore, the conditions (H_1)-(H_2) hold. According to Theorem 2, system (19) has a unique uniformly asymptotically stable almost periodic solution. Dynamic simulations of system (19), see Figures 1, 2 and 3.

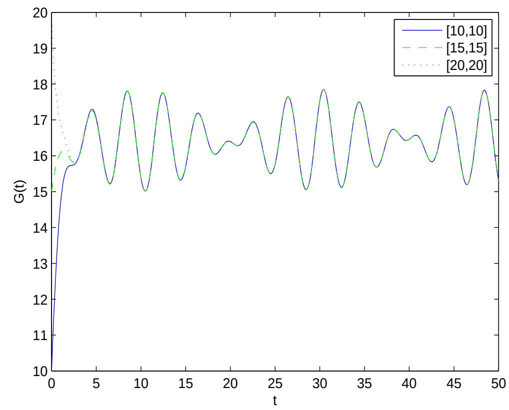

Fig. 1. Trajectory of $G(t)$ with initial condition $(x(0),y(0)) = \{(15,15),(17,17),(19,19)\}$.

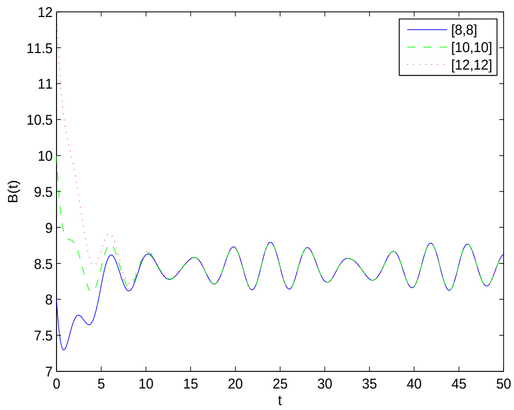

Fig. 2. Trajectory of $B(t)$ with initial condition $(x(0),y(0)) = \{(8,8),(10,10),(12,12)\}$.

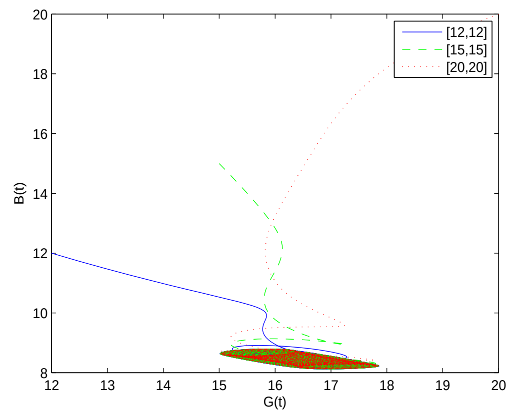

Fig. 3. The phase trajectory of $G(t)$ and $B(t)$ with initial condition $(x(0),y(0)) = \{(12,12),\ (15,15),\ (20,20)\}$.

VI. Conclusion

This paper is concerned with an almost predator-prey model with water level fluctuations. This study provides preliminary results of the evolution of the ecosystem based on water management of a lake. Permanence, stability, existence and uniqueness almost periodic solution are done. We showed that variations in water level of a lake has an important influence on the existence of almost periodic solution of system (1).

As we all know, aquatic ecosystem are often altered by human activities. One may consider many other types of predator-prey models with water level fluctuations. Models of predator-prey systems, see [11-13].

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