Permanence and Extinction in a Nonautonomous Discrete Competitive System with Delays

Lili Wang, Shilin Zhang, Meng Hu

Abstract—**A nonautonomous discrete competitive system with time delays is considered in this work. By using some analytical techniques, sufficient conditions on the coefficients are given to guarantee that one of the species will be driven to extinction while the other one will stabilize at a certain solution of a nonlinear single species model. Finally, numerical simulations are presented to illustrate the feasibility and effectiveness of the results.**

Index Terms—**Discrete competitive system; Permanence; Extinction; Global attractivity; Delay.**

I. Introduction

In the last decade, the application of theories of functional differential equations in mathematical ecology has developed rapidly. Various delayed models have been proposed in the study of population dynamics, ecology, and epidemic. In fact, more realistic population dynamics should take into account the effect of delay. Also, delay differential equations may exhibit much more complicated dynamic behaviors than ordinary differential equations since a delay could cause a stable equilibrium to become unstable and cause the population to fluctuate (see [1]). One of the famous models for dynamics of population is the delay Lotka-Volterra competitive system. Owing to its theoretical and practical significance, various delay competitive systems have been studied extensively (see [2-8]). Although much progress has been seen for Lotka-Volterra competitive systems, such systems are not well studied in the sense that most results are continuous time versions related. Many authors [9-11] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Therefore, the dynamic behaviors of population models governed by difference equations have been studied by many authors, see [12-18] and the references cited therein. Noting that some studies of the dynamics of natural populations indicate that the density-dependent population regulation probably takes place over many generations [19,20], many authors have discussed the influence of many past generations on the density of species population and discussed the dynamic behaviors of competitive, predator-prey, and cooperative systems (see [21-24]).

Motivated by the above work [19-24], in this paper we will investigate the following discrete time non-autonomous two-species competitive system with delays

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\Bigg[r_1(n)-\sum_{\tau=0}^{m}a_{1\tau}(n)x_1(n-\tau) \\ &\quad -\sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)}\Bigg], \\ x_2(n+1) &= x_2(n)\exp\Bigg[r_2(n)-\sum_{\tau=0}^{m}a_{2\tau}(n)x_2(n-\tau) \\ &\quad -\sum_{\tau=0}^{m}\frac{c_{1\tau}(n)x_1(n-\tau)}{1+x_1(n-\tau)}\Bigg], \end{aligned} \tag{1}$$

with the initial conditions

$$x_i(-\tau)>0, x_i(0)>0, \tau=0,1,2,\cdots,m, i=1,2, \tag{2}$$

where $x_i(n)$ represents the density of population x_i at the nth generation, $r_i(n)$ is the intrinsic growth rate of population x_i at the nth generation, $a_{i\tau}(n)$ measures the intraspecific influence of the $(n-\tau)$th generation of population x_i on the density of its own population, and $c_{j\tau}(n)$ stands for the interspecific influence of the $(n-\tau)$th generation of population x_j on population $x_i, i,j=1,2$ and $i\neq j$. The coefficients $r_i(n)$, $a_{i\tau}(n)$, and $c_{i\tau}(n)$, $i=1,2$ are bounded positive sequences. The exponential form of the equations in system (1) ensures that any forward trajectory $\{(x_1(n),x_2(n))^T\}$ of system (1) with initial conditions (2) remains positive for all $n\in\{0,1,2,\cdots\}$. For the investigations of some continuous versions of (1) we refer to [25,26] and the references cited therein.

Permanence (or extinction) of biotic population is a significant and comprehensive problem in biomathematics. Up to now, there are seldom results on the extinction and stability of species in a discrete population dynamic system, especially for a population dynamic system with delay. The main purpose of this paper is to study the extinction and stability of system (1), and derive some sufficient conditions which guarantee one of the species will be driven to extinction while the other one will be globally attractive with any positive solution of a discrete logistic equation.

For the simplicity and convenience of exposition, throughout this paper we let Z, Z^+, R^+ and R^2 denote the sets of all integers, nonnegative integers, nonnegative real numbers and two-dimensional Euclidian vector space, respectively. Meanwhile, we denote that $a^*=\sup\limits_{n\in z^+}a(n)$ and $a_*=\inf\limits_{n\in z^+}a(n)$ for any bounded sequence $\{a(n)\}$.

II. Preliminaries

In this section, we shall develop some preliminary results, which will be used to prove the main results.

Manuscript received February 24, 2019; revised May 19, 2019. This work was supported in part by the Key Project of Scientific Research in Colleges and Universities of Henan Province (No.18A110005), the College Students' Innovation Fund Project of Anyang Normal University.

L.L. Wang, S.L. Zhang and M. Hu are with the School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan, 455000 China e-mail: humeng2001@126.com.

Lemma 1. ([27]) *Assume that* $\{x(n)\}$ *satisfies*

$$x(n+1) \geq x(n)\exp\{r(n)(1-ax(n))\}, n \geq n_0,$$

$\limsup\limits_{n\to+\infty} x(n) \leq x^*$ *and* $x(n_0) > 0$*, where* a *is a positive constant such that* $ax^* > 1$ *and* $n_0 \in N$*. Then*

$$\liminf_{n\to+\infty} x(n) \geq \frac{1}{a}\exp\{r^*(1-ax^*)\}.$$

Lemma 2. ([27]) *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n+1) \leq x(n)\exp\{r(n)(1-ax(n))\}$$

for $n \in [n_1, +\infty)$*, where* a *is a positive constant. Then*

$$\limsup_{n\to+\infty} x(n) \leq \frac{1}{ar^*}\exp(r^*-1).$$

Lemma 3. *For every solution* $(x_1(n), x_2(n))^T$ *of (1) we have*

$$\limsup_{n\to+\infty} x_i(n) \leq x_i^*, i = 1, 2,$$

where $x_i^* = \frac{1}{a_{i0*}}\exp(r_i^*-1)$.

Proof: Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1). From the first equation of (1),

$$\begin{aligned} x_1(n+1) &\leq x_1(n)\exp[r_1(n) - a_{10}(n)x_1(n)] \\ &= x_1(n)\exp\left[r_1(n)\left(1-\frac{a_{10}(n)x_1(n)}{r_1(n)}\right)\right] \\ &\leq x_1(n)\exp\left[r_1(n)\left(1-\frac{a_{10*}x_1(n)}{r_1^*}\right)\right]. \end{aligned}$$

By Lemma 2, we have

$$\limsup_{n\to+\infty} x_1(n) \leq \frac{1}{a_{10*}}\exp(r_1^*-1).$$

Similarly, from the second equation of (1), we have

$$\limsup_{n\to+\infty} x_2(n) \leq \frac{1}{a_{20*}}\exp(r_2^*-1).$$

This completes the proof. ■

III. Extinction of x_2 and stability of x_1

In this section, we study the extinction of x_2 but x_1 of system (1).

Theorem 1. *Assume that the inequality*

$$\limsup_{n\to+\infty}\frac{r_2(n)}{r_1(n)} < \liminf_{n\to+\infty}\left\{\frac{c_{1\tau}(n)}{a_{1\tau}(n)(1+x_1^*)}, \frac{a_{2\tau}(n)}{c_{2\tau}(n)}\right\} \quad (3)$$

holds, then the species x_2 *will be driven to extinction, that is, for any positive solution* $(x_1(n), x_2(n))^T$ *of system (1),* $x_2(n) \to 0$ *exponentially as* $n \to +\infty$*.*

Proof: Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be a solution of system (1) with initial conditions (2). First we show that $x_2(n) \to 0$ exponentially as $n \to +\infty$.

From (1), we have

$$\begin{aligned} \ln x_1(n+1) - \ln x_1(n) &= r_1(n) - \sum_{\tau=0}^{m} a_{1\tau}(n)x_1(n-\tau) \\ &\quad - \sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)}, \\ \ln x_2(n+1) - \ln x_2(n) &= r_2(n) - \sum_{\tau=0}^{m} a_{2\tau}(n)x_2(n-\tau) \\ &\quad - \sum_{\tau=0}^{m}\frac{c_{1\tau}(n)x_1(n-\tau)}{1+x_1(n-\tau)}. \end{aligned} \quad (4)$$

By inequality (3), we can choose $\alpha, \beta, \varepsilon > 0$ such that

$$\begin{aligned} \limsup_{n\to+\infty}\frac{r_2(n)}{r_1(n)} &< \frac{\alpha}{\beta} - \varepsilon < \frac{\alpha}{\beta} \\ &< \liminf_{n\to+\infty}\left\{\frac{c_{1\tau}(n)}{a_{1\tau}(n)(1+x_1^*)}, \frac{a_{2\tau}(n)}{c_{2\tau}(n)}\right\}, \end{aligned} \quad (5)$$

then there exists an $N_1 > 0$ such that for all $n > N_1$,

$$r_2(n)\beta - r_1(n)\alpha < -\varepsilon\beta r_1(n) < -\varepsilon\beta r_{1*} < 0; \quad (6)$$

$$\alpha a_{1\tau}(n)(1+x_1^*) - \beta c_{1\tau}(n) < 0; \quad (7)$$

$$\alpha c_{2\tau}(n) - \beta a_{2\tau}(n) < 0. \quad (8)$$

It follows from (4), (6)-(8) that

$$\begin{aligned} &\beta(\ln x_2(n+1) - \ln x_2(n)) - \alpha(\ln x_1(n+1) \\ &- \ln x_1(n)) \\ = \ &(r_2(n)\beta - r_1(n)\alpha) \\ &- \sum_{\tau=0}^{m}\left(\beta a_{2\tau}(n) - \frac{\alpha c_{2\tau}(n)}{1+x_2(n-\tau)}\right)x_2(n-\tau) \\ &- \sum_{\tau=0}^{m}\left(\frac{\beta c_{1\tau}(n)}{1+x_1(n-\tau)} - \alpha a_{1\tau}(n)\right)x_1(n-\tau) \\ \leq \ &r_2(n)\beta - r_1(n)\alpha \\ < \ &-\varepsilon\beta r_{1*} < 0. \end{aligned} \quad (9)$$

Summating both side of inequality (9) from 0 to $n-1$, then

$$\begin{aligned} &\beta(\ln x_2(n) - \ln x_2(0)) - \alpha(\ln x_1(n) \\ &- \ln x_1(0)) < -\varepsilon\beta r_{1*}n. \end{aligned}$$

So, we can get

$$\begin{aligned} x_2(n) &< \left[\left(\frac{x_1(n)}{x_1(0)}\right)^{\alpha}(x_2(0))^{\beta}\right]^{\frac{1}{\beta}}\exp(-\varepsilon r_{1*}n) \\ &< \left[\left(\frac{x_1^*}{x_1(0)}\right)^{\alpha}(x_2(0))^{\beta}\right]^{\frac{1}{\beta}}\exp(-\varepsilon r_{1*}n). \end{aligned} \quad (10)$$

Therefore, we have $x_2(n) \to 0$ exponentially as $n \to +\infty$. This completes the proof. ■

Lemma 4. *Under the assumption of Theorem 1. Furthermore, assume that*

$$\mu = r_{1*} - \sum_{\tau=1}^{m} a_{1\tau}^* x_1^* > 0. \quad (11)$$

Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ *be any positive solution of system (1), then there exists a positive constant* x_{1*} *such that*

$$\liminf_{n\to+\infty} x_1(n) \geq x_{1*},$$

where

$$x_{1*}=\frac{\mu}{a_{10}^*}\exp\left[(r_1^*-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*)\left(1-\frac{a_{10}^*x_1^*}{\mu}\right)\right]$$

is a constant independent of any positive solution of system (1), i.e., the first species x_1 of system (1) is permanent.

Proof: By Lemma 3 and Theorem 1,

$$\limsup_{n\to+\infty}x_1(n)\le x_1^*,\ \lim_{n\to+\infty}x_2(n)=0,$$

for arbitrarily small positive constant $\varepsilon>0$, there exists an $N_2>0$ such that

$$x_1(n)<x_1^*+\varepsilon, x_2(n)<\varepsilon$$

for all $n>N_2$.

From the first equation of (1), for $n>N_2+m$,

$$\begin{aligned}x_1(n+1)&=x_1(n)\exp\left[r_1(n)-\sum_{\tau=0}^{m}a_{1\tau}(n)x_1(n-\tau)\right.\\&\quad\left.-\sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)}\right]\\&>x_1(n)\exp\left[r_1(n)-a_{10}^*x_1(n)\right.\\&\quad\left.-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*-\sum_{\tau=0}^{m}c_{2\tau}^*\varepsilon\right].\end{aligned}$$

Let $\varepsilon\to0$, then

$$\begin{aligned}x_1(n+1)&\ge x_1(n)\exp\left[(r_1(n)\right.\\&\quad\left.-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*)\left(1-\frac{a_{10}^*x_1(n)}{r_1(n)-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*}\right)\right]\\&\ge x_1(n)\exp\left[(r_1(n)\right.\\&\quad\left.-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*)\left(1-\frac{a_{10}^*x_1(n)}{\mu}\right)\right],\end{aligned}$$

where $\mu=r_{1*}-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*$. It is easy to check that the inequality $\frac{a_{10}^*x_1^*}{\mu}>1$ holds. By Lemma 1, we have

$$\begin{aligned}\liminf_{n\to+\infty}x_1(n)&\ge\frac{\mu}{a_{10}^*}\exp\left[(r_1^*-\sum_{\tau=1}^{m}a_{1\tau}^*x_1^*)\left(1-\frac{a_{10}^*x_1^*}{\mu}\right)\right]\\&\triangleq x_{1*}.\end{aligned}$$

This completes the proof. ■

Consider the following discrete logistic equation

$$x(n+1)=x(n)\exp\left[r_1(n)-\sum_{\tau=0}^{m}a_{1\tau}(n)x(n-\tau)\right].\quad(12)$$

Lemma 5. ([28]) *Assume that $\{r_1(n)\}$ and $\{a_{1\tau}(n)\}$ are bounded positive sequences, then any positive solution $\{x(n)\}$ of (12) satisfies*

$$x_{1*}<\liminf_{n\to+\infty}x(n)\le\limsup_{n\to+\infty}x(n)\le x_1^*.$$

Theorem 2. *Under the assumptions of Theorem 1, Lemmas 4 and 5. Furthermore, suppose that there exists a constant $\delta>0$ such that*

$$\min\left\{a_{10*},\frac{2}{x_1^*}-a_{10}^*\right\}-ma_1^*\ge\delta,\quad(13)$$

where $a_1^*=\max\{a_{1\tau}^*:\tau=1,2,\cdots,m\}$. Let $\tilde{x}(n)=(x_1(n),x_2(n))^T$ be any positive solution of system (1), then the species x_2 will be driven to extinction, that is, $x_2(n)\to0$ as $n\to+\infty$, and $x_1(n)\to x(n)$ as $n\to+\infty$, where $x(n)$ is any positive solution of equation (12).

Proof: Let $\tilde{x}(n)=(x_1(n),x_2(n))^T$ be a solution of system (1) with initial conditions (2). From Lemmas 3 and 4, $x_1(n)$ is bounded above and below by positive constants on $[0,+\infty)$. To finish the proof of Theorem 2, it is enough to show that $x_1(n)\to x(n)$ as $n\to+\infty$, where $x(n)$ is any positive solution of equation (12).

Let

$$V_1(n)=|\ln x_1(n)-\ln x(n)|,\quad(14)$$

then

$$x_1(n)=x(n)\exp\{y(n)\}.$$

From the first equation of (1) and equation (12), we have

$$\begin{aligned}\ln x_1(n+1)-\ln x_1(n)&=r_1(n)-\sum_{\tau=0}^{m}a_{1\tau}(n)x_1(n-\tau)\\&\quad-\sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)},\\\ln x(n+1)-\ln x(n)&=r_1(n)-\sum_{\tau=0}^{m}a_{1\tau}(n)x(n-\tau),\end{aligned}$$

then

$$\begin{aligned}V_1(n+1)&=|\ln x_1(n+1)-\ln x(n+1)|\\&=\Big|\ln x_1(n)-\ln x(n)\\&\quad-\sum_{\tau=0}^{m}a_{1\tau}(n)[x_1(n-\tau)-x(n-\tau)]\\&\quad-\sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)}\Big|\\&=\Big|\ln x_1(n)-\ln x(n)-a_{10}(n)[x_1(n)-x(n)]\\&\quad-\sum_{\tau=1}^{m}a_{1\tau}(n)[x_1(n-\tau)-x(n-\tau)]\\&\quad-\sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)}\Big|\\&\le|\ln x_1(n)-\ln x(n)-a_{10}(n)[x_1(n)-x(n)]|\\&\quad+\sum_{\tau=1}^{m}a_{1\tau}(n)|x_1(n-\tau)-x(n-\tau)|\\&\quad+\sum_{\tau=0}^{m}\frac{c_{2\tau}(n)x_2(n-\tau)}{1+x_2(n-\tau)}.\end{aligned}\quad(15)$$

Using the mean value theorem, then

$$\ln x_1(n)-\ln x(n)=\frac{1}{\zeta(n)}(x_1(n)-x(n)),$$

where $\zeta(n)$ lies between $x_1(n)$ and $x(n)$. Then we have

$$\begin{aligned}
&|\ln x_1(n)-\ln x(n)-a_{10}(n)[x_1(n)-x(n)]|\\
=&\ |\ln x_1(n)-\ln x(n)|-|\ln x_1(n)-\ln x(n)|\\
&+|\ln x_1(n)-\ln x(n)-a_{10}(n)[x_1(n)-x(n)]|\\
=&\ |\ln x_1(n)-\ln x(n)|-\frac{1}{\zeta(n)}|x_1(n)-x(n)|\\
&+\left|\frac{1}{\zeta(n)}[x_1(n)-x(n)]-a_{10}(n)[x_1(n)-x(n)]\right|\\
=&\ |\ln x_1(n)-\ln x(n)|\\
&-\left[\frac{1}{\zeta(n)}-\left|\frac{1}{\zeta(n)}-a_{10}(n)\right|\right]|x_1(n)-x(n)|. \qquad (16)
\end{aligned}$$

And hence it follows from (15) and (16) that

$$\begin{aligned}
\Delta V_1(n) =&\ V_1(n+1)-V_1(n)\\
\le&-\left[\frac{1}{\zeta(n)}-\left|\frac{1}{\zeta(n)}-a_{10}(n)\right|\right]|x_1(n)-x(n)|\\
&+\sum_{\tau=1}^{m}a_{1\tau}(n)|x_1(n-\tau)-x(n-\tau)|\\
&+\sum_{\tau=0}^{m}c_{2\tau}(n)x_2(n-\tau). \qquad (17)
\end{aligned}$$

Let

$$\begin{aligned}
V_2(n) =&\ \sum_{\tau=1}^{m}\sum_{s=n-\tau}^{n-1}a_{1\tau}(s+\tau)|x_1(s)-x(s)|\\
&+\sum_{\tau=0}^{m}\sum_{s=n-\tau}^{n-1}c_{2\tau}(s+\tau)x_2(s). \qquad (18)
\end{aligned}$$

By a direct calculation, it derives that

$$\begin{aligned}
\Delta V_2(n) =&\ V_2(n+1)-V_2(n)\\
=&\ \sum_{\tau=1}^{m}\sum_{s=n-\tau+1}^{n}a_{1\tau}(s+\tau)|x_1(s)-x(s)|\\
&+\sum_{\tau=0}^{m}\sum_{s=n-\tau+1}^{n}c_{2\tau}(s+\tau)x_2(s)\\
&-\sum_{\tau=1}^{m}\sum_{s=n-\tau}^{n-1}a_{1\tau}(s+\tau)|x_1(s)-x(s)|\\
&-\sum_{\tau=0}^{m}\sum_{s=n-\tau}^{n-1}c_{2\tau}(s+\tau)x_2(s)\\
=&\ \sum_{\tau=1}^{m}a_{1\tau}(n+\tau)|x_1(n)-x(n)|\\
&-\sum_{\tau=1}^{m}a_{1\tau}(n)|x_1(n-\tau)-x(n-\tau)|\\
&+\sum_{\tau=0}^{m}c_{2\tau}(n+\tau)x_2(n)\\
&-\sum_{\tau=0}^{m}c_{2\tau}(n)x_2(n-\tau). \qquad (19)
\end{aligned}$$

Let

$$V(n)=V_1(n)+V_2(n). \qquad (20)$$

Therefore, it follows from (17) and (19) that

$$\begin{aligned}
\Delta V(n) =&\ \Delta V_1(n)+\Delta V_2(n)\\
\le&-\left[\frac{1}{\zeta(n)}-\left|\frac{1}{\zeta(n)}-a_{10}(n)\right|\right]|x_1(n)-x(n)|\\
&+\sum_{\tau=1}^{m}a_{1\tau}(n+\tau)|x_1(n)-x(n)|\\
&+\sum_{\tau=0}^{m}c_{2\tau}(n+\tau)x_2(n)\\
=&-\left[\frac{1}{\zeta(n)}-\left|\frac{1}{\zeta(n)}-a_{10}(n)\right|-\sum_{\tau=1}^{m}a_{1\tau}(n+\tau)\right]\\
&\times|x_1(n)-x(n)|\\
&+\sum_{\tau=0}^{m}c_{2\tau}(n+\tau)x_2(n). \qquad (21)
\end{aligned}$$

From Lemmas 3, 4, 5 and Theorem 1, for arbitrarily small $\varepsilon>0$, there exists an $N_3>0$ such that

$$\begin{aligned}
&x_{1*}-\varepsilon<x_1(n)<x_1^*+\varepsilon,\\
&x_2(n)<\varepsilon, x_{1*}-\varepsilon<x(n)<x_1^*+\varepsilon
\end{aligned}$$

for all $n>N_3+m$. Therefore,

$$\begin{aligned}
\Delta V(n) <&-\left[\min\left\{a_{10*},\frac{2}{x_1^*+\varepsilon}-a_{10}^*\right\}-ma_1^*\right]\\
&\times|x_1(n)-x(n)|+\sum_{\tau=0}^{m}c_{2\tau}(n+\tau)\varepsilon. \qquad (22)
\end{aligned}$$

Let $\varepsilon\to 0$, then

$$\begin{aligned}
\Delta V(n) \le&-\left[\min\left\{a_{10*},\frac{2}{x_1^*}-a_{10}^*\right\}-ma_1^*\right]\\
&\times|x_1(n)-x(n)|\\
\le&-\delta|x_1(n)-x(n)|. \qquad (23)
\end{aligned}$$

Summating both side of (23) from N_4+m to n, we have

$$\sum_{j=N_4+m}^{n}(V(j+1)-V(j))\le-\delta\sum_{j=N_4+m}^{n}|x_1(j)-x(j)|,$$

then

$$V(n+1)+\delta\sum_{j=N_4+m}^{n}|x_1(j)-x(j)|\le V(N_4+m),$$

and so,

$$\sum_{j=N_4+m}^{n}|x_1(j)-x(j)|\le\frac{V(N_4+m)}{\delta}.$$

It follows from (20) that $V(N_4+m)$ is bounded. Hence,

$$\sum_{j=N_4+m}^{n}|x_1(j)-x(j)|\le\frac{V(N_4+m)}{\delta}<+\infty,$$

then

$$\limsup_{n\to+\infty}\sum_{j=N_4+m}^{n}|x_1(j)-x(j)|\le\frac{V(N_4+m)}{\delta}<+\infty.$$

This implies that

$$\lim_{n\to+\infty}|x_1(n)-x(n)|=0.$$

This completes the proof. ■

IV. Extinction of x_1 and stability of x_2

In this section, we study the extinction of x_1 but x_2 of system (1). Similar to the proofs in Section 3, we can obtain the following results.

Theorem 3. *Assume that the inequality*

$$\liminf_{n\to+\infty} \frac{r_2(n)}{r_1(n)} > \limsup_{n\to+\infty} \left\{ \frac{c_{1\tau}(n)}{a_{1\tau}(n)}, \frac{a_{2\tau}(n)(1+x_2^*)}{c_{2\tau}(n)} \right\} \quad (24)$$

holds, then the species x_1 will be driven to extinction, that is, for any positive solution $(x_1(n), x_2(n))^T$ of system (1), $x_1(n) \to 0$ exponentially as $n \to +\infty$.

Lemma 6. *Under the assumption of Theorem 3. Furthermore, assume that*

$$\eta = r_{2*} - \sum_{\tau=1}^{m} a_{2\tau}^* x_2^* > 0. \quad (25)$$

Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1), then there exists a positive constant x_{2*} such that

$$\liminf_{n\to+\infty} x_2(n) \geq x_{2*},$$

where

$$x_{2*} = \frac{\eta}{a_{20}^*} \exp\left[\left(r_2^* - \sum_{\tau=1}^{m} a_{2\tau}^* x_2^*\right)\left(1 - \frac{a_{20}^* x_2^*}{\eta}\right) \right]$$

is a constant independent of any positive solution of system (1), i.e., the first species x_2 of system (1) is permanent.

Consider the following discrete logistic equation

$$x(n+1) = x(n)\exp\left[r_2(n) - \sum_{\tau=0}^{m} a_{2\tau}(n) x(n-\tau) \right]. \quad (26)$$

Lemma 7. ([28]) *Assume that $\{r_2(n)\}$ and $\{a_{2\tau}(n)\}$ are bounded positive sequences, then any positive solution $\{x(n)\}$ of (12) satisfies*

$$x_{2*} < \liminf_{n\to+\infty} x(n) \leq \limsup_{n\to+\infty} x(n) \leq x_2^*.$$

Theorem 4. *Under the assumptions of Theorem 3, Lemmas 6 and 7. Furthermore, suppose that there exists a constant $\rho > 0$ such that*

$$\min\left\{ a_{20*}, \frac{2}{x_2^*} - a_{20}^* \right\} - m a_2^* \geq \rho, \quad (27)$$

where $a_2^* = \max\{a_{2\tau}^* : \tau = 1, 2, \cdots, m\}$. Let $\tilde{x}(n) = (x_1(n), x_2(n))^T$ be any positive solution of system (1), then the species x_1 will be driven to extinction, that is, $x_1(n) \to 0$ as $n \to +\infty$, and $x_2(n) \to x(n)$ as $n \to +\infty$, where $x(n)$ is any positive solution of equation (26).

V. Numerical examples and simulations

In this section, we give two examples to illustrate the feasibility of our results.

Example 1. Let $m = 1$, then system (1) can be written as

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\Big[r_1(n) - a_{10}(n)x_1(n) \\ &\quad - a_{11}(n)x_1(n-1) \\ &\quad - \frac{c_{20}(n)x_2(n)}{1+x_2(n)} - \frac{c_{21}(n)x_2(n-1)}{1+x_2(n-1)} \Big], \\ x_2(n+1) &= x_2(n)\exp\Big[r_2(n) - a_{20}(n)x_2(n) \\ &\quad - a_{21}(n)x_2(n-1) \\ &\quad - \frac{c_{10}(n)x_1(n)}{1+x_1(n)} - \frac{c_{11}(n)x_1(n-1)}{1+x_1(n-1)} \Big]. \end{aligned} \quad (28)$$

Choose the coefficients

$$\begin{aligned} & r_1(n) = 1 - 0.2\sin(n), a_{10}(n) = 0.5 - 0.1\cos(n), \\ & a_{11}(n) = 0.05 - 0.01\cos(n), \\ & c_{20}(n) = 0.6 + 0.2\sin(n), c_{21}(n) = 0.5 + 0.1\sin(n), \\ & r_2(n) = 0.1 - 0.05\cos(n), a_{20}(n) = 0.4 - 0.2\cos(n), \\ & a_{21}(n) = 0.4 - 0.1\sin(n), \\ & c_{10}(n) = 0.6 - 0.1\cos(n), c_{11}(n) = 0.5 - 0.1\cos(n). \end{aligned}$$

By a direct calculation, we can get

$$\begin{aligned} & x_1^* = 3.0535, x_2^* = 2.1369; \\ & \limsup_{n\to+\infty} \frac{r_2(n)}{r_1(n)} = 0.1593 < 0.2500 \\ & = \liminf_{n\to+\infty} \left\{ \frac{c_{1\tau}(n)}{a_{1\tau}(n)(1+x_1^*)}, \frac{a_{2\tau}(n)}{c_{2\tau}(n)} \right\}; \\ & \mu = r_{1*} - \sum_{\tau=1}^{m} a_{1\tau}^* x_1^* = 0.7905 > 0; \\ & \min\left\{ a_{10*}, \frac{2}{x_1^*} - a_{10}^* \right\} - m a_1^* = 0.4000 > 0; \end{aligned}$$

that is the conditions of Theorem 2 hold, and so species x_2 will be driven to extinction while species x_1 is asymptotically to any positive solution of

$$\begin{aligned} x(n+1) &= x(n)\exp[r_1(n) - a_{10}(n)x(n) \\ &\quad - a_{11}(n)x(n-1)]. \end{aligned} \quad (29)$$

The solutions of systems (28) and (29) corresponding to initial values are displayed in Figure 1.

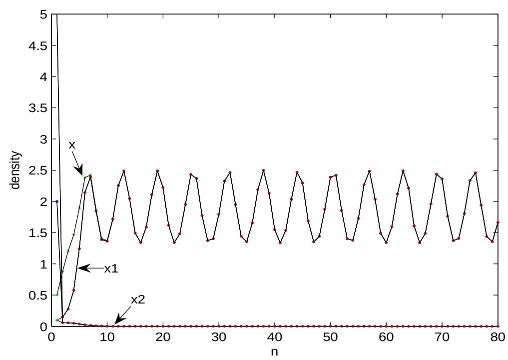

Fig. 1. Dynamic behaviors of species x_1 and x_2 in (28) with initial values $x_1(0) = 0.1, 2, 5$, $x_1(-1) = 1$, and $x_2(0) = 0.1, 2, 5$, $x_2(-1) = 1$, respectively; x is a solution of equation (29).

Example 2. Let $m = 2$, then system (1) can be written as

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\Big[r_1(n) - a_{10}(n)x_1(n) \\ &\quad -a_{11}(n)x_1(n-1) - a_{12}(n)x_1(n-2) \\ &\quad -\frac{c_{20}(n)x_2(n)}{1+x_2(n)} - \frac{c_{21}(n)x_2(n-1)}{1+x_2(n-1)} \\ &\quad -\frac{c_{22}(n)x_2(n-2)}{1+x_2(n-2)}\Big], \\ x_2(n+1) &= x_2(n)\exp\Big[r_2(n) - a_{20}(n)x_2(n) \\ &\quad -a_{21}(n)x_2(n-1) - a_{22}(n)x_2(n-2) \\ &\quad -\frac{c_{10}(n)x_1(n)}{1+x_1(n)} - \frac{c_{11}(n)x_1(n-1)}{1+x_1(n-1)} \\ &\quad -\frac{c_{12}(n)x_1(n-2)}{1+x_1(n-2)}\Big]. \end{aligned} \tag{30}$$

Choose the coefficients

$$\begin{aligned} &r_1(n) = 0.2 - 0.1\cos(n), a_{10}(n) = 0.4 - 0.2\sin(n), \\ &a_{11}(n) = 0.4 - 0.1\sin(n), a_{12}(n) = 0.4 - 0.1\sin(n), \\ &c_{20}(n) = 0.6 - 0.1\cos(n), c_{21}(n) = 0.5 - 0.1\cos(n), \\ &c_{22}(n) = 0.4 - 0.1\cos(n), \\ &r_2(n) = 0.6 - 0.2\sin(n), a_{20}(n) = 0.5 - 0.1\cos(n), \\ &a_{21}(n) = 0.05 - 0.01\cos(n), \\ &a_{22}(n) = 0.04 - 0.01\cos(n), \\ &c_{10}(n) = 0.6 + 0.2\sin(n), c_{11}(n) = 0.5 + 0.1\sin(n), \\ &c_{12}(n) = 0.4 + 0.1\sin(n). \end{aligned}$$

By a direct calculation, we can get

$$\begin{aligned} &x_1^* = 2.4827, x_2^* = 2.0467; \\ &\liminf_{n\to+\infty}\frac{r_2(n)}{r_1(n)} = 2.6633 > 2.5463 \\ &> \limsup_{n\to+\infty}\left\{\frac{c_{1\tau}(n)}{a_{1\tau}(n)}, \frac{a_{2\tau}(n)(1+x_2^*)}{c_{2\tau}(n)}\right\}; \\ &\eta = r_{2*} - \sum_{\tau=1}^{m} a_{2\tau}^* x_2^* = 0.1749 > 0; \\ &\min\left\{a_{20*}, \frac{2}{x_2^*} - a_{20}^*\right\} - m a_2^* = 0.2572 > 0; \end{aligned}$$

that is the conditions of Theorem 4 hold, and so species x_2 will be driven to extinction while species x_1 is asymptotically to any positive solution of

$$\begin{aligned} x(n+1) &= x(n)\exp[r_2(n) - a_{20}(n)x(n) \\ &\quad -a_{21}(n)x(n-1) - a_{22}(n)x(n-2)]. \end{aligned} \tag{31}$$

The solutions of systems (30) and (31) corresponding to initial values are displayed in Figure 2.

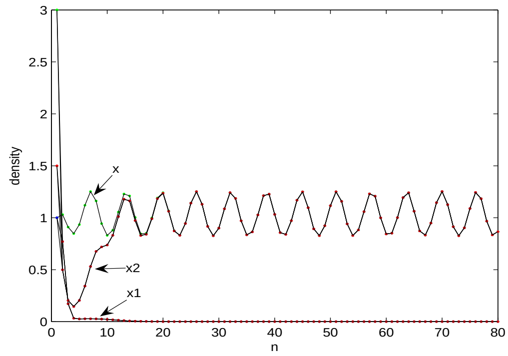

Fig. 2. Dynamic behaviors of species x_1 and x_2 in (30) with initial values $x_1(0) = 3, 1, 1.5$, $x_1(-2) = 0.5$, $x_1(-1) = 0.5$, and $x_2(0) = 3, 1, 1.5$, $x_2(-2) = 0.5$, $x_2(-1) = 0.5$, respectively; x is a solution of equation (31).

VI. Conclusion

This paper studied a nonautonomous discrete competitive system with delays. It is shown that if the coefficients are bounded above and below by positive constants and satisfy certain inequalities, then one of the species will be driven to extinction while the other one will be globally attractive with any positive solution of a discrete logistic equation.

This paper provided an effective method for the further study on permanence and extinction of population dynamic systems with time delay, one may see [29-31]. In fact, our techniques in this paper are applicable to a pure delayed discrete n-species competitive system. Furthermore, one may consider a discrete competitive system with infinite delay, which we leave for future work.

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