Parameter Estimation for Discretely Observed Cox-Ingersoll-Ross Model with Small Lévy Noises

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Abstract—This paper is concerned with the parameter estimation problem for Cox-Ingersoll-Ross model with small Lévy noises from discrete observations. The least squares method is used to obtain the parameter estimators and the explicit formula of the estimation error is given. The consistency of the estimators are derived when a small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously by using Cauchy-Schwarz inequality, Gronwall’s inequality, Markov inequality and dominated convergence. The asymptotic distribution of the estimation error is studied. The simulation is made to verify the effectiveness of the least squares estimators.

Index Terms—Least squares estimator, Lévy noises, discrete observations, consistency.

I. INTRODUCTION

Itô stochastic differential equations are important tools for studying random phenomena and are widely used in the modeling of stochastic phenomena in the fields of physics, chemistry, medicine and finance ([3], [12], [17]). However, part or all of the parameters in stochastic model are always unknown. In the past few decades, some popular methods have been put forward to estimate the parameters in Itô stochastic differential equations, such as maximum likelihood estimation ([1], [20], [21]), least squares estimation ([4], [16], [18]) and Bayes estimation ([7]–[9], [11]). But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. Lévy noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for stochastic differential equations with small Lévy noises. Recently, a number of literatures have been devoted to the parameter estimation for the models driven by small Lévy noises. When the coefficient of the Lévy jump term is constant, drift parameter estimation has been investigated by some authors ([13], [14]).

The Cox-Ingersoll-Ross model ([5], [6]), hereafter the CIR model, which was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model ([19]), describes the evolution of interest rates. It is known that parameter estimation for CIR model driven by Brownian motion has been well developed based on discrete observations ([2], [22]). However, some features of the financial processes cannot be captured by the CIR model, for example, discontinuous sample paths and heavy tailed properties. Therefore, it is natural to replace the Brownian motion by the Lévy process. Recently, the parameter estimation problems for CIR model driven by small Lévy noises have been studied by some authors. For example, Ma and Yang ([15]) established the central limit theorems, the deviation inequality and the moderate deviations for least squares estimators of parameters in the CIR type model driven by $\alpha$-stable noises; Li and Ma ([10]) derived the consistency and central limit theorems of the conditional least squares estimators in a stable Cox-Ingersoll-Ross model. But, the explicit formula of the estimators and the estimation error have not been given in these papers.

In this paper, we consider the parameter estimation problem for CIR model with small Lévy noises from discrete observations. The decomposition of the Lévy process is different from that in ([10], [15]), so the methods used to prove the asymptotic property of the estimators are different. The process is discretized based on Euler-Maruyama scheme, the least squares method is used to obtain the explicit formula of the estimators and the estimation errors are given as well. When the coefficient of the Lévy noises with decomposition $\nu(ds,dz)$, the least squares method is used to obtain the explicit formula of the estimators and the estimation errors are given. The asymptotic distribution of the estimation error is studied. The simulation is made to verify the effectiveness of the obtained estimators. Finally, the simulation result is provided to verify the effectiveness of the obtained estimators.

This paper is organized as follows. In Section 2, the CIR model driven by small Lévy noises is introduced, the contrast function is given and the explicit formula of the least squares estimators are obtained. In Section 3, the estimation errors are derived, the consistency of the estimators are proved and the asymptotic distribution of the estimation error are discussed. In Section 4, the results are extended to semimartingale noises. In Section 5, some simulation results are made. The conclusion is given in Section 6.

II. PROBLEM FORMULATION AND PRELIMINARIES

Let $\{\Omega,\mathcal{F},\mathbb{P}\}$ be a basic probability space equipped with a right continuous and increasing family of $\sigma$-algebras $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\{L_t, t \geq 0\}$ be an $\{\{\mathcal{F}_t\}\}$-adapted Lévy noises with decomposition

$$L_t = B_t + \int_0^t \int_{|z|>1} z N(ds,dz) + \int_0^t \int_{|z|\leq 1} z \tilde{N}(ds,dz),$$

(1)

where $(B_t, t \geq 0)$ is a standard Brownian motion, $N(ds,dz)$ is a Poisson random measure independent of $(B_t, t \geq 0)$ with characteristic measure $d\nu(ds,dz)$, and $\tilde{N}(ds,dz) = N(ds,dz) - \nu(ds,dz)$ is a martingale measure. We assume that $\nu(ds,dz)$ is a Lévy measure on $\mathbb{R}\setminus\{0\}$ satisfying $\int(|z|^2 \wedge 1)\nu(ds,dz) < \infty$.

In this paper, we study the parameter estimation for CIR model with small Lévy noises described by the following
stochastic differential equation:

\[
\begin{align*}
    dX_t &= (\alpha - \beta X_t)dt + \varepsilon \sigma \sqrt{X_t}dL_t, \quad t \in [0, 1] \\
    X_0 &= x_0,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are unknown parameters and \( \sigma \) is known constant. Without loss of generality, it is assumed that \( \varepsilon \in (0, 1] \).

Consider the following contrast function

\[
\rho_{n, \varepsilon}(\alpha, \beta) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1}|^2}{\varepsilon^2 \sigma^2 X_{t_{i-1}} \Delta t_{i-1}},
\]

where \( \Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n} \).

It is easy to obtain the estimators

\[
\begin{align*}
\hat{\alpha}_{n, \varepsilon} &= \frac{n \sum_{i=1}^{n} X_{t_i} - \sum_{i=1}^{n} X_{t_{i-1}} \sum_{i=1}^{n} X_{t_{i-1}}}{\Delta(n^2 - \sum_{i=1}^{n} X_{t_{i-1}}^2 + \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}})} \\
&\quad + \frac{n \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^{n} X_{t_{i-1}}}{\Delta(n^2 - \sum_{i=1}^{n} X_{t_{i-1}}^2 + \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}})} \\
\hat{\beta}_{n, \varepsilon} &= \frac{n^2 - \sum_{i=1}^{n} X_{t_i}^2 - \sum_{i=1}^{n} X_{t_{i-1}} \sum_{i=1}^{n} X_{t_{i-1}}}{\Delta(n^2 - \sum_{i=1}^{n} X_{t_{i-1}}^2 + \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}})} \\
&\quad + \frac{n \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^{n} X_{t_{i-1}}}{\Delta(n^2 - \sum_{i=1}^{n} X_{t_{i-1}}^2 + \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}})}.
\end{align*}
\]

Before giving the main results, we introduce some assumptions below.

Let \( X^0 = (X^0_t, t \geq 0) \) be the solution to the underlying ordinary differential equation under the true value of the parameter:

\[
dX^0_t = (\alpha - \beta X^0_t)dt, \quad X^0_0 = x_0.
\]

**Assumption 1:** \( \alpha \) and \( \beta \) are positive true values of the parameters and \( \sigma > 0 \).

**Assumption 2:** \( \inf_{0 \leq s \leq 1} \{ X_1 \} > 0 \).

**Assumption 3:** There exists \( L > 0 \) such that \( |\alpha - \beta x| \leq L(1 + |x|) \).

**Assumption 4:** \( \sup_t \mathbb{E}[|X_t|^p] < \infty \) and \( \mathbb{E}[\frac{1}{|X_t|^p}] < \infty \) for every \( p \geq 1 \).

In the next sections, the consistency of the least squares estimators are derived and the simulation is made to verify the effectiveness of the estimators.

### III. Main Result and Proofs

In the following theorem, the consistency in probability of the least squares estimators are proved by using Cauchy-Schwarz inequality, Gronwall’s inequality, Markov inequality and dominated convergence.

**Theorem 1:** The least squares estimators \( \hat{\alpha} \) and \( \hat{\beta} \) are consistent in probability, namely

\[
\hat{\alpha}_{n, \varepsilon} \xrightarrow{p} \alpha, \quad \hat{\beta}_{n, \varepsilon} \xrightarrow{p} \beta.
\]

**Proof:** By using the Euler-Maruyama scheme, from (2), we have

\[
X_{t_i} - X_{t_{i-1}} = (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1} + \varepsilon \sigma \sqrt{X_{t_{i-1}}} (L_{t_{i-1}} - L_{t_{i-1}}).
\]

Then, it is easy to see that

\[
\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) = \alpha - \frac{1}{n} \beta \sum_{i=1}^{n} X_{t_{i-1}} + \varepsilon \sigma \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_{i-1}} - L_{t_{i-1}}),
\]

\[
\sum_{i=1}^{n} \frac{X_{t_i}}{X_{t_{i-1}}} = n + \frac{1}{n} \alpha \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} - \beta
\]

Substituting (6) and (7) into the expression of \( \hat{\alpha} \), it follows that

\[
\hat{\alpha}_{n, \varepsilon} - \alpha = \frac{\varepsilon \sigma \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_{i-1}} - L_{t_{i-1}})}{n^2} + \frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} - \frac{\varepsilon \sigma \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}}}{n} \sum_{i=1}^{n} X_{t_{i-1}}.
\]

Let \( M^{n, \varepsilon} = X_{[nt]/n} \), in which \( [nt] \) denotes the integer part of \( nt \). We will prove that the sequence \( \{ M^{n, \varepsilon} \} \) converges to the deterministic process \( \{ X^0_t \} \) uniformly in probability as \( \varepsilon \to 0 \) and \( n \to \infty \).

Observe that

\[
X_t - X^0_t = \int_0^t (X^0_s - X_s)ds + \varepsilon \sigma \int_0^t \sqrt{X_s}dL_s.
\]

By using the Cauchy-Schwarz inequality, we have

\[
|X_t - X^0_t|^2 \leq 2 \int_0^t (X^0_s - X_s)ds^2 + 2\varepsilon^2 \sigma^2 \int_0^t \sqrt{X_s}dL_s^2
\]

\[
\leq 2t \int_0^t (X^0_s - X_s)^2ds + 2\varepsilon^2 \sigma^2 \int_0^t \sqrt{X_s}dL_s^2.
\]

According to the Gronwall’s inequality, we obtain

\[
|X_t - X^0_t|^2 \leq 2\varepsilon^2 \sigma^2 e^{2t} \int_0^t \sqrt{X_s}dL_s^2.
\]

Then, it follows that

\[
\sup_{0 \leq s \leq T} |X_t - X^0_t| < \varepsilon \sqrt{2\varepsilon \sigma^2} \sup_{0 \leq s \leq T} \int_0^t \sqrt{X_s}dL_s.
\]

Therefore, for each \( T > 0 \), it is easy to check that

\[
\sup_{0 \leq s \leq T} |X_t - X^0_t| \xrightarrow{p} 0.
\]

As \( [nt]/n \to t \) when \( n \to \infty \), we get that the sequence \( \{ M^{n, \varepsilon} \} \) converges to the deterministic process \( \{ X^0_t \} \) uniformly in probability as \( \varepsilon \to 0 \) and \( n \to \infty \).

Next we will prove that \( \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_{i-1}} - L_{t_{i-1}}) \xrightarrow{p} \int_0^t \sqrt{X^0_s}dL_s \).

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Note that
\[
\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}(L_{t_i} - L_{t_{i-1}})} = \int_{0}^{1} \sqrt{M_s^{n,\varepsilon}} dL_s. \tag{12}
\]

Then, it is elementary to see that
\[
\begin{align*}
&\int_{0}^{1} \left| \sqrt{M_s^{n,\varepsilon}} dL_s - \int_{0}^{1} \sqrt{X_s^{n}} dL_s \right| \\
&= \int_{0}^{1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| dB_s \\
&+ \int_{0}^{1} \int_{|z|>1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| z N(ds, dz) \\
&+ \int_{0}^{1} \int_{|z|\leq 1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| \tilde{N}(ds, dz) \\
&\leq \int_{0}^{1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| dB_s \\
&+ \int_{0}^{1} \int_{|z|>1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| z N(ds, dz) \\
&+ \int_{0}^{1} \int_{|z|\leq 1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| \tilde{N}(ds, dz) 
\end{align*}
\]

It can be easily checked that
\[
\begin{align*}
\int_{0}^{1} \int_{|z|>1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| z N(ds, dz) \\
\leq \int_{0}^{1} \int_{|z|>1} |\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}}| |z| N(ds, dz) \\
\leq \sup_{0 \leq s \leq 1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| \int_{0}^{1} \int_{|z|>1} |z| N(ds, dz) \\
\xrightarrow{P} 0
\end{align*}
\]

as \( \varepsilon \to 0 \) and \( n \to \infty \).

By using the Markov inequality and dominated convergence, we have
\[
\int_{0}^{1} \int_{|z|\leq 1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| \tilde{N}(ds, dz) \xrightarrow{P} 0
\]
and
\[
\int_{0}^{1} \int_{|z|\leq 1} \left| \sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^{n}} \right| \tilde{N}(ds, dz) \xrightarrow{P} 0.
\]

Thus, combining the previous results, it follows that
\[
\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}(L_{t_i} - L_{t_{i-1}})} \xrightarrow{P} \int_{0}^{1} \sqrt{X_s^{n}} dL_s. \tag{13}
\]

Then, it is easy to see that
\[
\sum_{i=1}^{n} \frac{1}{\sqrt{X_{t_{i-1}}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_{0}^{1} \frac{1}{\sqrt{X_s^{n}}} dL_s.
\]

Let
\[
X_N = \inf_{0 \leq t \leq 1} \{ X_{t_{i-1}} \}, \tag{14}
\]
and
\[
X_M = \sup_{0 \leq t \leq 1} \{ X_{t_{i-1}} \}. \tag{15}
\]

We make an assumption that \( X_N \neq X_M \).

From (15), it follows that
\[
\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \leq X_M < \infty.
\]

Therefore, when \( \varepsilon \to 0 \) and \( n \to \infty \), we have
\[
\varepsilon \sigma \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}(L_{t_i} - L_{t_{i-1}})} \xrightarrow{P} 0, \tag{16}
\]
and
\[
\varepsilon \sigma \sum_{i=1}^{n} \frac{1}{\sqrt{X_{t_{i-1}}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} 0. \tag{17}
\]

Finally, we will consider the boundedness of \( 1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} 0 \).

It is obviously that
\[
\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} 0 \geq 1,
\]

which, under the assumption that \( X_N \neq X_M \), implies
\[
1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} < 0. \tag{18}
\]

From (14) and (15), it follows that
\[
\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} < \frac{1}{X_M}. \tag{19}
\]

Then we have
\[
\frac{1}{\sum_{i=1}^{n} X_{t_{i-1}}} \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}} < \frac{1}{X_M} < \infty. \tag{20}
\]

Combining the previous arguments, when \( \varepsilon \to 0 \) and \( n \to \infty \), we have
\[
\hat{\alpha}_{n, \varepsilon} \xrightarrow{P} \alpha. \tag{22}
\]

From (4) and (6), we obtain
\[
\hat{\beta}_{n, \varepsilon} = \beta = \frac{(\hat{\alpha}_{n, \varepsilon} - \alpha) - \varepsilon \sigma \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}})}{\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} - \frac{1}{X_M}}. \tag{23}
\]

Since \( \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \geq X_N > 0 \), we get that \( \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \leq \frac{1}{X_M} < \infty \).

Together with the results that \( \hat{\alpha}_{n, \varepsilon} \xrightarrow{P} \alpha \) and \( \varepsilon \sigma \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} 0 \), it follows that
\[
\hat{\beta}_{n, \varepsilon} \xrightarrow{P} \beta, \tag{24}
\]

as \( \varepsilon \to 0 \) and \( n \to \infty \).

Therefore, \( \hat{\alpha}_{n, \varepsilon} \) and \( \hat{\beta}_{n, \varepsilon} \) are consistent in probability. The proof is complete.

In the following theorem, the asymptotic distribution of the estimation error is discussed.

**Theorem 2:** Under the conditions \( \varepsilon \to 0 \), \( n \to \infty \) and \( n \varepsilon \to \infty \), it follows that
\[
\varepsilon^{-1} (\hat{\alpha}_{n, \varepsilon} - \alpha) \xrightarrow{D} \frac{\sigma \int_{0}^{1} \sqrt{X_s^{n}} dL_s - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_s^{n}}} dL_s \int_{0}^{1} X_s^{n} ds}{1 - \int_{0}^{1} X_s^{n} ds \int_{0}^{1} \frac{1}{X_s^{n}} ds},
\]
and
\[
\varepsilon^{-1} (\hat{\beta}_{n, \varepsilon} - \beta) \xrightarrow{D} \frac{\sigma \int_{0}^{1} \sqrt{X_s^{n}} dL_s \int_{0}^{1} \frac{1}{X_s^{n}} dL_s - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_s^{n}}} dL_s \int_{0}^{1} \frac{1}{X_s^{n}} ds}{1 - \int_{0}^{1} X_s^{n} ds \int_{0}^{1} \frac{1}{X_s^{n}} ds}. \tag{25}
\]

**Proof:** Since
\[
\begin{align*}
\varepsilon^{-1} (\hat{\alpha}_{n, \varepsilon} - \alpha) &= \frac{\sigma \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}})}{1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} - \frac{1}{X_M} \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}}} \\
&- \frac{\sigma \sum_{i=1}^{n} \frac{1}{\sqrt{X_{t_{i-1}}}} (L_{t_i} - L_{t_{i-1}})}{1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} - \frac{1}{X_M} \sum_{i=1}^{n} \frac{1}{X_{t_{i-1}}}}.
\end{align*}
\]

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By the methods used in Theorem 1, it is easy to check that

$$\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \to \int_{0}^{1} X_{0}^{0} ds,$$

and

$$\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \to \int_{0}^{1} X_{0}^{0} ds.$$

Then, it is obviously that

$$1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \to 1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds.$$

Therefore, together with the results in Theorem 1, we obtain

$$\varepsilon^{-1}(\tilde{\alpha}_{n,\varepsilon} - \alpha) \to \frac{\sigma \int_{0}^{1} \sqrt{X_{0}^{0}} dL_{s} \frac{1}{\sqrt{X_{0}^{0}}} - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_{0}^{0}}} dL_{s} \int_{0}^{1} X_{0}^{0} ds}{1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds}.$$

As

$$\varepsilon^{-1}(\tilde{\beta}_{n,\varepsilon} - \beta) = \varepsilon^{-1}(\tilde{\alpha}_{n,\varepsilon} - \alpha) - \sigma \sum_{i=1}^{n} \int_{0}^{1} X_{t_{i-1}} (L_{t_{i}} - L_{t_{i-1}}).$$

Together with above results, it can be checked that

$$\varepsilon^{-1}(\tilde{\beta}_{n,\varepsilon} - \beta) \to \frac{\sigma \int_{0}^{1} \sqrt{X_{0}^{0}} dL_{s} \frac{1}{\sqrt{X_{0}^{0}}} - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_{0}^{0}}} dL_{s} \int_{0}^{1} X_{0}^{0} ds}{1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds}.$$

The proof is complete.

**Theorem 4:** Under the conditions $\varepsilon \to 0$, $n \to \infty$ and $n \varepsilon \to \infty$, it follows that

$$\varepsilon^{-1}(\tilde{\alpha}_{n,\varepsilon} - \alpha) \to \frac{\sigma \int_{0}^{1} \sqrt{X_{0}^{0}} dQ_{s} - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_{0}^{0}}} dQ_{s} \int_{0}^{1} X_{0}^{0} ds}{1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds},$$

and

$$\varepsilon^{-1}(\tilde{\beta}_{n,\varepsilon} - \beta) \to \frac{\sigma \int_{0}^{1} \sqrt{X_{0}^{0}} dQ_{s} \int_{0}^{1} \frac{1}{X_{0}^{0}} ds - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_{0}^{0}}} dQ_{s} \int_{0}^{1} X_{0}^{0} ds}{1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds}.$$

**Proof:** Since

$$\varepsilon^{-1}(\tilde{\alpha}_{n,\varepsilon} - \alpha) = \frac{\sigma \sum_{i=1}^{n} \int_{0}^{1} X_{t_{i-1}} (Q_{t_{i}} - Q_{t_{i-1}})}{1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \int_{0}^{1} X_{t_{i-1}} ds - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \int_{0}^{1} X_{t_{i-1}} ds}.$$

By the methods used in Theorem 1, it is easy to check that

$$\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \to \int_{0}^{1} X_{0}^{0} ds,$$

and

$$\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \to \int_{0}^{1} \frac{1}{X_{0}^{0}} ds.$$

Then, it is obviously that

$$1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \to 1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds.$$

Therefore, together with the results in Theorem 1, we obtain

$$\varepsilon^{-1}(\tilde{\beta}_{n,\varepsilon} - \beta) \to \frac{\sigma \int_{0}^{1} \sqrt{X_{0}^{0}} dQ_{s} - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_{0}^{0}}} dQ_{s} \int_{0}^{1} X_{0}^{0} ds}{1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds}.$$

As

$$\varepsilon^{-1}(\tilde{\alpha}_{n,\varepsilon} - \alpha) = \varepsilon^{-1}(\tilde{\alpha}_{n,\varepsilon} - \alpha) - \frac{\sigma \sum_{i=1}^{n} \int_{0}^{1} X_{t_{i-1}} (Q_{t_{i}} - Q_{t_{i-1}})}{1 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \int_{0}^{1} X_{t_{i-1}} ds - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} \int_{0}^{1} X_{t_{i-1}} ds}.$$

Together with above results, it can be checked that

$$\varepsilon^{-1}(\tilde{\beta}_{n,\varepsilon} - \beta) \to \frac{\sigma \int_{0}^{1} \sqrt{X_{0}^{0}} dQ_{s} \int_{0}^{1} \frac{1}{X_{0}^{0}} ds - \sigma \int_{0}^{1} \frac{1}{\sqrt{X_{0}^{0}}} dQ_{s} \int_{0}^{1} X_{0}^{0} ds}{1 - \int_{0}^{1} X_{0}^{0} ds \int_{0}^{1} \frac{1}{X_{0}^{0}} ds}.$$

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The proof is complete.

Remark 1: If the Cox-Ingersoll-Ross model is driven by \( \alpha \)-stable motion \( Z = \{ Zt, t \geq 0 \} \) with index \( \alpha \), where \( Z \) satisfies that for any \( 0 \leq s < t \), \( \mathbb{E}[e^{iu(Zt - Zs)}|F_s] = e^{-(t-s)\Psi_\alpha(u)} \), \( \Psi_\alpha(u) \) is the Lévy symbol of \( Z \). The model is described as follows:

\[
\begin{cases}
    dX_t = (\theta - \lambda X_t)dt + \sqrt{X_t}dZ_t, & t \geq 0 \\
    X_0 = x_0.
\end{cases}
\]

We introduce the following contrast function:

\[ \rho_n(\theta, \lambda) = \sum_{i=1}^{n} |X_{ti} - X_{ti-1} - (\theta - \lambda X_{ti-1})\Delta t_{i-1}|^2, \]

where \( \Delta t_{i-1} = t_i - t_{i-1} = h \).

Then, we can obtain the explicit expression of estimators

\[
\begin{align*}
    \hat{\theta}_n &= \frac{-\sum_{i=1}^{n} (X_{ti} - X_{ti-1})(\sum_{j=1}^{i} X_{tj})}{n\sum_{i=1}^{n} X_{ti}^2 - \sum_{i=1}^{n} (\sum_{j=1}^{i} X_{tj})^2} \\
    \hat{\lambda}_n &= \frac{-\sum_{i=1}^{n} X_{ti}^2 - \sum_{i=1}^{n} (\sum_{j=1}^{i} X_{tj})^2}{n\sum_{i=1}^{n} (X_{ti} - X_{ti-1})^2 - \sum_{i=1}^{n} (\sum_{j=1}^{i} X_{tj} - X_{ti-1})^2} \\
    \hat{\sigma}^2 &= \frac{1}{n\Delta} \sum_{i=1}^{n} X_{ti}^2 - X_{ti-1}^2 - (\hat{\alpha} - \hat{\beta} X_{ti-1}) \Delta t_{i-1}.
\end{align*}
\]

The expression of estimators are different from that in (4). Therefore, the methods used to discuss the consistency and asymptotic distribution of the estimators are different as well, and this is the further topics to consider.

V. COX-INGERSOLL-ROSS MODEL DRIVEN BY BROWNIAN MOTION

We consider Cox-Ingersoll-Ross model driven by Brownian motion, which is described as follows:

\[
\begin{cases}
    dX_t = (\alpha - \beta X_t)dt + \sigma \sqrt{X_t}dW_t \\
    X_0 = x_0,
\end{cases}
\]

where \( W_t \) is a Wiener process modeling the random market risk factor, \( \alpha, \beta \) and \( \sigma \) are unknown parameters.

It is assumed that the process is observed at times \( \{ t_0, t_1, \ldots, t_n \} \) where \( t_i = i\Delta, i = 1, 2, \ldots, n \) and \( 0 < \Delta < \frac{2}{\sigma} \).

Discretizing equation (37), it follows that

\[ X_{ti} - X_{ti-1} = (\alpha - \beta X_{ti-1})\Delta + \sigma \sqrt{X_{ti-1}} \Delta \varepsilon_t, \]

where \( \varepsilon_t \) is a i.i.d. \( N(0,1) \) sequence and for every \( i, \varepsilon_t \) is independent with \( \{ X_{t_j}, j < i \} \).

Then, it is easy to check that

\[
\begin{align*}
    \hat{\alpha} &= \frac{\sum_{i=1}^{n} X_{ti} - \sum_{i=1}^{n} X_{ti-1}}{n^2 - \sum_{i=1}^{n} X_{ti} - \sum_{i=1}^{n} X_{ti-1}} \\
    \hat{\beta} &= \frac{\sum_{i=1}^{n} (X_{ti} - X_{ti-1}) - \sum_{i=1}^{n} \frac{1}{X_{ti-1}}}{n^2 - \sum_{i=1}^{n} X_{ti} - \sum_{i=1}^{n} X_{ti-1}} + \frac{\sum_{i=1}^{n} X_{ti}^2 - X_{ti-1}^2}{n^2 - \sum_{i=1}^{n} X_{ti} - \sum_{i=1}^{n} X_{ti-1}} \\
    \hat{\sigma}^2 &= \frac{1}{n\Delta} \sum_{i=1}^{n} X_{ti}^2 - X_{ti-1}^2 - (\hat{\alpha} - \hat{\beta} X_{ti-1}) \Delta t_{i-1}.
\end{align*}
\]

Next we will prove the consistency of \( \hat{\sigma}^2 \).

Note that

\[
\begin{align*}
    (X_{ti} - X_{ti-1} - (\alpha - \beta X_{ti-1})\Delta)^2 &= (X_{ti} - X_{ti-1})^2 + [(\alpha - \beta X_{ti-1})\Delta]^2 \\
    &- 2(X_{ti} - X_{ti-1})(\alpha - \beta X_{ti-1})\Delta.
\end{align*}
\]

According to Itô’s lemma,

\[
\begin{align*}
    d(X_{ti} - X_{ti-1})^2 &= 2(X_{ti} - X_{ti-1})(\alpha - \beta X_{ti})dt_u \\
    &+ \sigma^2 X_{ti} dt_u + 2(X_{ti} - X_{ti-1})\sigma \sqrt{X_{ti}} dW_t,
\end{align*}
\]

it follows that

\[
\begin{align*}
    (X_{ti} - X_{ti-1})^2 &= 2\int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti}) dt_u \\
    &+ \sigma^2 \int_{t_{i-1}}^{t_i} X_{ti} dt_u \\
    &+ 2\int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})\sigma \sqrt{X_{ti}} dW_t.
\end{align*}
\]

Hence

\[
\begin{align*}
    |\hat{\sigma}^2 - \sigma^2|^2 &= \left| \frac{2}{n\Delta} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti}) dt_u \\
    &+ \frac{2}{n\Delta} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_{ti} \sigma \sqrt{X_{ti}} dW_t \\
    &+ \frac{\sigma^2}{n\Delta} \sum_{i=1}^{n} X_{ti} dt_u \\
    &+ \frac{1}{n\Delta} \sum_{i=1}^{n} ([\alpha - \beta X_{ti-1}]\Delta)^2 \\
    &- 2\int_{t_{i-1}}^{t_i} \frac{X_{ti} dt_u}{X_{ti-1}} \\
    &- \frac{2}{n^2\Delta} \sum_{i=1}^{n} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti-1})\Delta - \sigma^2|^2 \\
    &\leq \frac{16}{n^2\Delta} \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti}) dt_u \right|^2 \\
    &+ \frac{16\sigma^2}{n^2\Delta} \left| \int_{t_{i-1}}^{t_i} X_{ti} \sigma \sqrt{X_{ti}} dW_t \right|^2 \\
    &+ \frac{4\sigma^4}{n} \left| \int_{t_{i-1}}^{t_i} X_{ti} dt_u \right|^2 \\
    &+ \frac{4\Delta^2}{n} \left| \int_{t_{i-1}}^{t_i} ([\alpha - \beta X_{ti-1}]\Delta)^2 \\
    &+ \frac{16}{n\Delta} \left| \int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti-1})\Delta \right|^2.
\end{align*}
\]

Applying the Holder’s inequality and the Cauchy-Schwarz inequality,

\[
\begin{align*}
    &\leq \frac{16}{n^2\Delta} \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti}) dt_u \right|^2 \\
    &+ \frac{16\sigma^2}{n^2\Delta} \left| \int_{t_{i-1}}^{t_i} X_{ti} \sigma \sqrt{X_{ti}} dW_t \right|^2 \\
    &+ \frac{4\sigma^4}{n} \left| \int_{t_{i-1}}^{t_i} X_{ti} dt_u \right|^2 \\
    &+ \frac{4\Delta^2}{n} \left| \int_{t_{i-1}}^{t_i} ([\alpha - \beta X_{ti-1}]\Delta)^2 \\
    &+ \frac{16}{n\Delta} \left| \int_{t_{i-1}}^{t_i} (X_{ti} - X_{ti-1})(\alpha - \beta X_{ti-1})\Delta \right|^2.
\end{align*}
\]

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inequality, we have
\[
E\left[\int_{t_{i-1}}^{t_i} \frac{(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})}{X_{t_{i-1}}} dt_u\right]^2 \leq \Delta \int_{t_{i-1}}^{t_i} \frac{1}{X_{t_{i-1}}^2} \left(\frac{(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})}{X_{t_{i-1}}}\right)^2 dt_u,
\]
from Assumption 4 we get that \(E\left[\frac{1}{X_{t_{i-1}}^2}\right]^4\) is bounded. Furthermore, \(E\left[(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})\right]^4 \leq \Delta \int_{t_{i-1}}^{t_i} \frac{1}{X_{t_{i-1}}^2} \left(\frac{(X_{t_i} - X_{t_{i-1}})\right)^4 dt_u,\)
\[
\Delta \int_{t_{i-1}}^{t_i} \frac{1}{X_{t_{i-1}}^2} \left(\frac{(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})}{X_{t_{i-1}}}\right)^2 dt_u,
\]
and Assumption 4 we know that \(E[\alpha - \beta X|^0| \leq 28\Delta^2(1 + \sup \epsilon) < \infty.\)
Thus we obtain the result that
\[
E\left[\int_{t_{i-1}}^{t_i} \frac{(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})}{X_{t_{i-1}}} dt_u\right]^2 = O(\Delta^2). \quad (40)
\]
As
\[
E\left[\frac{(\hat{\alpha} - \beta X_{t_{i-1}})}{X_{t_{i-1}}}^2\right] = E\left[\frac{1}{X_{t_{i-1}}} \right]^4 \frac{(\hat{\alpha} - \beta X_{t_{i-1}})}{X_{t_{i-1}}} \right]\frac{2}{\Delta \int_{t_{i-1}}^{t_i} \frac{1}{X_{t_{i-1}}^2} \left(\frac{(X_{t_i} - X_{t_{i-1}})}{X_{t_{i-1}}}\right)^4 dt_u,\)
\]
it is easy to check that \(E\left[\frac{(\hat{\alpha} - \beta X_{t_{i-1}})}{X_{t_{i-1}}}^2\right]\) is bounded. Then look at \(E\left[(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})\right]^4 = O(\Delta^2),\) it follows that
\[
E\left[\frac{(X_{t_i} - X_{t_{i-1}})(\alpha - \beta X_{t_{i-1}})}{X_{t_{i-1}}}\right]^2 = O(\Delta). \quad (41)
\]
Hereafter, we consider \(E\left[\int_{t_{i-1}}^{t_i} \frac{X_n}{X_{t_{i-1}}} dt_u - 1\right]^2\). As
\[
E\left[\int_{t_{i-1}}^{t_i} \frac{X_n}{X_{t_{i-1}}} dt_u - 1\right]^2 = \frac{1}{\Delta^2} E\left[\int_{t_{i-1}}^{t_i} \frac{X_n}{X_{t_{i-1}}} dt_u - 1\right]^2 \leq \frac{1}{\Delta^2} E\Delta \int_{t_{i-1}}^{t_i} \left|\frac{X_n}{X_{t_{i-1}}}\right|^2 dt_u,
\]
\[
\leq \frac{1}{\Delta^2} \int_{t_{i-1}}^{t_i} \left(\frac{1}{X_{t_{i-1}}}\right)^2 \frac{E|X_n - X_{t_{i-1}}|^4}{dt_u}, \quad (42)
\]
and \(E|X_n - X_{t_{i-1}}|^4 = O(\Delta^2),\) we obtain that
\[
E\left[\int_{t_{i-1}}^{t_i} \frac{X_n}{X_{t_{i-1}}} dt_u - 1\right]^2 = O(\Delta). \quad (41)
\]
Finally, we study \(E\left[\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{X_n - X_{t_{i-1}}}{X_{t_{i-1}}} \sqrt{X_{t_{i-1}}} dW_{t_{i-1}}^2\right].\) Let
\[
N_t = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{X_n - X_{t_{i-1}}}{X_{t_{i-1}}} \sqrt{X_{t_{i-1}}} dW_{t_{i-1}}. \quad (43)
\]
It is obvious that \(N_t\) is a martingale. From the martingale moment inequality, there exists a constant C such that
\[
E|N_t|^2 \leq CE(N)^t, \quad (44)
\]
where \(\langle N\rangle_t\) denotes the quadratic variation of \(N_t\). Note that
\[
\langle N\rangle_t = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{X_n - X_{t_{i-1}}}{X_{t_{i-1}}} \sqrt{X_{t_{i-1}}} dW_{t_{i-1}}, \quad (45)
\]
and
\[
E(N)^t = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{X_n}{X_{t_{i-1}}}\right)^2 \frac{E|X_n - X_{t_{i-1}}|^4}{dt_u}, \quad (46)
\]
which is bounded and
\[
E|X_n - X_{t_{i-1}}|^4 = O(|t_n - t_{i-1}|^2), \quad (47)
\]
then we have
\[
E(\Delta^2) \Rightarrow \Delta \rightarrow \infty. \quad (48)
\]
As a result
\[
E|\frac{\sigma - \sigma_0|^2} = O(\Delta^2) \Rightarrow \Delta \rightarrow \infty. \quad (48)
\]
Therefore, \(E|\frac{\sigma - \sigma_0|^2} = O(\Delta) \Rightarrow \Delta \rightarrow \infty. \quad (48)
\]
VI. SIMULATION
In this experiment, we generate a discrete sample \((X_{i})_{i=0,1,...,n}\) and compute \(\hat{\alpha}_n, \hat{\beta}_n, \hat{\alpha}_n, \hat{\beta}_n,\) and \(\hat{\alpha}_n, \hat{\beta}_n,\) from the sample. We let \(\sigma = 0.5, x_0 = 0.1.\) For every given true value of the parameters \(\alpha, \beta,\) the size of the sample is represented as “Size n” and given in the first column of the table. In Table 1, \(\epsilon = 0.05,\) the size is increasing from 500 to 3000. In Table 2, \(\epsilon = 0.001,\) the size is increasing from 5000 to 30000. The tables list the value of \(\alpha_0 - LSE; \beta_0 - LSE;\) and the absolute errors (AE) of LSE. LSE means least squares estimator.
Two tables illustrate that when \(n\) is large enough and \(\epsilon\) is small enough, the obtained estimators are very close to the true parameter value. Therefore, the methods used in this paper are effective and the obtained estimators are good.
Next we give some simulation results of the confidence interval of \(\alpha_0, \beta_0\) and \(\alpha_0, \beta_0\) under 0.95 confidence level. In Table 3 and Table 4, We let \(\sigma = 0.5, x_0 = 0.1.\) For every given true value of \(\alpha_0, \beta_0,\) the size of the sample is increasing from 5000 to 10000. These tables list the value of \(\alpha_0 - LSE; \beta_0 - LSE;\) and the last column of the table list the confidence interval of \(\alpha_0, \beta_0.\) Table 3 and Table 4 illustrate that the length of the confidence interval is becoming small when the size of the sample is increasing.
### TABLE I
LSE SIMULATION RESULTS OF $\alpha_0$ AND $\beta_0$

<table>
<thead>
<tr>
<th>True</th>
<th>Aver</th>
<th>AE</th>
<th>Size n</th>
<th>$\alpha_0$</th>
<th>$\beta_0$</th>
<th>$\alpha_0 - \text{LSE}$</th>
<th>$\beta_0 - \text{LSE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha_0, \beta_0)$</td>
<td></td>
<td></td>
<td>500</td>
<td>0.9542</td>
<td>1.9452</td>
<td>0.0458</td>
<td>0.0548</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td>0.9628</td>
<td>2.0467</td>
<td>0.0372</td>
<td>0.0467</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3000</td>
<td>0.9765</td>
<td>2.0326</td>
<td>0.0235</td>
<td>0.0326</td>
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<tr>
<td>$(1,2)$</td>
<td></td>
<td></td>
<td>500</td>
<td>1.9586</td>
<td>2.9610</td>
<td>0.0414</td>
<td>0.0390</td>
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<tr>
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<td></td>
<td></td>
<td>1000</td>
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<td>3.0224</td>
<td>0.0371</td>
<td>0.0224</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>3000</td>
<td>2.0189</td>
<td>3.0123</td>
<td>0.0189</td>
<td>0.0123</td>
</tr>
<tr>
<td>$(2,3)$</td>
<td></td>
<td></td>
<td>500</td>
<td>2.9496</td>
<td>3.9438</td>
<td>0.0504</td>
<td>0.0562</td>
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<tr>
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<td></td>
<td></td>
<td>1000</td>
<td>3.0377</td>
<td>4.0397</td>
<td>0.0377</td>
<td>0.0397</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>3000</td>
<td>3.0214</td>
<td>4.0205</td>
<td>0.0214</td>
<td>0.0205</td>
</tr>
<tr>
<td>$(3,4)$</td>
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<td></td>
<td>500</td>
<td>1.9965</td>
<td>2.0005</td>
<td>0.0035</td>
<td>0.0021</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1000</td>
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<td>2.0001</td>
<td>0.0004</td>
<td>0.0004</td>
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<td>2.0004</td>
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<td>0.0026</td>
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### TABLE III
SIMULATION RESULTS OF CONFIDENCE INTERVAL OF $\alpha_0$

<table>
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<tr>
<th>True</th>
<th>Aver</th>
<th>AE</th>
<th>Size n</th>
<th>$\alpha_0$</th>
<th>$\alpha_0 - \text{LSE}$</th>
<th>Confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2000</td>
<td>0.9963</td>
<td>[0.9825, 1.2246]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td>1.0018</td>
<td>[0.9917, 1.1148]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>2000</td>
<td>1.9951</td>
<td>[1.9792, 2.3423]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td>2.0014</td>
<td>[1.9845, 2.2247]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2000</td>
<td>2.9954</td>
<td>[2.9763, 3.3592]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
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<td>3.0015</td>
<td>[2.9932, 3.1221]</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE IV
LSE SIMULATION RESULTS OF $\alpha_0$ AND $\beta_0$

<table>
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<th>Aver</th>
<th>AE</th>
<th>Size n</th>
<th>$\alpha_0$</th>
<th>$\beta_0$</th>
<th>$\alpha_0 - \text{LSE}$</th>
<th>$\beta_0 - \text{LSE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha_0, \beta_0)$</td>
<td></td>
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<td>2.0032</td>
<td>2.0005</td>
<td>[1.9875, 2.2354]</td>
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<td>2.0000</td>
<td>[2.0001, 2.1032]</td>
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</tr>
<tr>
<td></td>
<td></td>
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<td>3.0009</td>
<td>3.0005</td>
<td>[2.9986, 3.1105]</td>
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</tr>
<tr>
<td></td>
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<td></td>
<td>10000</td>
<td>4.0021</td>
<td>4.0010</td>
<td>[3.9875, 4.3889]</td>
<td></td>
</tr>
</tbody>
</table>

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VII. CONCLUSION

In this paper, the parameter estimation for CIR model with small Lévy noises has been studied from discrete observations. The least squares method has been used to obtain the estimators. The explicit formula of the estimation error has been given and the consistency of the least squares estimators has been proved. The asymptotic distribution of the estimation error has been discussed as well. Further research topics will include the parameter estimation for general nonlinear stochastic differential equations driven by Lévy noises.

REFERENCES


