Analytical Approximate Solutions for a Class of Non-homogeneous Time-fractional Partial Differential Equations by a Hybrid Series Method

Jianke Zhang, Luyang Yin, Linna Li and Qiongdan Huang

Abstract—**The purpose of this paper is to obtain the analytical approximate solutions for a class of non-homogeneous time-fractional partial differential equations. A new kind of hybrid series method is proposed depending on the Fourier series and power series. Firstly, the Fourier series is used to transform the time-fractional partial differential equations to the time-fractional ordinary differential equations. Then, the power series is presented to get the analytical approximate solutions with the polynomial least squares method. The fractional derivatives are in Caputo sense. Several instances are given in this paper. The results are shown in the form of data and graphs, which represent that the hybrid series method is effective and convenient to solve this class of non-homogeneous time-fractional partial differential equations.**

Index Terms—**hybrid series method, non-homogeneous time-fractional differential equation, Caputo fractional derivative.**

I. Introduction

The fractional derivatives have attracted lots of attention recently, which can describe the dynamic process of the systematic function. In addition, fractional derivative model is better than classical integer order model, which obtains better results by using fewer parameters. Nowadays, fractional differential equations have been widely used in many fields of engineering, physics and mathematics [1-3].

A class of non-homogeneous time-fractional partial differential equations in this paper consists of many different equations. Liao et al. [4] studied the time-fractional sub-diffusion equations by the modified Du Fort-Frankel schemes. Califano et al. [5] solved the fractional diffusion-wave equations, using the domain decomposition methods. Huang [6] discussed the analytical solutions of the time-fractional telegraph equation. Srivastava et al. [7] solved the hyperbolic one-dimensional time fractional telegraph equation by the reduced differential transformation method.

Lots of methods have been proposed to solve the fractional differential equations, including numerical methods, semi-analytical methods and all-analytical methods. Zeng et al. [8] used the finite difference/element approaches to solve the time-fractional sub-diffusion equation. Bhrawy et al. [9] presented a Chebyshev-Laguerre Gauss-Radau collocation scheme for solving the time-fractional sub-diffusion equation. In [10], two finite difference schemes were used by Huang et al. for the time fractional diffusion-wave equation. Feng [37] proposed a Crank-Nicolson finite difference scheme for a class of space fractional differential equations. In [38], a numerical method based on the Legendre polynomials was developed to solve the variable order time fractional diffusion equation. Kazem et al. [11] proposed a semi-analytical solution for the time-fractional diffusion equation. Pandey et al. [12] developed a semi-analytic numerical method to solve the time-space fractional heat and the wave type equations. In [13], the space-time fractional diffusion equation was solved by Elsaid et al. to get the semi-analytic solution. Yıldırım et al. [14] used the homotopy perturbation methods to get the analytical solution for the space-time fractional advection-dispersion equation. Wang et al. [15] obtained the solutions of a class of time-fractional telegraph equations by the reproducing kernel method. However, we can't make a theoretical analysis for the solutions solved by the numerical method. In addition, each of numerical method and semi-analytical method are complicated.

In this paper, a new kind of all-analytical method is introduced, which combines the Fourier series and power series. The Fourier series is presented to transform the time-fractional partial differential equations(TFPDEs) into the time-fractional ordinary differential equations(TFODEs). Besides, the power series is proposed to solve a class of non-homogeneous time-fractional ordinary differential equations with the polynomial least squares method.

The structure of this paper is as follows. In Section 2, the basic definitions and notations are presented, including the definition of Caputo fractional derivative, introduction of a class of non-homogeneous TFPDEs and series approximation method. In Section 3, A new kind of hybrid series method combining Fourier series with power series is introduced in detail. In Section 4, several numerical examples have been solved, where the absolute errors are shown in graphes and tables. At last, the conclusion is proposed in Section 5.

II. Preliminaries

In this section, we introduce some basic knowledge with respect to the definition of Caputo fractional derivative and a class of non-homogeneous time-fractional partial differential equations in this paper.

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A. Definition of Caputo fractional derivative

Definition 1([16]). The Caputo time-fractional derivative operator of order α of $u(x,t)$ is defined as follows

$$D^{\alpha}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t(t-\tau)^{n-\alpha-1}\frac{\partial^n u(x,\tau)}{\partial\tau^n}d\tau, \\ \qquad n-1<\alpha<n \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, \quad \alpha=n\in N.\end{cases} \tag{1}$$

Theorem 1([17]). By the Caputo derivative, we get

$$D^{\alpha}t^p=\begin{cases}\dfrac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}t^{p-\alpha}, \quad \alpha\le p, \\ 0, \quad \alpha>p.\end{cases} \tag{2}$$

B. Introduction of a class of non-homogeneous TFPDEs

Consider a series of non-homogeneous TFPDEs in the following form [18]:

$$\begin{aligned}&D_t^{\alpha}u(x,t)+\sum_{i=1}^{l}\rho_i(t)D_t^{\alpha_i}u(x,t)\\&\quad=\Big[\sum_{i=I+1}^{K}\rho_i(t)D_t^{\alpha_i}\Big]\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\\&0\le x\le a,\quad 0\le t\le T.\end{aligned} \tag{3}$$

where the initial conditions are

$$u(x,0)=h_0(t),\quad \frac{\partial u(x,0)}{\partial t}=h_1(t), \tag{4}$$

and the boundary conditions are

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t). \tag{5}$$

Here $0<\alpha\le 2$, $0\le\alpha_i<\alpha$, $i=1,2,\ldots,K$ are fractional or integer numbers, and $\rho_i(t),h_0(t),h_1(t),g_0(t),g_1(t)$ are all known smooth functions.

This class of non-homogeneous time-fractional partial differential equations have many forms, including the following equations:

1. Time-fractional sub-diffusion equations [8], [9], [19]:

$$\begin{aligned}&D_t^{\alpha}u(x,t)=b_1\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\\&\quad 0<\alpha<1,\quad b_1>0.\end{aligned} \tag{6}$$

2. Multi-term time-fractional diffusion and diffusion-wave equations [5], [10], [20]:

$$\begin{aligned}&D_t^{\alpha}u(x,t)+\sum_{i=1}^{n}\rho_iD_t^{\alpha_i}u(x,t)=b_2\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\\&\quad 0<\alpha_i<\alpha<2,\quad \rho_i,b_2>0.\end{aligned} \tag{7}$$

3. Time-fractional modified anomalous sub-diffusion equations [21], [22], [23]:

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t}=[\rho_1D_t^{1-\alpha_1}+\rho_2D_t^{1-\alpha_2}]\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\\&\quad 0<\alpha_1,\alpha_2<1,\quad \rho_1,\rho_2>0.\end{aligned} \tag{8}$$

4. Time-fractional telegraph equations [6], [24]:

$$\begin{aligned}&D_t^{\alpha}u(x,t)+\rho_1D_t^{\alpha-1}u(x,t)+\rho_2u(x,t)\\&\quad=\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\\&\quad 1<\alpha\le 2,\quad \rho_1,\rho_2>0.\end{aligned} \tag{9}$$

5. Time-fractional second-order hyperbolic telegraph equations [7]:

$$\begin{aligned}&D_t^{2\alpha}u(x,t)+(\rho+\sigma)D_t^{\alpha}u(x,t)+(\rho\sigma)u(x,t)\\&\quad=\eta^2\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t),\\&0<\alpha\le 1,\quad \rho+\sigma,\rho\sigma,\eta>0.\end{aligned} \tag{10}$$

C. Series approximation method

Firstly, the power series [25] can be defined by

$$\omega(t)=\sum_{l=0}^{m}a_lt^l,\quad m\in N.$$

Secondly, the introduction of the polynomial least squares method is as follows.

Definition 2([26]). One calls an ϵ-approximate polynomial solution $\tilde\omega(t)$ of problem (3), (4) and (5) on the $[0,b]$ interval, which satisfies the following inequality:

$$|Res(t,\tilde\omega(t))|<\epsilon,\quad \tilde\omega(0)=c_0,\quad \frac{d\tilde\omega(0)}{dt}=c_1, \tag{11}$$

where $Res(t)$ represents the residual function.

Definition 3([26]). One calls a weak δ-approximate polynomial solution $\tilde\omega(t)$ of equation (11), which satisfies the following condition:

$$\int_0^1 Res^2(t,\tilde\omega(t))dt\le\delta, \tag{12}$$

together with the initial condition.

We call the sequence of polynomials $\omega(t)$ convergent to the solution of this problem if $\lim\limits_{n\to\infty}Res(\omega(t))=0$.

We compute a weak ϵ-approximate solution of the type[26], [27]:

$$\tilde\omega(t)=\sum_{l=0}^{m}d_lt^l, \tag{13}$$

where the constants $d_0,d_0,\ldots,d_m$ are calculated using the steps outlined in the following:

1. Let us substitute the approximate solution (13) in (3), we can get

$$Res(t,d_0,d_1,\ldots,d_m)=Res(t,\tilde\omega(t)). \tag{14}$$

2. Attach to residual function the following real functional:

$$J(d_2,d_3,\ldots,d_m)=\int_0^1 Res^2(t,d_0,d_1,\ldots,d_m)dt, \tag{15}$$

where d_0,d_1 are computed as functions of $d_2,d_3,\ldots,d_m$ by using the initial condition.

3. Compute $d_2^0,d_3^0,\ldots,d_m^0$ as the values which give the minimum of the functional (15).

4. Consider the following polynomial:

$$P_m(t)=\sum_{l=0}^{m}d_l^0t^l. \tag{16}$$

Then, the following convergence theorem holds.

Theorem 2([27]). The necessary condition for this problem to admit a sequence of polynomials $\omega(t)$ convergent to the solution of this problem is:

$$\lim_{m\to\infty}\int_0^1 Res^2(t,P_m(t))dt=0. \tag{17}$$

Moreover, $\forall \epsilon > 0, \exists m_0 \in N$ such that $\forall m \in N, m > m_0$, it follows that $P_m(t)$ is a weak ϵ-approximate polynomial solution of this problem.

The set of the weak approximate solutions of this problem also contains the approximate solutions of the problem. Taking into account the above condition, we will first determine weak approximate polynomial solutions, $\tilde{\omega}(t)$. If $|Res(t,\tilde{\omega}(t))| < \epsilon$ then $\tilde{\omega}(t))$ is also an ϵ-approximate polynomial solution of the problem.

There are a lot of methods, which can solve the fractional differential equations, such as numerical methods, semi-analytical methods and all-analytical methods. All-analytical methods can make the qualitative analysis for the approximate solutions. The residual power series method, one of the simplest all-analytical methods, has been considered to solve homogeneous fractional differential equations [28], [29]. However, it is difficult to solve non-homogeneous fractional differential equations by residual power series method. Therefore, a new kind of hybrid series method is proposed in the following section.

III. A NEW KIND OF HYBRID SERIES METHOD

In this section, we create a new hybrid series method, which combines the Fourier series and power series.

Then, the specific steps of solving this class of non-homogeneous TFPDEs are as follows.

Step 1. In order to transform the non-homogeneous boundary conditions (5) into homogeneous boundary conditions, we can suppose

$$u(x,t) = \nu(x,t) + s(x,t). \tag{18}$$

Then, $\nu(x,t)$ should satisfy the following equation

$$\begin{aligned} &D_t^{\alpha}\nu(x,t) + \sum_{i=1}^{l}\rho_i(t)D_t^{\alpha_i}\nu(x,t) \\ &\qquad\qquad - \Big[\sum_{i=I+1}^{K}\rho_i(t)D_t^{\alpha_i}\Big]\frac{\partial^2 u(x,t)}{\partial x^2} \\ &= f(x,t) - D_t^{\alpha}s(x,t) - \sum_{i=1}^{l}\rho_i(t)D_t^{\alpha_i}s(x,t) \\ &= \Theta(x,t). \end{aligned} \tag{19}$$

The homogeneous boundary conditions are

$$\nu(0,t) = \nu(a,t) = 0. \tag{20}$$

Then, we can obtain

$$s(0,t) = u(0,t) = g_0(t), \quad s(a,t) = u(a,t) = g_1(t). \tag{21}$$

The derivation process of $s(x,t)$ is as follows.

We can suppose

$$s(x,t) = A(t)x + B(t). \tag{22}$$

Substitute the conditions (21) into (22), we can get

$$s(x,t) = \frac{x}{a}(g_1(t) - g_0(t)) + g_0(t). \tag{23}$$

The initial conditions (4) now are

$$\begin{aligned} \nu(x,0) &= u(x,0) - s(x,0), \\ \frac{\partial\nu(x,0)}{\partial t} &= \frac{\partial u(x,0)}{\partial t} - \frac{\partial s(x,0)}{\partial t}. \end{aligned} \tag{24}$$

Then, the time-fractional partial differential equations can be transformed to the time-fractional ordinary differential equations by Fourier series. Therefore, we obtain

$$\nu(x,t) = \sum_{n=1}^{\infty}\omega_n(t)\sin(\frac{n\pi}{a}x). \tag{25}$$

Therefore, we attain the following time-fractional ordinary differential equation(TFODE):

$$\begin{aligned} &D_t^{\alpha}\omega_n(t) + \sum_{i=1}^{l}\rho_i(t)D_t^{\alpha_i}\omega_n(t) \\ &\qquad + \Big(\frac{n\pi}{a}\Big)^2\sum_{i=I+1}^{K}\rho_i(t)D_t^{\alpha_i}\omega_n(t) = \phi_n(t), \end{aligned} \tag{26}$$

where

$$\phi_n(t) = \frac{2}{a}\int_0^a \Theta(x,t)\sin(\frac{n\pi}{a}x)dx. \tag{27}$$

Then, the initial conditions of the TFODE are:

$$\begin{aligned} \omega_n(0) &= \frac{2}{a}\int_0^a \nu(x,0)\sin(\frac{n\pi}{a}x)dx, \\ \frac{d\omega_n(0)}{dt} &= \frac{2}{a}\int_0^a \frac{\partial\nu(x,0)}{\partial t}\sin(\frac{n\pi}{a}x)dx. \end{aligned} \tag{28}$$

Step 2. Let us consider the power series $\omega_n(t) = \sum_{l=0}^{m} a_l t^l$ to solve the TFODEs.

Then, the residual function is defined by

$$\begin{aligned} Res_n(t) &= D_t^{\alpha}\omega_n(t) + \sum_{i=1}^{l}\rho_i(t)D_t^{\alpha_i}\omega_n(t) \\ &+ \Big(\frac{n\pi}{a}\Big)^2\sum_{i=I+1}^{K}\rho_i(t)D_t^{\alpha_i}\omega_n(t) - \phi_n(t), t \in [0,1] \end{aligned} \tag{29}$$

Step 3. Then, we use the polynomial least squares method to get the analytical approximate solutions.

$$\min \quad \int_0^1 Res_n^2(t,\omega_n(t))dt, \tag{30}$$

with the initial conditions:

$$\begin{aligned} \omega_n(0) &= \frac{2}{a}\int_0^a \nu(x,0)\sin(\frac{n\pi}{a}x)dx, \\ \frac{d\omega_n(0)}{dt} &= \frac{2}{a}\int_0^a \frac{\partial\nu(x,0)}{\partial t}\sin(\frac{n\pi}{a}x)dx. \end{aligned} \tag{31}$$

Step 4. At last, we substitute the solutions $\omega_n(t)$ we got from Step 3 into the equation (25). Then, we can get $\nu(x,t)$. Substitute $\nu(x,t)$ into equation (18), then the analytical approximate solutions can be attained. Compare with the exact solutions, we can get the absolute errors and analyze the results. Then, the analytical approximate solution $u_{HSM}(x,t)$ is

$$\begin{aligned} u_{HSM}(x,t) &= \nu(x,t) + s(x,t) \\ &= \sum_{n=1}^{\infty}\omega_n(t)\sin(\frac{n\pi}{a}x) + \Big(\frac{x}{a}(g_1(t) - g_0(t)) + g_0(t)\Big). \end{aligned}$$

IV. Numerical examples

In this section, several examples are presented to get the analytical approximate solutions for the following non-homogeneous TFPDEs. All the data are received by Maple 18 on a Intel Core i3-2350M, 2.30GHz CPU. Then, the maximum absolute error at $t = T$ is defined by

$$E_{max} = max|u_{exact}(x,T) - u_{HSM}(x,T)|. \quad (32)$$

where $u_{excat}(x,T)$ represents the exact solution and $u_{HSM}(x,T)$ represents the analytical approximate solution at $t = T$ in this paper.

Example 1. Consider the following time-fractional partial differential equation(TFPDE):

$$D_t^{\alpha}u(x,t) + D_t^{\alpha_1}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$1 < \alpha < 2, \quad 1 < \alpha_1 < 2, \quad 0 \le x \le 1, \quad 0 \le t \le T, \quad (33)$$

where the initial conditions are

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad (34)$$

and the boundary conditions are

$$u(0,t) = u(1,t) = 0. \quad (35)$$

Here

$$f(x,t) = \Big(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{6t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + \pi^2 t^3\Big)\sin(\pi x),$$

corresponds to the exact solution $u_{excat}(x,t) = \sin(\pi x)t^3$.

Consider the substitution $u(x,t) = \omega(t)\sin(\pi x)$, and this TFPDE can be transformed into the following TFODE:

$$D_t^{\alpha}\omega(t) + D_t^{\alpha_1}\omega(t) + \pi^2\omega(t) = \phi(t), \quad 0 \le t \le 1, \quad (36)$$

where the initial conditions are

$$\omega(0) = \frac{d\omega(0)}{dt} = 0, \quad (37)$$

and at this time, we have

$$\phi(t) = \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{6t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + \pi^2 t^3.$$

Therefore, we can obtain

$$\begin{aligned} &u_{HSM}(x,t) = \omega(t)\sin(\pi x) \\ &= (2.466654065714307700 \times 10^{-16}t^2 \\ &\quad +9.99999999999999796 \times 10^{-1}t^3)sin(\pi x). \end{aligned}$$

Figure 1 presents the absolute error when $\alpha = 1.9$, $\alpha_1 = 1.3$ at the final time $T = 1$ and the number of the power harmonics $M = 4$. From Figure 1, we can find that when $x = 0.5$, the maximum absolute error is 4.0×10^{-17}.

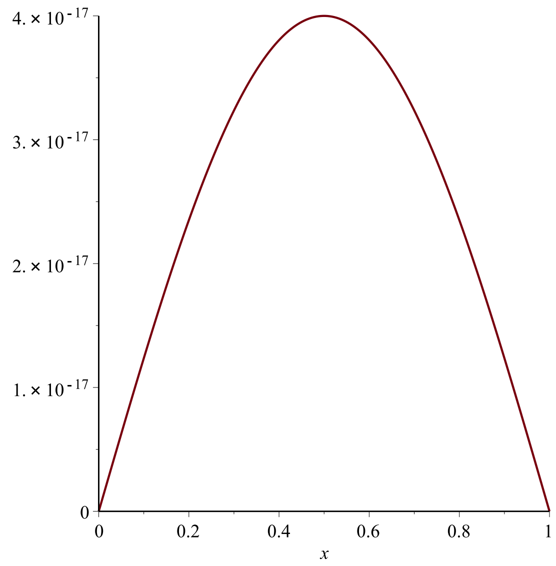

Fig. 1. Absolute error for Example 1 when $\alpha = 1.9, \alpha_1 = 1.3$

Table 1 shows the maximum absolute error when $\alpha = 1.9$, $\alpha_1 = 1.3$ and $T = 1$. Reutskiy [18] used a semi-analytical collocation method to solve this problem. Dehghan et al. [30] obtained the analytical approximate solution by the compact finite difference procedure and Galerkin spectral method. Compared with the errors in [18] and [30], the maximum absolute error in this paper is much smaller than the others when $M = 4$.

TABLE I
Maximum absolute errors for Example 1 with $\alpha = 1.9$, $\alpha_1 = 1.3$

Present method		[18], $\delta = 0.15$		[30], $h = 1/100$	
M	E_{max}	M	E_{max}	$1/\tau$	E_{max}
4	4.0×10^{-17}	4	1.3×10^{-2}	1/20	1.3×10^{-2}
		16	1.6×10^{-13}	1/80	3.2×10^{-3}
		64	2.2×10^{-16}	1/320	7.0×10^{-4}

Example 2. Let's consider the following two-term wave-diffusion equation:

$$D_t^{\alpha}u(x,t) + \tfrac{\partial u(x,t)}{\partial t} = \tfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$0 \le x \le 1, \quad 0 \le t \le T, \quad (38)$$

where the initial conditions are

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad (39)$$

and the boundary conditions are

$$u(0,t) = t^3, \quad u(1,t) = et^3. \quad (40)$$

Here

$$f(x,t) = \Big(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + 3t^2 - t^3\Big)e^x,$$

corresponds to the exact solution $u_{excat}(x,t) = e^x t^3$.

Consider the substitution $u(x,t) = \nu(x,t) + t^3[1 + x(e-1)]$, we attain the TFPDE:

$$D_t^\alpha \nu(x,t) + \frac{\partial \nu(x,t)}{\partial t} - \frac{\partial^2 \nu(x,t)}{\partial x^2}$$
$$= f(x,t) - \Big(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + 3t^2\Big)[1 + x(e-1)] = \Theta(x,t),$$

where homogeneous conditions are

$$\nu(x,0) = \frac{\partial \nu(x,0)}{\partial t} = 0, \quad \nu(0,t) = \nu(1,t) = 0.$$

Then, we can obtain the following transformation:

$$\begin{aligned} D_t^\alpha \omega_n(t) + \tfrac{d\omega_n(t)}{dt} + (n\pi)^2 \omega_n(t) = \phi_n(t), \\ \omega_n(0) = \tfrac{d\omega_n(0)}{dt} = 0. \end{aligned} \tag{41}$$

Therefore, we can get the following solutions when $M = 4$ and $N = 4$

$$\begin{aligned} \omega_1(t) &= -2.9882377206 \times 10^{-19} t^2 \\ &\quad -2.1777533422 \times 10^{-1} t^3 \end{aligned} \tag{42}$$
$$\begin{aligned} \omega_2(t) &= -3.6847687029 \times 10^{-20} t^2 \\ &\quad +1.3512042358 \times 10^{-2} t^3 \end{aligned} \tag{43}$$
$$\begin{aligned} \omega_3(t) &= -4.3374322440 \times 10^{-21} t^2 \\ &\quad -8.7840942359 \times 10^{-3} t^3 \end{aligned} \tag{44}$$
$$\begin{aligned} \omega_4(t) &= -3.8600866221 \times 10^{-22} t^2 \\ &\quad +1.7208906314 \times 10^{-3} t^3. \end{aligned} \tag{45}$$

Then, the analytical approximate solution is

$$\begin{aligned} u_{HSM}(x,t) &= \nu(x,t) + s(x,t) \\ &= \sum_{n=1}^{4} \omega_n(t) \sin(\frac{n\pi}{a} x) + t^3(1 + x(e-1)), \end{aligned}$$

where the $\omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t)$ are given in (42)-(45).

Figure 2, Figure 3 and Figure 4 show the absolute errors when $\alpha = 1.85$ at the final time $T = 1$. As is shown from Figure 2, 3 and 4, with the increase of N and fixed $M = 4$, the maximum absolute error is getting smaller and smaller. Besides, the maximum absolute error in Figure 2 is 3.55×10^{-3} with $N = 4$, while in Figure 3, it is 9.69×10^{-4} with $N = 8$ and in Figure 4, it is 2.53×10^{-4} with $N = 16$.

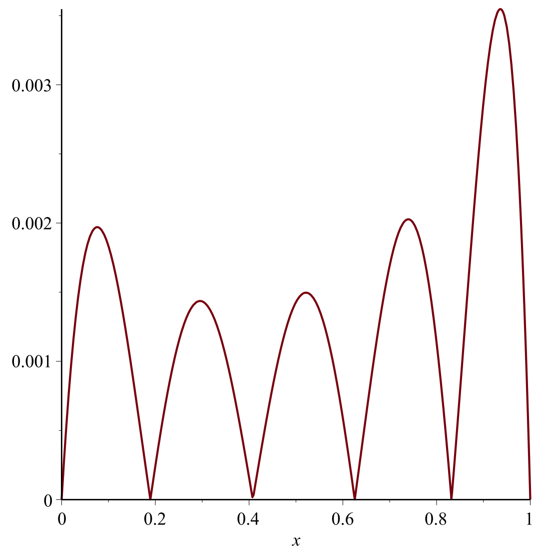

Fig. 2. Absolute errors for Example 2 when $\alpha = 1.85$ at $M = 4, N = 4$

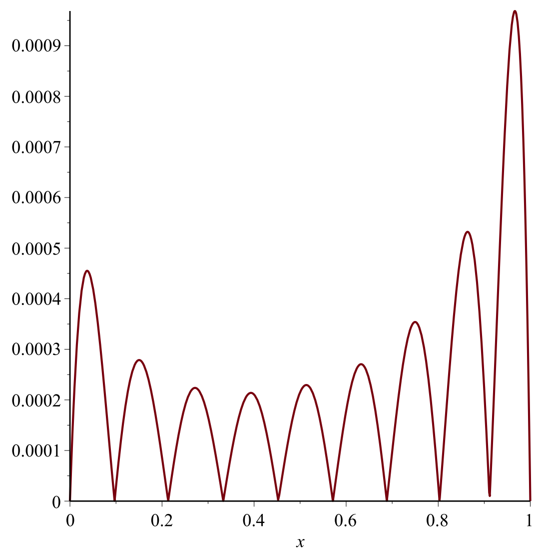

Fig. 3. Absolute errors for Example 2 when $\alpha = 1.85$ at $M = 4, N = 8$

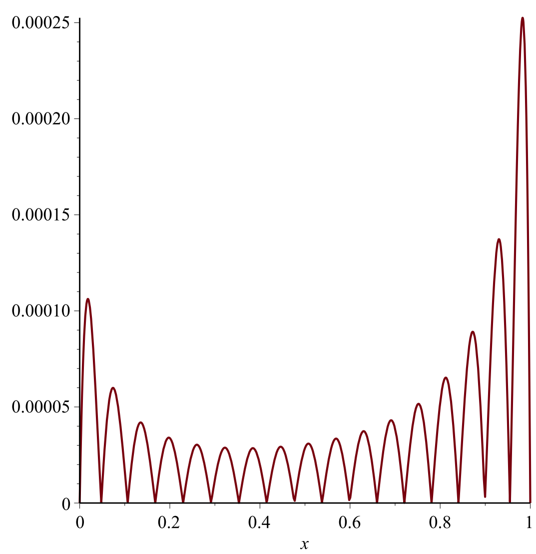

Fig. 4. Absolute errors for Example 2 when $\alpha = 1.85$ at $M = 4, N = 16$

Table 2 shows the maximum absolute errors when $\alpha = 1.85$ with $T = 1$. Liu et al. [31] considered this problem by the finite difference method. The maximum absolute errors in this paper are smaller than that in [31] with the same N and the value of M is also smaller than that in [31]. While the maximum absolute errors between [18] and this paper are about the same, the value of M of this paper is smaller than that in [18]. Therefore, the hybrid series method in this paper is more simple and convenient.

TABLE II
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 2 WITH $\alpha = 1.85$

Present method			[18], Example 3		[31], Example 2	
M	N	E_{max}	$M=N$	E_{max}	$M=N$	E_{max}
	4	3.55×10^{-3}	4	3.55×10^{-3}	4	1.09×10^{-1}
4	8	9.69×10^{-4}	8	9.69×10^{-4}	8	2.76×10^{-2}
	16	2.53×10^{-4}	16	2.53×10^{-4}	16	6.72×10^{-3}

Example 3. Consider the linear TFPDE:

$$D_t^{\alpha}u(x,t) + D_t^{\alpha_1}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \tag{46}$$

where the initial conditions are

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \tag{47}$$

and the boundary conditions are

$$u(0,t) = u_{exact}(0,t), \quad u(1,t) = u_{exact}(1,t). \tag{48}$$

Here

$$f(x,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{2-\alpha_1}}{\Gamma(3-\alpha_1)}\right)exp\left(\frac{-(x-0.2)^2}{0.01}\right) - t^2\left(\frac{exp\left(\frac{-(x-0.2)^2}{0.01}\right)(2x-0.4)^2}{0.01^2} - 2\frac{exp\left(\frac{-(x-0.2)^2}{0.01}\right)}{0.01}\right),$$

corresponds to the exact solution $u_{excat}(x,t) = exp\left(\frac{-(x-0.2)^2}{0.01}\right)t^2$.

Consider the substitution

$$u(x,t) = \nu(x,t) + s(x,t) = \nu(x,t) + t^2[exp(\frac{-0.2^2}{0.01}) + x(exp(\frac{-0.8^2}{0.01}) - exp(\frac{-0.2^2}{0.01}))],$$

then, we get the following TFPDE:

$$D_t^{\alpha}\nu(x,t) + D_t^{\alpha_1}\nu(x,t) - \frac{\partial^2 \nu(x,t)}{\partial x^2} = f(x,t) - (D_t^{\alpha}s(x,t) + D_t^{\alpha_1}s(x,t)) = \Theta(x,t),$$

where homogeneous conditions are

$$\nu(x,0) = \frac{\partial \nu(x,0)}{\partial t} = 0, \quad \nu(0,t) = \nu(1,t) = 0.$$

Then, we can obtain the following transformation:

$$D_t^{\alpha}\omega_n(t) + D_t^{\alpha_1}\omega_n(t) + (n\pi)^2\omega_n(t) = \phi_n(t), \quad \omega_n(0) = \frac{d\omega_n(0)}{dt} = 0. \tag{49}$$

Therefore, we can get the following solutions when $M = 3$ and $N = 16$

$$\omega_1(t) = 1.9168033620976242143 \times 10^{-1}t^2 \tag{50}$$

$$\omega_2(t) = 2.9973294501796186408 \times 10^{-1}t^2 \tag{51}$$

$$\omega_3(t) = 2.6627492675068672567 \times 10^{-1}t^2 \tag{52}$$

$$\vdots \tag{53}$$

Then, the analytical approximate solution is

$$u_{HSM}(x,t) = \nu(x,t) + s(x,t) = \sum_{n=1}^{16}\omega_n(t)\sin(\frac{n\pi}{a}x) + t^2[exp(\frac{-0.2^2}{0.01}) + x(exp(\frac{-0.8^2}{0.01}) - exp(\frac{-0.2^2}{0.01}))],$$

where the $\omega_1(t), \omega_2(t), \omega_3(t), \ldots, \omega_{16}(t)$ are given in (50)-(53) when $\alpha = 1.9$, $\alpha_1 = 1.6$.

Figure 5, Figure 6 and Figure 7 show the absolute errors when $\alpha = 1.9$, $\alpha_1 = 1.6$ at the final time $T = 1$. As is shown from Figure 5, 6 and 7, with the increase of N and fixed $M = 3$, the maximum absolute error is getting smaller and smaller. In addition, the maximum absolute error in Figure 5 is 2.34×10^{-3} with $N = 16$, while in Figure 6, it is 5.82×10^{-4} with $N = 32$ and in Figure 7, it is 1.51×10^{-4} with $N = 64$.

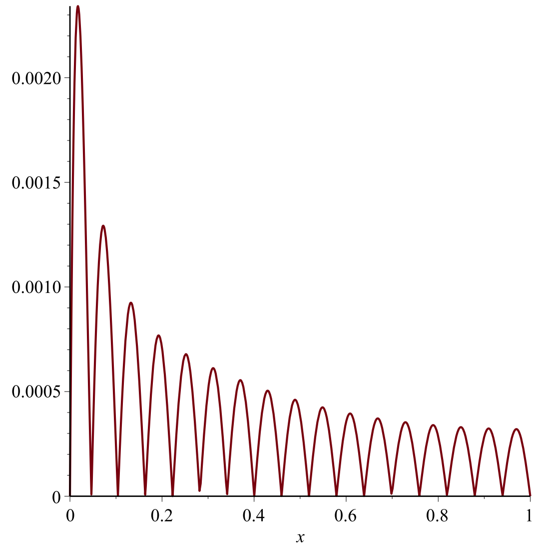

Fig. 5. Absolute errors for Example 3 when $\alpha = 1.9$, $\alpha_1 = 1.6$ at $M = 3, N = 16$

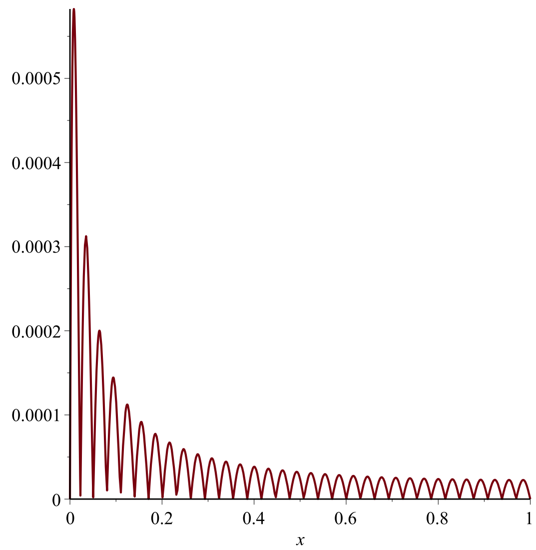

Fig. 6. Absolute errors for Example 3 when $\alpha = 1.9$, $\alpha_1 = 1.6$ at $M = 3, N = 32$

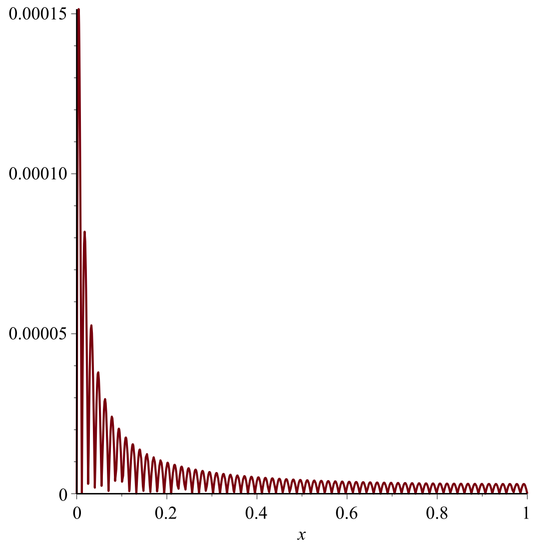

Fig. 7. Absolute errors for Example 3 when $\alpha = 1.9$, $\alpha_1 = 1.6$ at $M = 3, N = 64$

Table 3 shows the maximum absolute errors when $\alpha = 1.9$, $\alpha_1 = 1.6$ with $T = 1$. The maximum absolute errors in this paper are smaller than that in [30] with the smaller M. The maximum absolute errors between [18] and this paper are similar, but the value of M is smaller than that in [18].

TABLE III
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 3 WITH $\alpha = 1.9$, $\alpha_1 = 1.6$

Present method			[18], Example 4			[30], Test problem 1	
M	N	E_{max}	M	N	E_{max}	τ	E_{max}
	16	2.34×10^{-3}		16	2.33×10^{-3}	1/16	6.87×10^{-3}
3	32	5.82×10^{-4}	10	32	5.78×10^{-4}	1/32	6.56×10^{-4}
	64	1.51×10^{-4}		64	1.51×10^{-4}	1/64	3.00×10^{-4}

Example 4. Consider the following fractional modified anomalous sub-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\left(D_t^{1-\alpha} + D_t^{1-\beta}\right)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$0 \le x \le 1, \quad 0 \le t \le T, \tag{54}$$

where the initial condition is

$$u(x,0) = 0, \tag{55}$$

and the boundary conditions are

$$u(0,t) = t^{1+\alpha} + t^{1+\beta}, \quad u(1,t) = e(t^{1+\alpha} + t^{1+\beta}). \tag{56}$$

Here

$$f(x,t) = e^x[(1+\alpha)t^\alpha - \frac{\Gamma(2+\alpha)}{\Gamma(1+2\alpha)}t^{2\alpha}]$$
$$+e^x[(1+\beta)t^\beta - \frac{\Gamma(2+\beta)}{\Gamma(1+2\beta)}t^{2\beta}]$$

corresponds to the exact solution $u_{excat}(x,t) = e^x(t^{1+\alpha} + t^{1+\beta})$.

Consider the substitution

$$\begin{aligned} u(x,t) &= \nu(x,t) + s(x,t) \\ &= \nu(x,t) + (t^{1+\alpha} + t^{1+\beta})[1 + x(e-1)], \end{aligned}$$

then, we get the following TFPDE:

$$\frac{\partial \nu(x,t)}{\partial t} - \frac{1}{2}\left(D_t^{1-\alpha} + D_t^{1-\beta}\right)\frac{\partial^2 \nu(x,t)}{\partial x^2}$$
$$= f(x,t) - \frac{\partial s(x,t)}{\partial t} = \Theta(x,t),$$

where homogeneous conditions are

$$\nu(x,0) = 0, \quad \nu(0,t) = \nu(1,t) = 0.$$

Then, we can obtain the following transformation:

$$\frac{d\omega_n(t)}{dt} + \frac{1}{2}(n\pi)^2\left(D_t^{1-\alpha} + D_t^{1-\beta}\right)\omega_n(t) = \phi_n(t),$$
$$\omega_n(0) = 0. \tag{57}$$

Therefore, we can get the following solutions when $M = 6$ and $N = 16$

$$\begin{aligned} \omega_1(t) &= -1.77171745524417 \times 10^{-1}t - 5.85403685279693 \times 10^{-1}t^2 \\ &+7.53444405578777 \times 10^{-1}t^3 - 6.29457764700512 \times 10^{-1}t^4 \\ &+2.0741446256227 \times 10^{-1}t^5 \end{aligned} \tag{58}$$

$$\begin{aligned} \omega_2(t) &= 1.13854263641094 \times 10^{-2}t + 3.42950774703081 \times 10^{-2}t^2 \\ &-4.27985828874540 \times 10^{-2}t^3 + 0.355147172609665 \times 10^{-2}t^4 \\ &-0.116789394621282 \times 10^{-2}t^5 \end{aligned} \tag{59}$$

$$\vdots \tag{60}$$

Then, the analytical approximate solution is

$$u_{HSM}(x,t) = \nu(x,t) + s(x,t)$$
$$= \sum_{n=1}^{16} \omega_n(t)\sin(\frac{n\pi}{a}x) + (t^{1+\alpha} + t^{1+\beta})[1 + x(e-1)],$$

where the $\omega_1(t), \omega_2(t), \ldots, \omega_{16}(t)$ are given in (58)-(60) when $\alpha = 0.5$, $\beta = 0.2$.

Figure 8 shows the absolute errors when $\alpha = 0.5$, $\beta = 0.2$ at the final time $T = 1$. In Figure 8, the maximum absolute error is 4.27×10^{-3} with $M = 6$, $N = 16$.

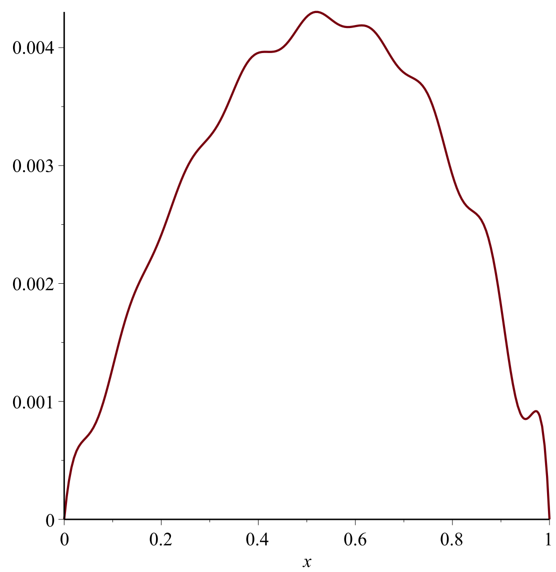

Fig. 8. Absolute errors for Example 4 when $\alpha = 0.5$, $\beta = 0.2$

Table 4 shows the absolute errors when $\alpha = 0.5$, $\beta = 0.2$ with $T = 1$. Liu et al. [32] solved this equation by using the implicit difference method. The absolute errors in this paper are smaller than that in [32] with the smaller M and N.

TABLE IV
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 4 WITH $\alpha = 0.5$, $\beta = 0.2$

	Present method		[32], Example 1	
x		Errors		Errors
0.1		1.29×10^{-3}		2.05×10^{-3}
0.3		3.25×10^{-3}		5.04×10^{-3}
0.5	$M = 6, N = 16$	4.26×10^{-3}	$h = 0.1, \tau = 0.01$	6.42×10^{-3}
0.7		3.79×10^{-3}		5.83×10^{-3}
0.9		1.81×10^{-3}		2.73×10^{-3}

Example 5. Let's consider the fractional sub-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \left(D_t^{\alpha} + D_t^{\beta}\right)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$0 \le x \le 1, \quad 0 \le t \le T, \tag{61}$$

where the initial condition is

$$u(x,0) = 0, \tag{62}$$

and the boundary conditions are

$$u(0,t) = 0, \quad u(1,t) = \sin(1)t^{3-\alpha-\beta}. \tag{63}$$

Here

$$f(x,t) = \sin(x)\Big[(3-\alpha-\beta)t^{2-\alpha-\beta} + \frac{\Gamma(4-\alpha-\beta)}{\Gamma(4-2\alpha-\beta)}t^{3-2\alpha-\beta} + \frac{\Gamma(4-\alpha-\beta)}{\Gamma(4-\alpha-2\beta)}t^{3-\alpha-2\beta}\Big],$$

corresponds to the exact solution $u_{excat}(x,t) = \sin(x)t^{3-\alpha-\beta}$.

Consider the following substitution

$$\begin{aligned} u(x,t) &= \nu(x,t) + s(x,t) \\ &= \nu(x,t) + x\sin(1)t^{3-\alpha-\beta}, \end{aligned}$$

then we can attain

$$\frac{\partial \nu(x,t)}{\partial t} - \left(D_t^{\alpha} + D_t^{\beta}\right)\frac{\partial^2 \nu(x,t)}{\partial x^2} = f(x,t) - \frac{\partial s(x,t)}{\partial t} = \Theta(x,t),$$

where homogeneous conditions are

$$\nu(x,0) = 0, \quad \nu(0,t) = \nu(1,t) = 0.$$

Then. we can obtain the following transformation:

$$\frac{d\omega_n(t)}{dt} + (n\pi)^2\left(D_t^{\alpha} + D_t^{\beta}\right)\omega_n(t) = \phi_n(t), \quad \omega_n(0) = 0. \tag{64}$$

Therefore, we can get the following solutions when $M = 5$ and $N = 16$

$$\begin{aligned} \omega_1(t) &= -9.86189594776539 \times 10^{-4}t + 1.78854950515071 \times 10^{-2}t^2 \\ &+5.21741627279073 \times 10^{-2}t^3 - 8.68649516474632 \times 10^{-3}t^4 \quad (65) \\ \omega_2(t) &= 1.131779350916144 \times 10^{-4}t - 2.05519921599097 \times 10^{-3}t^2 \\ &-6.02879462558576 \times 10^{-3}t^3 + 1.01149878666679 \times 10^{-3}t^4 \quad (66) \\ \omega_3(t) &= -3.29653470637822 \times 10^{-5}t + 5.99569550815599 \times 10^{-4}t^2 \\ &+1.76239043192235 \times 10^{-3}t^3 - 2.96364658323121 \times 10^{-4}t^4 \quad (67) \\ &\vdots \quad (68) \end{aligned}$$

Then, the analytical approximate solution is

$$u_{HSM}(x,t) = \nu(x,t) + s(x,t) = \sum_{n=1}^{16}\omega_n(t)\sin(\frac{n\pi}{a}x) + x\sin(1)t^{3-\alpha-\beta},$$

where the $\omega_1(t), \omega_2(t), \omega_3(t), \ldots, \omega_{16}(t)$ are given in (65)-(68) when $\alpha = 0.35$, $\beta = 0.05$.

Figure 9-11 show the absolute errors when $\alpha = 0.35$, $\beta = 0.05$ at the final time $T = 1$. As is shown in Figure 9-11, the maximum absolute error is getting smaller and smaller with the increase of N and fixed $M = 5$. Besides, the maximum absolute error in Figure 9 is 7.74×10^{-5} with $N = 16$, while in Figure 10, it is 2.37×10^{-5} with $N = 30$ and in Figure 11, it is 1.24×10^{-5} with $N = 50$.

Table 5 shows the maximum absolute errors when $\alpha = 0.35$, $\beta = 0.05$ with $T = 1$. Mohebbi et al. [22] solved the modified anomalous fractional sub-diffusion equation by the compact finite difference and Fourier analysis. Wang et al. [33] solved the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation in using the compact finite difference schemes. The maximum absolute errors in this paper are smaller than that in [22] and [33]. The maximum absolute errors for Example 5 in this paper are slightly larger than that in [18].

TABLE V
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 5 WITH $\alpha = 1.9$, $\alpha_1 = 1.6$

Present method			[18], Example 8			[22], Test problem 1		[33], Example 5.1	
M	N	E_{max}	M	N	E_{max}	τ	E_{max}	τ	E_{max}
	16	7.74×10^{-5}	5	30	2.20×10^{-5}	1/10	1.74×10^{-2}	1/5	8.21×10^{-4}
5	30	2.37×10^{-5}	6	50	8.10×10^{-6}	1/20	9.00×10^{-3}	1/10	1.99×10^{-4}
	50	1.24×10^{-5}	8	40	1.30×10^{-5}	1/40	4.58×10^{-3}	1/20	5.01×10^{-5}

Example 6. Consider the following fractional modified anomalous fractional sub-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \left(D_t^{1-\alpha} + D_t^{1-\beta}\right)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$0 \le x \le 1, \quad 0 \le t \le T, \tag{69}$$

where the initial condition is

$$u(x,0) = 0, \tag{70}$$

and the boundary conditions are

$$u(0,t) = t^{1+\alpha+\beta}, \quad u(1,t) = et^{1+\alpha+\beta}. \tag{71}$$

Here

$$f(x,t) = e^x[(1+\alpha)t^{\alpha} - \frac{\Gamma(2+\alpha+\beta)}{\Gamma(1+2\alpha+\beta)}t^{2\alpha+\beta} - \frac{\Gamma(2+\alpha+\beta)}{\Gamma(1+\alpha+2\beta)}t^{\alpha+2\beta}]$$

corresponds to the exact solution $u_{excat}(x,t) = e^x(t^{1+\alpha+\beta})$.

Consider the substitution

$$\begin{aligned} u(x,t) &= \nu(x,t) + s(x,t) \\ &= \nu(x,t) + (t^{1+\alpha+\beta})[1 + x(e-1)], \end{aligned}$$

then we obtain the following TFPDE:

$$\frac{\partial \nu(x,t)}{\partial t} - \left(D_t^{1-\alpha} + D_t^{1-\beta}\right)\frac{\partial^2 \nu(x,t)}{\partial x^2} = f(x,t) - \frac{\partial s(x,t)}{\partial t} = \Theta(x,t),$$

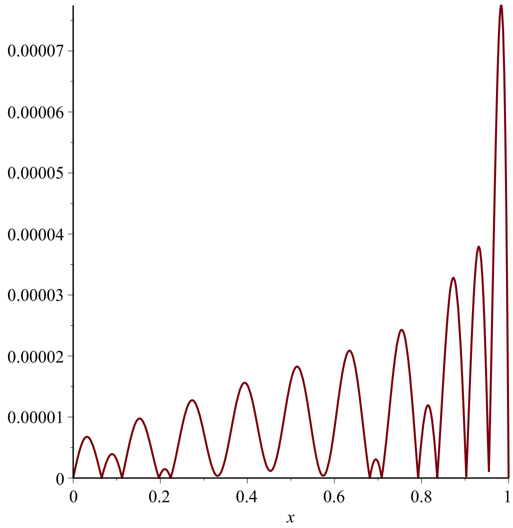

Fig. 9. Absolute errors for Example 5 when $\alpha = 0.35$, $\beta = 0.05$ at $M = 5, N = 16$

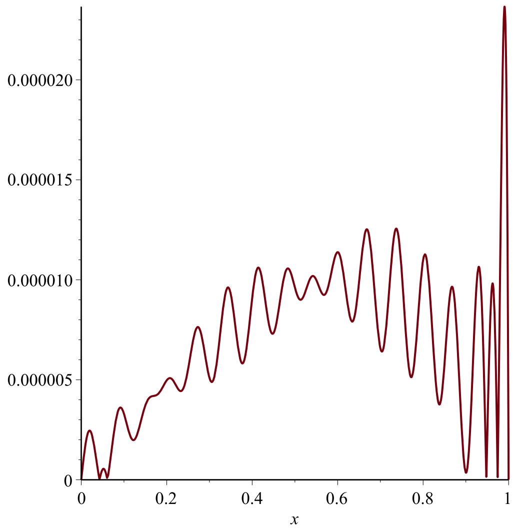

Fig. 10. Absolute errors for Example 5 when $\alpha = 0.35$, $\beta = 0.05$ at $M = 5, N = 30$

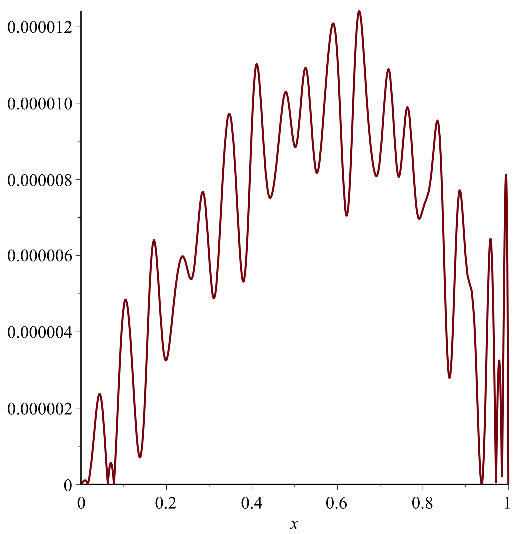

Fig. 11. Absolute errors for Example 5 when $\alpha = 0.35$, $\beta = 0.05$ at $M = 5, N = 50$

where homogeneous conditions are

$$\nu(x, 0) = 0, \quad \nu(0, t) = \nu(1, t) = 0.$$

Then, we can obtain the following transformation:

$$\frac{d\omega_n(t)}{dt} + (n\pi)^2 \Big(D_t^{1-\alpha} + D_t^{1-\beta}\Big)\omega_n(t) = \phi_n(t), \\ \omega_n(0) = 0. \tag{72}$$

Therefore, we can get the following solutions when $M = 10$ and $N = 10$

$$\begin{aligned} \omega_1(t) = {} & 7.68535704501287 \times 10^{-2} t - 1.538638423975210 t^2 \\ & + 7.79719926193383 t^3 - 28.84450237802211 t^4 \\ & + 67.57623646989848 t^5 - 98.58606620128812 t^6 \\ & + 86.73639399791612 t^7 - 42.07758068090534 t^8 \\ & + 8.63864943482636 t^9 \end{aligned} \tag{73}$$

$$\begin{aligned} \omega_2(t) = {} & -5.30877558421330 \times 10^{-3} t + 1.00911050742783 \times 10^{-1} t^2 \\ & - 5.18570503133282 \times 10^{-1} t^3 + 1.92643111896331 t^4 \\ & - 4.52221136665213 t^5 + 6.60539987194772 t^6 \\ & - 5.81647510036073 t^7 + 2.82361615999036 t^8 \\ & - 5.80031812461746 \times 10^{-1} t^9 \end{aligned} \tag{74}$$

$$\vdots \tag{75}$$

Then, the analytical approximate solution is

$$\begin{aligned} u_{HSM}(x, t) &= \nu(x, t) + s(x, t) \\ &= \sum_{n=1}^{10} \omega_n(t) \sin(\frac{n\pi}{a} x) + (t^{1+\alpha+\beta})[1 + x(e - 1)], \end{aligned}$$

where the $\omega_1(t), \omega_2(t), \ldots, \omega_{10}(t)$ are given in (73)-(75) when $\alpha = 0.1$, $\beta = 0.3$.

Figure 12-14 show the absolute errors when $\alpha = 0.1$, $\beta = 0.3$ at the final time $T = 1$. As is shown in Figure 12-14, the maximum absolute error is getting smaller and smaller with the increase of N and the fixed $M = 10$. Moreover, the maximum absolute error in Figure 12 is 3.68×10^{-3} with $N = 10$, while in Figure 13, it is 3.59×10^{-3} with $N = 20$ and in Figure 14, it is 3.57×10^{-3} with $N = 30$.

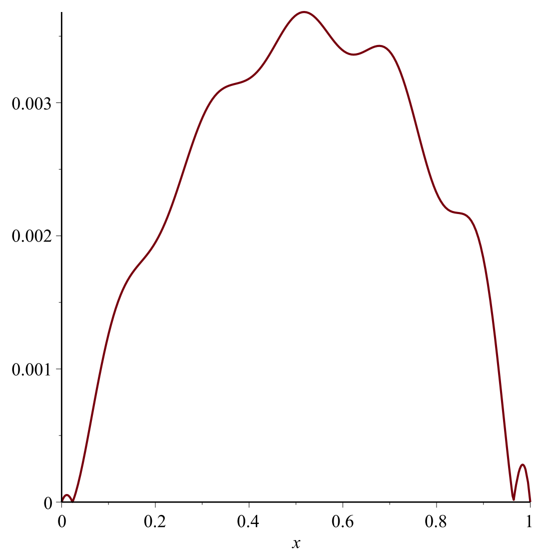

Fig. 12. Absolute errors for Example 6 when $\alpha = 0.1$, $\beta = 0.3$ at $M = 10, N = 10$

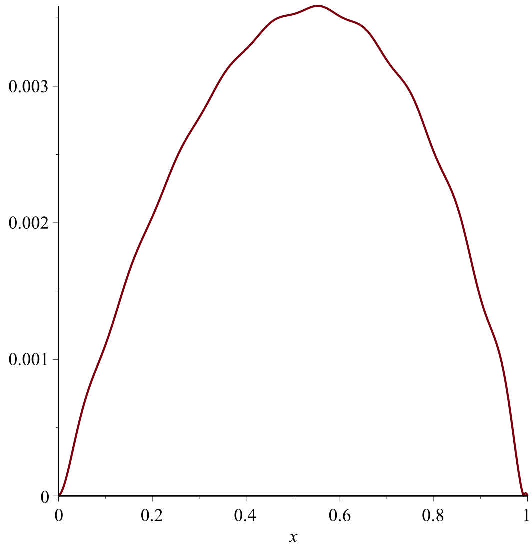

Fig. 13. Absolute errors for Example 6 when $\alpha = 0.1$, $\beta = 0.3$ at $M = 10, N = 20$

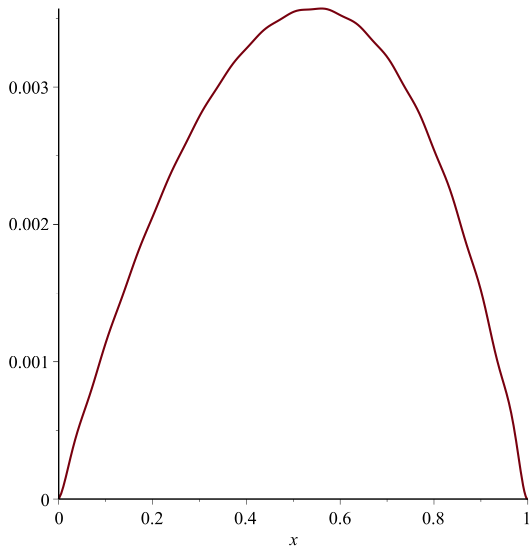

Fig. 14. Absolute errors for Example 6 when $\alpha = 0.1$, $\beta = 0.3$ at $M = 10, N = 30$

Table 6 shows the absolute errors when $\alpha = 0.1$, $\beta = 0.3$ with $T = 1$. Dehghan et al. [34] solved the time fractional modified anomalous sub-diffusion equation by the Legendre spectral element method. The absolute errors in this paper are larger than that in [18] and [34]. However, in [18], the values of N are much bigger, which represents that this equation is difficult and complex to solve.

TABLE VI
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 6 WITH $\alpha = 0.1, \beta = 0.3$

Present method			[18], Example 6		[34], Test problem 1	
M	N	E_{max}	N	E_{max}	$1/\tau$	E_{max}
	10	3.68×10^{-3}	500	2.68×10^{-7}	1/100	1.79×10^{-4}
10	20	3.59×10^{-3}	1000	6.70×10^{-8}	1/200	1.13×10^{-4}
	30	3.57×10^{-3}	2000	1.68×10^{-8}	1/400	6.74×10^{-5}

Example 7. Let us consider the following TFPDE:

$$\frac{\partial u(x,t)}{\partial t} = D_t^{\alpha}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t),$$
$$0 \le x \le 1, \quad 0 \le t \le T, \tag{76}$$

where the initial condition is

$$u(x,0) = 0, \tag{77}$$

and the boundary conditions are

$$u(0,t) = 0, \quad u(1,t) = 0. \tag{78}$$

Here

$$f(x,t) = \left[2t + \frac{8\pi^2}{\Gamma(3-\alpha)}t^{2-\alpha}\right]\sin(2\pi x)$$

corresponds to the exact solution $u_{excat}(x,t) = t^2\sin(2\pi x)$.

Consider the substitution

$$u(x,t) = \nu(x,t) + s(x,t) = \nu(x,t) + 0 = \nu(x,t)$$

then we obtain the following TFPDE:

$$\frac{\partial \nu(x,t)}{\partial t} - D_t^{\alpha}\left[\frac{\partial^2 \nu(x,t)}{\partial x^2}\right] = f(x,t) - \frac{\partial s(x,t)}{\partial t} = \Theta(x,t),$$

where homogeneous conditions are

$$\nu(x,0) = 0, \quad \nu(0,t) = \nu(1,t) = 0.$$

Then, we can obtain the following transformation:

$$\frac{d\omega_n(t)}{dt} + (n\pi)^2 D_t^{\alpha}\omega_n(t) = \phi_n(t), \quad \omega_n(0) = 0. \tag{79}$$

Therefore, we can attain

$$u_{HSM}(x,t) = (2.179485940561139 \times 10^{-20}t$$
$$+1.000000000000000t^2)sin(2\pi x).$$

Table 7 shows the absolute errors when $\alpha = 0.5$ with $T = 1$. Cui [35] solved this equation by the compact finite difference method. The absolute errors in this paper are much smaller than that in [33] and [35]. The maximum absolute error when $M = 3$ and $N = 2$ is 2.18×10^{-20}, which is about equal to zero.

TABLE VII
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 7 WITH $\alpha = 0.5$

Present method			[33], Example 5.2		[35], Example 2		
M	N	E_{max}	τ	E_{max}	h	τ	E_{max}
			1/5	1.04×10^{-2}	1/4	1/16	1.92×10^{-1}
3	2	2.18×10^{-20}	1/20	6.56×10^{-4}	1/8	1/256	1.14×10^{-2}
			1/80	4.14×10^{-5}	1/16	1/4096	7.10×10^{-4}

Example 8. Consider the following fractional modified anomalous fractional sub-diffusion equation:

$$D_t^{\alpha}u(x,t) + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$0 \le x \le 1, \quad 0 \le t \le T, \tag{80}$$

where the initial condition is

$$u(x,0) = 0, \tag{81}$$

and the boundary conditions are

$$u(0,t) = t^2, \quad u(1,t) = et^2. \tag{82}$$

Here

$$f(x,t) = e^x[2t - t^2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}]$$

corresponds to the exact solution $u_{excat}(x,t) = e^x t^2$.

Consider the substitution

$$\begin{aligned} u(x,t) &= \nu(x,t) + s(x,t) \\ &= \nu(x,t) + t^2[1 + x(e-1)], \end{aligned}$$

then we can get the TFPDE:

$$\begin{aligned} &D_t^\alpha \nu(x,t) + \frac{\partial \nu(x,t)}{\partial t} - \frac{\partial^2 \nu(x,t)}{\partial x^2} \\ &\quad = f(x,t) - \frac{\partial s(x,t)}{\partial t} = \Theta(x,t), \end{aligned}$$

where homogeneous conditions are

$$\nu(x,0) = 0, \quad \nu(0,t) = \nu(1,t) = 0.$$

Then, we can obtain the following transformation:

$$\begin{aligned} D_t^\alpha \omega_n(t) + \tfrac{d\omega_n(t)}{dt} + (n\pi)^2 \omega_n(t) = \phi_n(t), \\ \omega_n(0) = 0. \end{aligned} \tag{83}$$

Therefore, we can get the following solutions when $M = 3$ and $N = 24$

$$\begin{aligned} \omega_1(t) &= 3.7622203509136 \times 10^{-20} t \\ &\quad -2.1777533421500 \times 10^{-1} t^2 && (84) \\ \omega_2(t) &= -2.2991962724973 \times 10^{-21} t \\ &\quad +1.3512042357841 \times 10^{-2} t^2 && (85) \\ \omega_3(t) &= 6.3928185209204 \times 10^{-21} t \\ &\quad -8.7840942359312 \times 10^{-3} t^2 && (86) \\ &\vdots && (87) \end{aligned}$$

Then, the analytical approximate solution is

$$\begin{aligned} &u_{HSM}(x,t) = \nu(x,t) + s(x,t) \\ &= \sum_{n=1}^{24} \omega_n(t) \sin(\frac{n\pi}{a}x) + t^2[1 + x(e-1)], \end{aligned}$$

where the $\omega_1(t), \omega_2(t), \omega_3(t), \ldots, \omega_{24}(t)$ are given in (84)-(87) when $\alpha = 0.75$.

Figure 15-16 show the absolute errors when $\alpha = 0.75$ at the final time $T = 1$. As is shown in Figure 15-16, the maximum absolute error is getting smaller and smaller with the increase of N and the fixed $M = 3$. What's more, the maximum absolute error in Figure 15 is 4.43×10^{-4} with $N = 12$, while in Figure 16, it is 1.14×10^{-4} with $N = 24$.

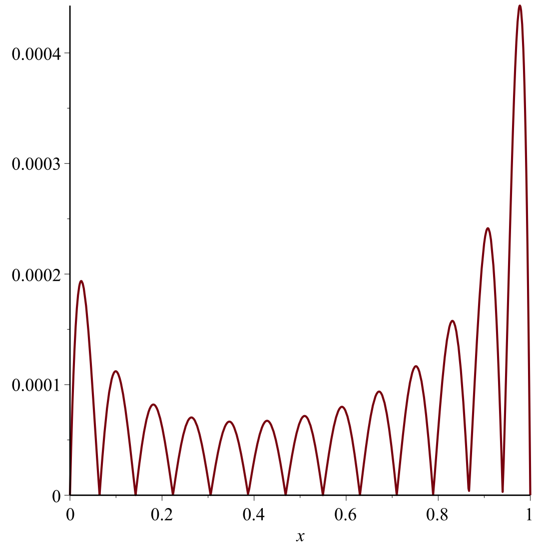

Fig. 15. Absolute errors for Example 8 when $\alpha = 0.75$ at $M = 3, N = 12$

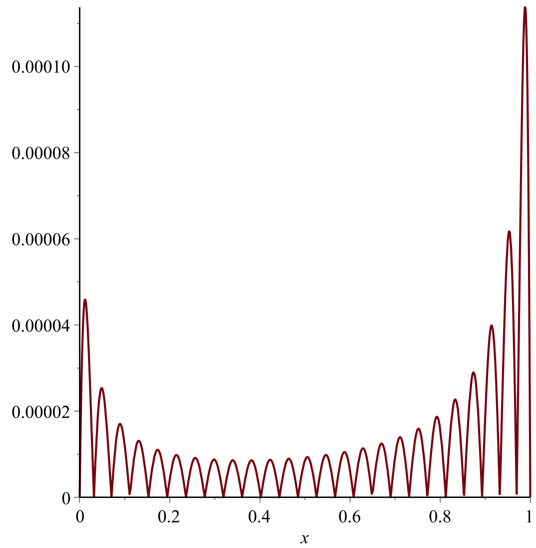

Fig. 16. Absolute errors for Example 8 when $\alpha = 0.75$ at $M = 3, N = 24$

Table 8 shows the absolute errors when $\alpha = 0.75$ with $T = 1$ when $M = 3$ and $N = 24$. Sarwara et al. [36] extended the optimal homotopy asymptotic method (OHAM) to solve the two-term fractional-order wave-diffusion equations. The absolute errors in this paper are smaller than that in [36] when $M = 3$ and $N = 24$. From Table 8, we can see that the hybrid series method in this paper is effective.

TABLE VIII
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 8 WITH $\alpha = 0.75$

	Present method	[36], Example 1
x	Errors	Errors
0.3	8.68×10^{-6}	2.32×10^{-5}
0.6	6.13×10^{-6}	3.13×10^{-5}
0.9	1.87×10^{-5}	4.22×10^{-5}

V. CONCLUSION

In this paper, the non-homogeneous time-fractional differential equations are solved by a new kind of hybrid series method with respect to the Fourier series and power series. The time-fractional partial differential equations are transformed into the time-fractional ordinary differential equations by the Fourier series. Next we obtain the analytical approximate solutions by using the power series with the polynomial least squares method. Besides, the analytical approximate solutions of the examples above are received, which are presented in the form of data and pictures. The results show that this hybrid series method is an effective and reliable method for solving this class of non-homogeneous time-fractional partial differential equations.

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