Parameter Estimation for Discretely Observed Cox-Ingersoll-Ross Model with Small Lévy Noises

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Abstract—**This paper is concerned with the parameter estimation problem for Cox-Ingersoll-Ross model with small Lévy noises from discrete observations. The least squares method is used to obtain the parameter estimators and the explicit formula of the estimation error is given. The consistency of the estimators are derived when a small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. The asymptotic distribution of the estimation error is studied. The simulation is made to verify the effectiveness of the least squares estimators.**

Index Terms—**Least squares estimator, Lévy noises, discrete observations, consistency.**

I. Introduction

Itô stochastic differential equations are important tools for studying random phenomena and are widely used in the modeling of stochastic phenomena in the fields of physics, chemistry, medicine and finance([3], [12], [17]). However, part or all of the parameters in stochastic model are always unknown. In the past few decades, some popular methods have been put forward to estimate the parameters in Itô stochastic differential equations, such as maximum likelihood estimation([1], [20], [21]), least squares estimation([4], [16], [18]) and Bayes estimation([7]–[9], [11]). But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. Lévy noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for stochastic differential equations with small Lévy noises. Recently, a number of literatures have been devoted to the parameter estimation for the models driven by small Lévy noises. When the coefficient of the Lévy jump term is constant, drift parameter estimation has been investigated by some authors [13], [14].

The Cox-Ingersoll-Ross model([5], [6]), hereafter the CIR model, which was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model ([19]), describes the evolution of interest rates. It is known that parameter estimation for CIR model driven by Brownian motion has been well developed based on discrete observations([2], [22]). However, some features of the financial processes cannot be captured by the CIR model, for example, discontinuous sample paths and heavy tailed properties. Therefore, it is natural to replace the Brownian motion by the Lévy process. Recently, the parameter estimation problems for CIR model driven by small Lévy noises have been studied by some authors. For example, Ma and Yang([15]) established the central limit theorems, the deviation inequality and the moderate deviations for least squares estimators of parameters in the CIR type model driven by α-stable noises; Li and Ma([10]) derived the consistency and central limit theorems of the conditional least squares estimators in a stable Cox-Ingersoll-Ross model. But, the explicit formula of the estimators and the estimation error have not been given in these papers.

In this paper, we consider the parameter estimation problem for CIR model with small Lévy noises from discrete observations. The decomposition of the Lévy process is different from that in ([10], [15]), so the methods used to prove the asymptotic property of the estimators are different. The process is discreted based on Euler-Maruyama scheme, the least squares method is used to obtain the explicit formula of the estimators and the estimation errors are given as well. when the small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously, the consistency of the least squares estimators are proved by applying the Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. Finally, the simulation result is provided to verify the effectiveness of the obtained estimators.

This paper is organized as follows. In Section 2, the CIR model driven by small Lévy noises is introduced, the contrast function is given and the explicit formula of the least squares estimators are obtained. In Section 3, the estimation errors are derived, the consistency of the estimators are proved and the asymptotic distribution of the estimation error are discussed. In Section 4, the results are extended to semimartingale noises. In Section 5, some simulation results are made. The conclusion is given in Section 6.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $(\{\mathscr{F}_t\}_{t\geq 0})$. Let $(L_t, t \geq 0)$ be an $(\{\mathscr{F}_t\})$-adapted Lévy noises with decomposition

$$L_t = B_t + \int_0^t \int_{|z|>1} zN(ds,dz) + \int_0^t \int_{|z|\leq 1} z\widetilde{N}(ds,dz), \tag{1}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion, $N(ds,dz)$ is a Poisson random measure independent of $(B_t, t \geq 0)$ with characteristic measure $dt\nu(dz)$, and $\widetilde{N}(ds,dz) = N(ds,dz) - \nu(dz)$ is a martingale measure. We assume that $\nu(dz)$ is a Lévy measure on $\mathbb{R}\backslash 0$ satisfying $\int(|z|^2 \wedge 1)\nu(dz) < \infty$.

In this paper, we study the parameter estimation for CIR model with small Lévy noises described by the following

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stochastic differential equation:

$$\begin{cases} dX_t = (\alpha - \beta X_t)dt + \varepsilon\sigma\sqrt{X_t}dL_t, \quad t \in [0,1] \\ X_0 = x_0, \end{cases} \tag{2}$$

where α and β are unknown parameters and σ is known constant. Without loss of generality, it is assumed that $\varepsilon \in (0,1]$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha,\beta) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1}|^2}{\varepsilon^2\sigma^2 X_{t_{i-1}}\Delta t_{i-1}}, \tag{3}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the estimators

$$\begin{cases} \widehat{\alpha}_{n,\varepsilon} = \dfrac{n\sum_{i=1}^n X_{t_i} - \sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}}\sum_{i=1}^n X_{t_{i-1}}}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})} \\ \qquad + \dfrac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})} \\ \widehat{\beta}_{n,\varepsilon} = \dfrac{n^2 - n\sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}}}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})} \\ \qquad + \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})}. \end{cases} \tag{4}$$

Before giving the main results, we introduce some assumptions below.

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameter:

$$dX_t^0 = (\alpha - \beta X_t^0)dt, \quad X_0^0 = x_0.$$

Assumption 1: α and β are positive true valves of the parameters and $\sigma > 0$.

Assumption 2: $\inf_{0 \leq t \leq 1}\{X_t\} > 0$.

Assumption 3: There exists $L > 0$ such that$|\alpha - \beta x| \leq L(1 + |x|)$.

Assumption 4: $\sup_t \mathbb{E}[|X_t|^p] < \infty$ and $\sup_t \mathbb{E}[\frac{1}{|X_t|^p}] < \infty$ for every $p \geq 1$.

In the next sections, the consistency of the least squares estimators are derived and the simulation is made to verify the effectiveness of the estimators.

III. Main Result and Proofs

In the following theorem, the consistency in probability of the least squares estimators are proved by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence.

Theorem 1: The least squares estimators $\widehat{\alpha}$ and $\widehat{\beta}$ are consistent in probability, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha, \quad \widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta.$$

Proof: By using the Euler-Maruyama scheme, from (2), we have

$$X_{t_i} - X_{t_{i-1}} = (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1} + \varepsilon\sigma\sqrt{X_{t_{i-1}}}(L_{t_i} - L_{t_{i-1}}). \tag{5}$$

Then, it is easy to see that

$$\begin{aligned} &\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \\ = \ & \alpha - \frac{1}{n}\beta\sum_{i=1}^n X_{t_{i-1}} + \varepsilon\sigma\sum_{i=1}^n \sqrt{X_{t_{i-1}}}(L_{t_i} - L_{t_{i-1}}), \end{aligned} \tag{6}$$

$$\begin{aligned} &\sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}} \\ = \ & n + \frac{1}{n}\alpha\sum_{i=1}^n \frac{1}{X_{t_{i-1}}} - \beta \\ + \ & \varepsilon\sigma\sum_{i=1}^n \frac{1}{\sqrt{X_{t_{i-1}}}}(L_{t_i} - L_{t_{i-1}}). \end{aligned} \tag{7}$$

Substituting (6) and (7) into the expression of $\widehat{\alpha}$, it follows that

$$\begin{aligned} &\widehat{\alpha}_{n,\varepsilon} - \alpha \\ = \ & \frac{\varepsilon\sigma\sum_{i=1}^n \sqrt{X_{t_{i-1}}}(L_{t_i} - L_{t_{i-1}})}{1 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}} \\ - \ & \frac{\varepsilon\sigma\sum_{i=1}^n \frac{1}{\sqrt{X_{t_{i-1}}}}(L_{t_i} - L_{t_{i-1}})\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{1 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}. \end{aligned}$$

Let $M_t^{n,\varepsilon} = X_{[nt]/n}$, in which $[nt]$ denotes the integer part of nt. We will prove that the sequence $\{M_t^{n,\varepsilon}\}$ converges to the deterministic process $\{X_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Observe that

$$X_t - X_t^0 = \int_0^t (X_s^0 - X_s)ds + \varepsilon\sigma\int_0^t \sqrt{X_s}dL_s. \tag{8}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &|X_t - X_t^0|^2 \\ \leq \ & 2|\int_0^t (X_s^0 - X_s)ds|^2 + 2\varepsilon^2\sigma^2|\int_0^t \sqrt{X_s}dL_s|^2 \\ \leq \ & 2t\int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2\sigma^2|\int_0^t \sqrt{X_s}dL_s|^2 \end{aligned}$$

.

According to the Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2\sigma^2 e^{2t^2}|\int_0^t \sqrt{X_s}dL_s|^2. \tag{9}$$

Then, it follows that

$$\sup_{0 \leq t \leq T}|X_t - X_t^0| \leq \sqrt{2}\varepsilon\sigma e^{T^2}\sup_{0 \leq t \leq T}|\int_0^t \sqrt{X_s}dL_s|. \tag{10}$$

Therefore, for each $T > 0$, it is easy to check that

$$\sup_{0 \leq t \leq T}|X_t - X_t^0| \xrightarrow{P} 0. \tag{11}$$

As $[nt]/n \to t$ when $n \to \infty$, we get that the sequence $\{M_t^{n,\varepsilon}\}$ converges to the deterministic process $\{X_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Next we will prove that $\sum_{i=1}^n \sqrt{X_{t_{i-1}}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \sqrt{X_s^0}dL_s$.

Note that

$$\sum_{i=1}^{n}\sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})=\int_0^1\sqrt{M_s^{n,\varepsilon}}dL_s. \quad (12)$$

Then, it is elementary to see that

$$\begin{aligned}
&|\int_0^1\sqrt{M_s^{n,\varepsilon}}dL_s-\int_0^1\sqrt{X_s^0}dL_s|\\
=&\ |\int_0^1(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})dB_s\\
+&\ \int_0^1\int_{|z|>1}(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})zN(ds,dz)\\
+&\ \int_0^1\int_{|z|\le 1}(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})z\widetilde{N}(ds,dz)|\\
\le&\ |\int_0^1(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})dB_s|\\
+&\ |\int_0^1\int_{|z|>1}(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})zN(ds,dz)|\\
+&\ |\int_0^1\int_{|z|\le 1}(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})z\widetilde{N}(ds,dz)|.
\end{aligned}$$

It can be easily to check that

$$\begin{aligned}
&|\int_0^1\int_{|z|>1}(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})zN(ds,dz)|\\
\le&\ \int_0^1\int_{|z|>1}|\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0}||z|N(ds,dz)\\
\le&\ \sup_{0\le s\le 1}|\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0}|\int_0^1\int_{|z|>1}|z|N(ds,dz)\\
\xrightarrow{P}&\ 0,
\end{aligned}$$

as $\varepsilon\to 0$ and $n\to\infty$.

By using the Markov inequality and dominated convergence, we have $|\int_0^1(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})dB_s|\xrightarrow{P}0$ and $|\int_0^1\int_{|z|\le 1}(\sqrt{M_s^{n,\varepsilon}}-\sqrt{X_s^0})z\widetilde{N}(ds,dz)|\xrightarrow{P}0$.

Thus, combining the previous results, it follows that

$$\sum_{i=1}^{n}\sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})\xrightarrow{P}\int_0^1\sqrt{X_s^0}dL_s. \quad (13)$$

Then, it is easy to see that $\sum_{i=1}^{n}\frac{1}{\sqrt{X_{t_{i-1}}}}(L_{t_i}-L_{t_{i-1}})\xrightarrow{P}\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s$.

Let

$$X_N=\inf_{0\le t_{i-1}\le 1}\{X_{t_{i-1}}\}, \quad (14)$$

and

$$X_M=\sup_{0\le t_{i-1}\le 1}\{X_{t_{i-1}}\}. \quad (15)$$

.

We make an assumption that $X_N\neq X_M$.

From (15), it follows that

$$\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\le X_M<\infty.$$

Therefore, when $\varepsilon\to 0$ and $n\to\infty$, we have

$$\varepsilon\sigma\sum_{i=1}^{n}\sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})\xrightarrow{P}0, \quad (16)$$

and

$$\varepsilon\sigma\sum_{i=1}^{n}\frac{1}{\sqrt{X_{t_{i-1}}}}(L_{t_i}-L_{t_{i-1}})\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\xrightarrow{P}0. \quad (17)$$

Finally, we will consider the boundedness of $1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}$.

It is obviously that

$$\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\ge 1, \quad (18)$$

which, under the assumption that $X_N\neq X_M$, implies

$$1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}<0. \quad (19)$$

From (14) and (15), it follows that

$$\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}<X_M\frac{1}{X_N}. \quad (20)$$

Then we have

$$\frac{1}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}<\frac{1}{1-X_M\frac{1}{X_N}}<\infty. \quad (21)$$

Combining the previous arguments, when $\varepsilon\to 0$ and $n\to\infty$, we have

$$\widehat{\alpha}_{n,\varepsilon}\xrightarrow{P}\alpha. \quad (22)$$

From (4) and (6), we obtain

$$\widehat{\beta}_{n,\varepsilon}-\beta=\frac{(\widehat{\alpha}_{n,\varepsilon}-\alpha)-\varepsilon\sigma\sum_{i=1}^{n}\sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}. \quad (23)$$

Since $\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\ge X_N>0$, we get that $\frac{1}{\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}\le\frac{1}{X_N}<\infty$.

Together with the results that $\widehat{\alpha}_{n,\varepsilon}-\alpha\xrightarrow{P}0$ and $\varepsilon\sigma\sum_{i=1}^{n}\sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})\xrightarrow{P}0$, it follows that

$$\widehat{\beta}_{n,\varepsilon}-\beta\xrightarrow{P}0, \quad (24)$$

as $\varepsilon\to 0$ and $n\to\infty$.

Therefore, $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are consistent in probability. The proof is complete. ∎

In the following theorem, the asymptotic distribution of the estimation error are discussed.

Theorem 2: Under the conditions $\varepsilon\to 0$, $n\to\infty$ and $n\varepsilon\to\infty$, it follows that

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha)\xrightarrow{P}\frac{\sigma\int_0^1\sqrt{X_s^0}dL_s-\sigma\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s\int_0^1X_s^0ds}{1-\int_0^1X_s^0ds\int_0^1\frac{1}{X_s^0}ds},$$

and

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta)\xrightarrow{P}\frac{\sigma\int_0^1\sqrt{X_s^0}dL_s\int_0^1\frac{1}{X_s^0}ds-\sigma\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s}{1-\int_0^1X_s^0ds\int_0^1\frac{1}{X_s^0}ds}.$$

Proof: Since

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha)\\
=&\ \frac{\sigma\sum_{i=1}^{n}\sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}\\
-&\ \frac{\sigma\sum_{i=1}^{n}\frac{1}{\sqrt{X_{t_{i-1}}}}(L_{t_i}-L_{t_{i-1}})\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}.
\end{aligned}$$

By the methods used in Theorem 1, it is easy to check that

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_s^0 ds, \tag{25}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} \xrightarrow{P} \int_0^1 \frac{1}{X_s^0} ds. \tag{26}$$

Then, it is obviously that

$$1-\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} \xrightarrow{P} 1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds. \tag{27}$$

Therefore, together with the results in Theorem 1, we obtain

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) \xrightarrow{P} \frac{\sigma\int_0^1 \sqrt{X_s^0} dL_s - \sigma\int_0^1 \frac{1}{\sqrt{X_s^0}} dL_s \int_0^1 X_s^0 ds}{1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds}. \tag{28}$$

As

$$\begin{aligned} &\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) \\ &= \frac{\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) - \sigma\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(L_{t_i}-L_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}. \end{aligned}$$

Together with above results, it can be checked that

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) \xrightarrow{P} \frac{\sigma\int_0^1 \sqrt{X_s^0} dL_s \int_0^1 \frac{1}{X_s^0} ds - \sigma\int_0^1 \frac{1}{\sqrt{X_s^0}} dL_s}{1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds}. \tag{29}$$

The proof is complete. ∎

IV. Generalization to semi-martingale noises

In this section, we discuss the extension of our main results in Section 3 to the general case when the driving noise is a semi-martingale. Let $Q_t = Q_0 + M_t + A_t$ be a semi-martingale, where M_t is a local martingale and A_t is a finite variation process. Then, we can replace the driving Lévy process L_t by the semi-martingale Q_t to get

$$\begin{cases} dX_t = (\alpha-\beta X_t)dt + \varepsilon\sigma\sqrt{X_t} dQ_t, \quad t\in[0,1] \\ X_0 = x_0. \end{cases} \tag{30}$$

Now we state the new results as follows.

Theorem 3: Under Assumptions $1-2$, the least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are consistent, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha,$$

and

$$\widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta.$$

Proof: According to the results in Section 3, it is easy to get the error of estimation

$$\begin{aligned} &\widehat{\alpha}-\alpha \\ &= \frac{\varepsilon\sigma\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(Q_{t_i}-Q_{t_{i-1}})}{1-\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \frac{1}{n} \frac{1}{X_{t_{i-1}}}} \\ &- \frac{\varepsilon\sigma\sum_{i=1}^{n} \frac{1}{\sqrt{X_{t_{i-1}}}}(Q_{t_i}-Q_{t_{i-1}})\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{1-\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \frac{1}{n} \frac{1}{X_{t_{i-1}}}}, \end{aligned}$$

and

$$\widehat{\beta}_{n,\varepsilon}-\beta = \frac{(\widehat{\alpha}_{n,\varepsilon}-\alpha_0) - \varepsilon\sigma\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(Q_{t_i}-Q_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}. \tag{31}$$

By applying the same methods, it can be checked that

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha,$$

and

$$\widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta.$$

The proof is complete. ∎

Theorem 4: Under the conditions $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$, it follows that

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) \xrightarrow{P} \frac{\sigma\int_0^1 \sqrt{X_s^0} dQ_s - \sigma\int_0^1 \frac{1}{\sqrt{X_s^0}} dQ_s \int_0^1 X_s^0 ds}{1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds},$$

and

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) \xrightarrow{P} \frac{\sigma\int_0^1 \sqrt{X_s^0} dQ_s \int_0^1 \frac{1}{X_s^0} ds - \sigma\int_0^1 \frac{1}{\sqrt{X_s^0}} dQ_s}{1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds}.$$

Proof: Since

$$\begin{aligned} &\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) \\ &= \frac{\sigma\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(Q_{t_i}-Q_{t_{i-1}})}{1-\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}}} \\ &- \frac{\sigma\sum_{i=1}^{n} \frac{1}{\sqrt{X_{t_{i-1}}}}(Q_{t_i}-Q_{t_{i-1}})\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{1-\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}}}. \end{aligned}$$

By the methods used in Theorem 1, it is easy to check that

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_s^0 ds, \tag{32}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} \xrightarrow{P} \int_0^1 \frac{1}{X_s^0} ds. \tag{33}$$

Then, it is obviously that

$$1-\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} \xrightarrow{P} 1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds. \tag{34}$$

Therefore, together with the results in Theorem 1, we obtain

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) \xrightarrow{P} \frac{\sigma\int_0^1 \sqrt{X_s^0} dQ_s - \sigma\int_0^1 \frac{1}{\sqrt{X_s^0}} dQ_s \int_0^1 X_s^0 ds}{1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds}. \tag{35}$$

As

$$\begin{aligned} &\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) \\ &= \frac{\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) - \sigma\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(Q_{t_i}-Q_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}. \end{aligned}$$

Together with above results, it can be checked that

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) \xrightarrow{P} \frac{\sigma\int_0^1 \sqrt{X_s^0} dQ_s \int_0^1 \frac{1}{X_s^0} ds - \sigma\int_0^1 \frac{1}{\sqrt{X_s^0}} dQ_s}{1-\int_0^1 X_s^0 ds \int_0^1 \frac{1}{X_s^0} ds}. \tag{36}$$

The proof is complete. ■

Remark 1: If the Cox-Ingersoll-Ross model is driven by α-stable motion $Z = \{Z_t, t \geq 0\}$ with index α, where Z satisfies that for any $0 \leq s < t$, $\mathbb{E}[e^{iu(Z_t - Z_s)}|\mathcal{F}_s] = e^{-(t-s)\Psi_\alpha(u)}$, $\Psi_\alpha(u)$ is the Lévy symbol of Z. The model is described as follows:

$$\begin{cases} dX_t = (\theta - \lambda X_t)dt + \sqrt{X_t}dZ_t, \quad t \geq 0 \\ X_0 = x_0, \end{cases}$$

We introduce the following contrast function:

$$\rho_n(\theta, \lambda) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} - (\theta - \lambda X_{t_{i-1}})\Delta t_{i-1}|^2,$$

where $\Delta t_{i-1} = t_i - t_{i-1} = h$.

Then, we can obtain the explicit expression of estimators

$$\begin{cases} \widehat{\theta}_n = \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}^2}{h(n\sum_{i=1}^n X_{t_{i-1}}^2 - \sum_{i=1}^n (\sum_{i=1}^n X_{t_{i-1}})^2)} \\ \qquad - \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}}}{h(n\sum_{i=1}^n X_{t_{i-1}}^2 - \sum_{i=1}^n (\sum_{i=1}^n X_{t_{i-1}})^2)} \\ \widehat{\lambda}_n = \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}}{h(n\sum_{i=1}^n X_{t_{i-1}}^2 - \sum_{i=1}^n (\sum_{i=1}^n X_{t_{i-1}})^2)} \\ \qquad - \dfrac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}}}{h(n\sum_{i=1}^n X_{t_{i-1}}^2 - \sum_{i=1}^n (\sum_{i=1}^n X_{t_{i-1}})^2)}. \end{cases}$$

The expression of estimators are different from that in (4). Therefore, the methods used to discuss the consistency and asymptotic distribution of the estimators are different as well, and this is the further topics to consider.

V. Cox-Ingersoll-Ross Model Driven by Brownian Motion

We consider Cox-Ingersoll-Ross model driven by Brownian motion, which is described as follows:

$$\begin{cases} dX_t = (\alpha - \beta X_t)dt + \sigma\sqrt{X_t}dW_t \\ X_0 = x_0, \end{cases} \tag{37}$$

where W_t is a Wiener process modeling the random market risk factor, α, β and σ are unknown parameters.

It is assumed that the process is observed at times $\{t_0, t_1, ..., t_n\}$ where $t_i = i\Delta, i = 1, 2, ..., n$ and $0 < \Delta < \frac{2}{\beta}$. Discretizing equation (37), it follows that

$$X_{t_i} - X_{t_{i-1}} = (\alpha - \beta X_{t_{i-1}})\Delta + \sigma\sqrt{X_{t_{i-1}}\Delta}\varepsilon_{t_i}, \tag{38}$$

where ε_{t_i} is a i.i.d. N(0,1) sequence and for every i, ε_{t_i} is independent with $\{X_{t_j}, j < i\}$.

Then, it is easy to check that

$$\begin{cases} \widehat{\alpha} = \dfrac{n\sum_{i=1}^n X_{t_i} - \sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}} \sum_{i=1}^n X_{t_{i-1}}}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}})} \\ \widehat{\beta} = \dfrac{n^2 - n\sum_{i=1}^n \frac{X_{t_i}}{X_{t_{i-1}}}}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}})} \\ \qquad + \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n \frac{1}{X_{t_{i-1}}}}{\Delta(n^2 - \sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \frac{1}{X_{t_{i-1}}})} \\ \widehat{\sigma^2} = \dfrac{1}{n\Delta} \sum_{i=1}^n (\dfrac{X_{t_i} - X_{t_{i-1}} - (\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})\Delta}{\sqrt{X_{t_{i-1}}}})^2. \end{cases} \tag{39}$$

Next we will prove the consistency of $\widehat{\sigma^2}$.

Note that

$$\begin{aligned} &(X_{t_i} - X_{t_{i-1}} - (\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})\Delta)^2 \\ &= (X_{t_i} - X_{t_{i-1}})^2 + [(\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})\Delta]^2 \\ &-2(X_{t_i} - X_{t_{i-1}})(\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})\Delta. \end{aligned}$$

According to Itô's lemma,

$$\begin{aligned} &d(X_{t_u} - X_{t_{i-1}})^2 \\ &= 2(X_{t_u} - X_{t_{i-1}})(\alpha - \beta X_{t_u})dt_u \\ &+\sigma^2 X_{t_u} dt_u + 2(X_{t_u} - X_{t_{i-1}})\sigma\sqrt{X_{t_u}}dW_{t_u}, \end{aligned}$$

it follows that

$$\begin{aligned} &(X_{t_i} - X_{t_{i-1}})^2 \\ &= 2\int_{t_{i-1}}^{t_i} (X_{t_u} - X_{t_{i-1}})(\alpha - \beta X_{t_u})dt_u \\ &+\sigma^2 \int_{t_{i-1}}^{t_i} X_{t_u} dt_u \\ &+2\int_{t_{i-1}}^{t_i} (X_{t_u} - X_{t_{i-1}})\sigma\sqrt{X_{t_u}}dW_{t_u}. \end{aligned}$$

Hence

$$\begin{aligned} &|\widehat{\sigma^2} - \sigma^2|^2 \\ = &|\frac{2}{n\Delta}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(X_{t_u} - X_{t_{i-1}})(\alpha - \beta X_{t_u})}{X_{t_{i-1}}} dt_u \\ &+\frac{2}{n\Delta}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(X_{t_u} - X_{t_{i-1}})\sigma\sqrt{X_{t_u}}}{X_{t_{i-1}}} dW_{t_u} \\ &+\frac{\sigma^2}{n\Delta}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_u}}{X_{t_{i-1}}} dt_u \\ &+\frac{1}{n\Delta}\sum_{i=1}^n \frac{[(\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})\Delta]^2}{X_{t_{i-1}}} \\ &-\frac{2}{n\Delta}\sum_{i=1}^n \frac{(X_{t_i} - X_{t_{i-1}})(\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})\Delta}{X_{t_{i-1}}} - \sigma^2|^2 \\ \leq &\frac{16}{n\Delta^2}\sum_{i=1}^n |\int_{t_{i-1}}^{t_i} \frac{(X_{t_u} - X_{t_{i-1}})(\alpha - \beta X_{t_u})}{X_{t_{i-1}}} dt_u|^2 \\ &+\frac{16\sigma^2}{n^2\Delta^2}|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(X_{t_u} - X_{t_{i-1}})\sqrt{X_{t_u}}}{X_{t_{i-1}}} dW_{t_u}|^2 \\ &+\frac{4\sigma^4}{n}\sum_{i=1}^n |\int_{t_{i-1}}^{t_i} \frac{X_{t_u}}{X_{t_{i-1}}\Delta} dt_u - 1|^2 \\ &+\frac{4\Delta^2}{n}\sum_{i=1}^n |\frac{(\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})^2}{X_{t_{i-1}}}|^2 \\ &+\frac{16}{n}\sum_{i=1}^n |\frac{(X_{t_i} - X_{t_{i-1}})(\widehat{\alpha} - \widehat{\beta} X_{t_{i-1}})}{X_{t_{i-1}}}|^2. \end{aligned}$$

Applying the Holder's inequality and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mathbb{E}|\int_{t_{i-1}}^{t_i}\frac{(X_{t_u}-X_{t_{i-1}})(\alpha-\beta X_{t_u})}{X_{t_{i-1}}}dt_u|^2\\
\leq\ &\mathbb{E}[\Delta\int_{t_{i-1}}^{t_i}|\frac{(X_{t_u}-X_{t_{i-1}})(\alpha-\beta X_{t_u})}{X_{t_{i-1}}}|^2dt_u\\
\leq\ &\Delta\int_{t_{i-1}}^{t_i}(\mathbb{E}|\frac{1}{X_{t_{i-1}}}|^4)^{\frac{1}{2}}(\mathbb{E}|(X_{t_u}-X_{t_{i-1}})\\
&(\alpha-\beta X_{t_u})|^4)^{\frac{1}{2}}dt_u,
\end{aligned}
$$

from *Assumption 4* we get that $\mathbb{E}|\frac{1}{X_{t_{i-1}}}|^4$ is bounded. Furthermore, $\mathbb{E}|(X_{t_u}-X_{t_{i-1}})(\alpha-\beta X_{t_u})|^4\leq(\mathbb{E}|X_{t_u}-X_{t_{i-1}}|^8)^{\frac{1}{2}}(\mathbb{E}|\alpha-\beta X_{t_u}|^8)^{\frac{1}{2}}$. According to lemma 1, it shows that $\mathbb{E}|X_{t_u}-X_{t_{i-1}}|^8=O(\Delta^4)$ and with *Assumption 3* and *Assumption 4* we know that $\mathbb{E}|\alpha-\beta X_t|^8\leq 128L^8(1+\sup_t\mathbb{E}|X_t|^8)<\infty$.

Thus we obtain the result that

$$
\mathbb{E}|\int_{t_{i-1}}^{t_i}\frac{(X_{t_u}-X_{t_{i-1}})(\alpha-\beta X_{t_u})}{X_{t_{i-1}}}dt_u|^2=O(\Delta^3). \quad (40)
$$

As

$$
\begin{aligned}
&\mathbb{E}|\frac{(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})^2}{X_{t_{i-1}}}|^2\\
=\ &\mathbb{E}\frac{(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})^4}{X_{t_{i-1}}^2}\\
\leq\ &(\mathbb{E}|\frac{1}{X_{t_{i-1}}}|^4)^{\frac{1}{2}}(\mathbb{E}|\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}}|^8)^{\frac{1}{2}},
\end{aligned}
$$

it is easy to check that $\mathbb{E}|\frac{(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})^2}{X_{t_{i-1}}}|^2$ is bounded.

Then look at $\mathbb{E}|\frac{(X_{t_i}-X_{t_{i-1}})(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})}{X_{t_{i-1}}}|^2$.

Since

$$
\begin{aligned}
&\mathbb{E}|\frac{(X_{t_i}-X_{t_{i-1}})(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})}{X_{t_{i-1}}}|^2\\
\leq\ &(\mathbb{E}|\frac{1}{X_{t_{i-1}}}|^4)^{\frac{1}{2}}(\mathbb{E}|(X_{t_i}-X_{t_{i-1}})(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})|^4)^{\frac{1}{2}},
\end{aligned}
$$

and $\mathbb{E}|(X_{t_i}-X_{t_{i-1}})(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})|^4=O(\Delta^2)$, it follows that

$$
\mathbb{E}|\frac{(X_{t_i}-X_{t_{i-1}})(\widehat{\alpha}-\widehat{\beta}X_{t_{i-1}})}{X_{t_{i-1}}}|^2=O(\Delta). \quad (41)
$$

Hereafter, we consider $\mathbb{E}|\int_{t_{i-1}}^{t_i}\frac{X_{t_u}}{X_{t_{i-1}}\Delta}dt_u-1|^2$.

As

$$
\begin{aligned}
&\mathbb{E}|\int_{t_{i-1}}^{t_i}\frac{X_{t_u}}{X_{t_{i-1}}\Delta}dt_u-1|^2\\
=\ &\frac{1}{\Delta^2}\mathbb{E}|\int_{t_{i-1}}^{t_i}\frac{X_{t_u}-X_{t_{i-1}}}{X_{t_{i-1}}}dt_u|^2\\
\leq\ &\frac{1}{\Delta^2}\mathbb{E}|\Delta\int_{t_{i-1}}^{t_i}|\frac{X_{t_u}-X_{t_{i-1}}}{X_{t_{i-1}}}|^2dt_u|\\
\leq\ &\frac{1}{\Delta}\int_{t_{i-1}}^{t_i}(\mathbb{E}|\frac{1}{X_{t_{i-1}}}|^4)^{\frac{1}{2}}(\mathbb{E}|X_{t_u}-X_{t_{i-1}}|^4)^{\frac{1}{2}}dt_u,
\end{aligned}
$$

and $\mathbb{E}|X_{t_u}-X_{t_{i-1}}|^4=O(\Delta^2)$, we obtain that

$$
\mathbb{E}|\int_{t_{i-1}}^{t_i}\frac{X_{t_u}}{X_{t_{i-1}}\Delta}dt_u-1|^2=O(\Delta). \quad (42)
$$

Finally, we study $\mathbb{E}|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\frac{(X_{t_u}-X_{t_{i-1}})\sqrt{X_{t_u}}}{X_{t_{i-1}}}dW_{t_u}|^2$. Let

$$
N_t=\sum_{i=1}^n\int_{t\wedge t_{i-1}}^{t\wedge t_i}\frac{(X_{t_u}-X_{t_{i-1}})\sqrt{X_{t_u}}}{X_{t_{i-1}}}dW_{t_u}. \quad (43)
$$

It is obvious that N_t is a martingale. From the martingale moment inequality, there exists a constant C such that

$$
\mathbb{E}|N_t|^2\leq C\mathbb{E}\langle N\rangle_t, \quad (44)
$$

where $\langle N\rangle_t$ denotes the quadratic variation of N_t.

Note that

$$
\langle N\rangle_t=\sum_{i=1}^n\int_{t\wedge t_{i-1}}^{t\wedge t_i}\frac{(X_{t_u}-X_{t_{i-1}})^2X_{t_u}}{X_{t_{i-1}}^2}dt_u, \quad (45)
$$

and

$$
\begin{aligned}
&\mathbb{E}\langle N\rangle_t\\
=\ &\mathbb{E}\sum_{i=1}^n\int_{t\wedge t_{i-1}}^{t\wedge t_i}\frac{(X_{t_u}-X_{t_{i-1}})^2X_{t_u}}{X_{t_{i-1}}^2}dt_u\\
\leq\ &\sum_{i=1}^n\int_{t\wedge t_{i-1}}^{t\wedge t_i}(\mathbb{E}|\frac{X_{t_u}}{X_{t_{i-1}}^2}|^2)^{\frac{1}{2}}(\mathbb{E}|X_{t_u}-X_{t_{i-1}}|^4)^{\frac{1}{2}}dt_u.
\end{aligned}
$$

Since $\mathbb{E}|\frac{X_{t_u}}{X_{t_{i-1}}^2}|^2$ is bounded and

$$
\mathbb{E}|X_{t_u}-X_{t_{i-1}}|^4=O(|t_u-t_{i-1}|^2), \quad (46)
$$

it follows that

$$
\mathbb{E}\langle N\rangle_t=nO(\Delta^2). \quad (47)
$$

As a result

$$
\mathbb{E}|\widehat{\sigma^2}-\sigma^2|^2=O(\Delta)+O(\frac{1}{n}). \quad (48)
$$

Therefore, $\mathbb{E}|\widehat{\sigma^2}-\sigma^2|^2\to 0$ when $\Delta\to 0$ and $n\Delta\to\infty$ as $n\to\infty$. This means that $\widehat{\sigma^2}\to\sigma^2$ in the L_2 norm.

VI. Simulation

In this experiment, we generate a discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$ and compute $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ from the sample. We let $\sigma=0.5$, $x_0=0.1$. For every given true value of the parameters-(α_0,β_0), the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, $\varepsilon=0.05$, the size is increasing from 500 to 3000. In Table 2, $\varepsilon=0.001$, the size is increasing from 5000 to 30000. The tables list the value of "α_0-LSE","β_0-LSE"and the absolute errors (AE) of LSE, LSE means least squares estimator.

Two tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. Therefore, the methods used in this paper are effective and the obtained estimators are good.

Next we give some simulation results of the confidence interval of α_0 and β_0 under 0.95 confidence level. In Table 3 and Table 4, We let $\sigma=0.5$, $x_0=0.1$. For every given true value of α_0 and β_0, let $\varepsilon=0.01$, the size of the sample is increasing from 2000 to 10000. These tables list the value of α_0-LSE and β_0-LSE and in the last column of the table list the confidence interval of α_0 and β_0. Table 3 and Table 4 illustrate that the length of the confidence interval is becoming small when the size of the sample is increasing.

TABLE I
LSE SIMULATION RESULTS OF α_0 AND β_0

True		Aver		AE	
(α_0, β_0)	Size n	$\alpha_0 - LSE$	$\beta_0 - LSE$	α_0	β_0
(1,2)	500	0.9542	1.9452	0.0458	0.0548
	1000	0.9628	2.0467	0.0372	0.0467
	3000	0.9765	2.0326	0.0235	0.0326
(2,3)	500	1.9586	2.9610	0.0414	0.0390
	1000	2.0371	3.0224	0.0371	0.0224
	3000	2.0189	3.0123	0.0189	0.0123
(3,4)	500	2.9496	3.9438	0.0504	0.0562
	1000	3.0377	4.0397	0.0377	0.0397
	3000	3.0214	4.0205	0.0214	0.0205

TABLE II
LSE SIMULATION RESULTS OF α_0 AND β_0

True		Aver		AE	
(α_0, β_0)	Size n	$\alpha_0 - LSE$	$\beta_0 - LSE$	α_0	β_0
(1,2)	5000	0.9965	1.9979	0.0035	0.0021
	10000	0.9982	2.0013	0.0018	0.0013
	30000	0.9997	2.0004	0.0003	0.0004
(2,3)	5000	1.9972	2.9974	0.0028	0.0026
	10000	2.0014	3.0015	0.0014	0.0015
	30000	2.0002	3.0003	0.0002	0.0003
(3,4)	5000	2.9968	3.9978	0.0032	0.0022
	10000	3.0017	4.0012	0.0017	0.0012
	30000	3.0004	4.0005	0.0004	0.0005

TABLE III
SIMULATION RESULTS OF CONFIDENCE INTERVAL OF α_0

True	Aver		0.95
α_0	Size n	$\alpha_0 - LSE$	confidence interval
1	2000	0.9963	[0.9825,1.2246]
	5000	1.0018	[0.9917,1.1148]
	10000	1.0007	[0.9956,1.1032]
2	2000	1.9951	[1.9792,2.3423]
	5000	2.0027	[1.9845,2.2247]
	10000	2.0011	[1.9983,2.1162]
3	2000	2.9954	[2.9763,3.3592]
	5000	3.0028	[2.9894,3.2334]
	10000	3.0015	[2.9932,3.1221]

TABLE IV
SIMULATION RESULTS OF CONFIDENCE INTERVAL OF β_0

True	Aver		0.95
β_0	Size n	$\alpha_0 - LSE$	confidence interval
2	2000	2.0032	[1.9875,2.2354]
	5000	2.0019	[1.9986,2.1241]
	10000	2.0008	[2.0001,2.1032]
3	2000	3.0043	[2.9721,3.3246]
	5000	3.0025	[2.9875,3.2314]
	10000	3.0009	[2.9986,3.1105]
4	2000	4.0035	[3.9749,4.3368]
	5000	4.0021	[3.9875,4.2389]
	10000	4.0010	[3.9983,4.1568]

VII. Conclusion

In this paper, the parameter estimation for CIR model with small Lévy noises has been studied from discrete observations. The least squares method has been used to obtain the estimators. The explicit formula of the estimation error has been given and the consistency of the least squares estimators has been proved. The asymptotic distribution of the estimation error has been discussed as well. Further research topics will include the parameter estimation for general nonlinear stochastic differential equations driven by lévy noises.

References

[1] M. Barczy, G. Pap. "Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes", *Journal of Statistical Planning and Inference*, vol. 140, no. 1, pp. 1576-1593, 2010.

[2] B. Bibby, M. Sqrensen, "Martingale estimation functions for discretely observed diffusion processes", *Bernoulli*, vol. 1, no. 1, pp. 017-039, 1995.

[3] J. P. N. Bishwal, *Parameter Estimation in Stochastic Differential Equations*, Springer-Verlag, Berlin, 2008.

[4] J. W. Cai, P. Chen, X. Mei, "Range-based threshold spot volatility estimation for jump diffusion models", *IAENG International Journal of Applied Mathematics*, vol. 47, no. 1, pp. 43-48, 2017.

[5] Cox. J.,Ingersoll. J., S. Ross. "An intertemporal general equilibrium model of asset prices", *Econometrica*, vol. 53, no. 1, pp. 363-384. 1985.

[6] Cox. J.,Ingersoll. J., S. Ross. "A theory of the term structure of interest rates", *Econometrica*. vol. 53, no. 1, pp. 385-408. 1985.

[7] T. Deck, "Asymptotic properties of Bayes estimators for Gaussian Itô processes with noisy observations", *Journal of Multivariate Analysis*, vol. 97, no. 5, pp. 563-573, 2006.

[8] X. Kan, H. S. Shu, Y. Che, "Asymptotic parameter estimation for a class of linear stochastic systems using Kalman-Bucy filtering", *Mathematical Problems in Engineering*, Article ID: 342705, DOI: 10.1155/2012/342705, 2012.

[9] Y A Kutoyants. *Statistical Inference for Ergodie Diffusion Processes*, Springer-Verlag, London, 2004.

[10] Z. H. Li, C. H. Ma, "Asymptotic properties of estimators in a stable Cox-Ingersoll-Ross model", *Stochastic Processes and Their Applications*, vol. 125, no. 1, pp. 3196-3233, 2015.

[11] C. P. Li, H. B. Hao, "Likelihood and Bayesian estimation in stress strength model from generalized exponential distribution containing outliers," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp. 155-159, 2016.

[12] L. Liu, et al, "Modelling and Estimation for Uncertain Systems with Transmission Delays, Packet Dropouts, and Out-of-Order Packets", *Complexity*, Article ID 7531547, 15 pages, 2018.

[13] H. W. Long, "Least squares estimator for discretely observed Ornstein-Uhlenbeck processes with small Lévy noises", *Statistics and Probability Letters*, vol. 79, no. 1, pp. 2076-2085, 2009.

[14] H. W. Long, Y. Shimizu, W. Sun, "Least squares estimators for discretely observed stochastic processes driven by small Lévy noises", *Journal of Multivariate Analysis*. vol. 116, no. 1, pp. 422-439, 2013.

[15] C. H. Ma, X. Yang, "Small noise fluctuations of the CIR model driven by α-stable noises", *Statistics and Probability Letters*, vol. 94, no. 1, pp. 1-11, 2014.

[16] I. Mendy, "Parametric estimation for sub-fractional OrnsteinCUhlenbeck process", *Journal of Statistical Planning and Inference*, vol. 143, no. 3, pp. 663-674, 2013.

[17] P. E. Protter, *Stochastic Integration and Differential Equations: Stochastic Modelling and Applied Probability*, 2nd ed, Applications of Mathematics(New York) 21, Springer, Berlin, 2004.

[18] K. Skouras, "Strong consistency in nonlinear stochastic regression models", *The Annals of Statistics*, vol. 28, no. 1, pp. 871-879, 2000.

[19] O. Vasicek, "An equilibrium characterization of the term structure", *Journal of Financial Economics*. vol. 5, no. 2, pp. 177-186, 1977.

[20] J. H. Wen, X. J. Wang, S. H. Mao, X. P. Xiao, "Maximum likelihood estimation of McKeanCVlasov stochastic differential equation and its application", *Applied Mathematics and Computation*, vol. 274, no. 1, pp. 237-246, 2015.

[21] C. Wei, H. S. Shu, "Maximum likelihood estimation for the drift parameter in diffusion processes", *Stochastics: An International Journal of Probability and Stochastic Processes*, vol. 88, no. 5, pp. 699-710, 2016.

[22] C. Wei, H. S. Shu, Y. R. Liu, "Gaussian estimation for discretely observed Cox-Ingersoll-Ross model", *International Journal of General Systems*, vol. 45, no. 5, pp. 561-574, 2016.