Dynamic Analysis of a Logarithmic Population Model with Piecewise Constant Arguments

Yongzhi Liao, Qilin Tang, Guofang Wu and Tianwei Zhang

Abstract—**In this paper, we consider a logarithmic population model with piecewise constant arguments. First, we study the uniqueness and existence range of the equilibrium point of the model. After that, by using the linearized stability theorem, the semicycle property and a suitable Lyapunov function, some sufficient conditions are obtained for the local and global asymptotic stability of the equilibrium point and the damped oscillation of positive solutions of the model. Finally, some examples with computer simulations are given to illustrate the main results in this paper.**

Index Terms—**Logarithmic population model, Stability, Boundedness, Semicycle, Damped oscillation.**

I. Introduction

It is well known that studies of differential equations with a piecewise constant argument are motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both the differential and difference equations. These equations play an important role in investigating the existence, uniqueness and the asymptotic behavior of the solutions of the equations in numerous applications and other aspects, referred to [1-7].

Because of the existence of many population models in real world, the logarithmic population model has recently attracted the attention of many mathematicians and biologists, see various types of logarithmic models in Refs. [8-15]. One can easily see that all equations considered in the above mentioned papers are subject to the existence and stability of their periodic solutions or almost periodic solutions by using the methods of some fixed point theorems and Lyapunov functions. However, there are few papers concerning the existence and stability of the equilibrium points of logarithmic population models. Especially, for the logarithmic population models with piecewise constant arguments, very few results can be found in the literature.

Due to the linearized stability theorem in [16] and the known result about semicycle property in [17], we shall study the stability of a unique equilibrium point and the semicycle property of positive solutions of the following logarithmic population model with piecewise constant arguments:

$$\begin{aligned} x'(t) = {} & x(t)\{r - a_0x(t) \\ & -a_1\ln(x([t])) - a_2\ln(x([t-1]))\}, \end{aligned} \quad (1.1)$$

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Yongzhi Liao is with the School of Mathematics and Computer Science, Panzhihua University, Panzhihua, Sichuan 617000, China. (mathyzliao@163.com).

Qilin Tang is with the School of Mathematics and Computer Science, Panzhihua University, Panzhihua, Sichuan 617000, China. (mathtang@126.com).

Guofang Wu is with the Affiliated Hospital, Panzhihua University, Panzhihua, Sichuan 617000, China. (627097605@qq.com).

Tianwei Zhang is with the City College, Kunming University of Science and Technology, Kunming 650051, China. (zhang@kmust.edu.cn).

Correspondence author: Tianwei Zhang. (zhang@kmust.edu.cn).

where $x(t)$ is the size of population, the parameters r, a_0, a_1 and a_2 are positive real numbers and $[t]$ denotes the integer part of $t \in [0, +\infty)$. We emphasize that these parameters play important roles to determine the local and global asymptotic stability of a positive equilibrium point of Eq. (1.1).

Throughout this paper, we assume that the initial conditions $x(-1)$, $x(0)$ of Eq. (1.1) are positive numbers.

This paper is organized as follows: In Section 2, we study the uniqueness and existence range of the equilibrium point of Eq.(1.1) and prove that the equilibrium point is locally asymptotically stable and the solutions of Eq. (1.1) are bounded. In Section 3, the semicycle and oscillation of a discrete solution (which is positive) of Eq. (1.1) is investigated. Several illustrative example is shown in Section 4.

II. Preliminaries

In this section, we shall state the following definitions and lemmas, which will be useful in proving our main result.

By [6] and [16], let $I \subseteq \mathbb{R}$ be an interval, bounded or not, and let

$$f : I \times I \longrightarrow I$$

be a continuously differentiable function. For every set of initial conditions $\{x_{-1}, x_0\} \subset I$ the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 1, 2, \ldots \quad (2.1)$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Let $\bar{x}$ be an equilibrium point of Eq. (2.1), i.e., $\bar{x} = f(\bar{x}, \bar{x})$. If we replace x_n and x_{n-1} in Eq. (2.1) by the variables u and v respectively, then we have

$$p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}), \quad q = \frac{\partial f}{\partial v}(\bar{x}, \bar{x}).$$

The equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 1, 2, \ldots \quad (2.2)$$

is called the linearized equation associated with Eq. (2.2) about the equilibrium point $\bar{x}$. Its characteristic equation is

$$\lambda^2 - p\lambda - q = 0. \quad (2.3)$$

Lemma 1. (*Linearized Stability* [16]).

(1) *If both roots of the quadratic Eq.* (2.3) *lie in the open disk* $|\lambda| < 1$, *then the equilibrium point* $\bar{x}$ *of Eq.* (2.1) *is locally asymptotically stable.*

(2) *A necessary and sufficient condition for both roots of Eq.* (2.3) *to lie in the open unit disk* $|\lambda| < 1$ *is*

$$|p| < 1 - q < 2.$$

In this case $\bar{x}$ *is locally asymptotically stable, where the locally asymptotically stable* $\bar{x}$ *is also called a sink.*

(3) *A necessary and sufficient condition for both roots of Eq.* (2.3) *lying outside the open disk* $|\lambda| < 1$ *is*

$$|q| > 1, \quad |p| < |1 - q|.$$

In this case $\bar{x}$ *is unstable and called a repeller point.*

(4) *A necessary and sufficient condition for one of the roots of Eq.* (2.3) *lying outside the open disk* $|\lambda| < 1$ *and the other root inside is*

$$p^2 + 4q > 0, \quad |p| > |1 - q|.$$

In this case $\bar{x}$ *is unstable and called a saddle point.*

Lemma 2. [17] *Assuming that* $f \in C[(0,\infty) \times (0,\infty),(0,\infty)]$ *and that* $f(x,y)$ *is decreasing in both arguments. Let* $\bar{x}$ *be a positive equilibrium point of Eq.* (2.1). *Then even oscillatory solutions of Eq.* (2.1) *has semicycle of length at most two.*

One can know that the simplest logistic differential equation with piecewise constant arguments can be written as

$$x'(t) = x(t)\left\{r - \frac{x([t])}{k}\right\}, \quad t \geqslant 0, \quad r,\ k \in (0,+\infty). \quad (2.4)$$

By integrating Eq. (2.4) on any interval of the form $[n, n+1)$ for $n = 1, 2, \ldots$, and taking limits as $t \to n+1$, we get

$$x(n+1) = x(n)\big(e^{r - \frac{x(n)}{k}}\big), \quad n = 1, 2, \ldots. \quad (2.5)$$

The asymptotic behavior of the solutions of Eq. (2.5) was considered by May [18], and May and Oster [19].

Following the same idea and the same method in Eq. (2.4), one can easily derive the following discrete analogues of Eq. (1.1):

$$x(t) = x(n)\big(e^{\int_n^t (r - a_0 x(s) - a_1 \ln(x(n)) - a_2 \ln(x(n-1))\, \mathrm{d}s}\big). \quad (2.6)$$

Distinctly, the initial conditions $x([0]) = x(0) > 0$, and the solutions of Eq. (1.1) are also positive.

Therefore, from (1.1), we have

$$\begin{aligned} & x'(t) - x(t)[r - a_1 \ln(x(n)) - a_2 \ln(x(n-1))] \\ = \ & -a_0 x^2(t), \quad n \leqslant t < n+1. \end{aligned} \quad (2.7)$$

Let $m_1 = r - a_1 \ln(x(n)) - a_2 \ln(x(n-1))$, $x = x(t)$, then Eq. (2.7) is changed to

$$-x' + m_1 x = a_0 x^2, \quad n \leqslant t < n+1. \quad (2.8)$$

If $m_1 = 0$, on the one hand, the Eq. (2.8) is changed to $-x' = a_0 x^2$, its solution is $x(t) = \frac{1}{a_0 t + x(0)}$. On the other hand, $r - a_1 \ln(x(n)) - a_2 \ln(x(n-1)) = 0$, we get $x^{a_1}(n) x^{a_2}(n-1) = e^r$. Therefore, we have

$$\frac{1}{(a_0 n + x(0))^{a_1}} \frac{1}{(a_0(n-1) + x(0))^{a_2}} = e^r. \quad (2.9)$$

However, when $n \to \infty$, the left of equation (2.9) converges to 0, which is contrary. Therefore, there is a natural number N_0, when $n > N_0$, $m_1 \neq 0$.

In the following investigation of this paper, we assume $m_1 \neq 0$.

From (2.8), we get

$$\frac{d}{dt}\left(\frac{e^{m_1 t}}{x}\right) = a_0 e^{m_1 t}, \quad n \leqslant t < n+1. \quad (2.10)$$

Using (2.10) and by letting $t \to n+1$ for $n = 1, 2, \ldots$, we obtain the solution of (2.7) as

$$x(n+1) = \frac{x(n) e^{m_1}}{1 + a_0 x(n) \frac{e^{m_1} - 1}{m_1}}. \quad (2.11)$$

To investigate the solution of Eq. (1.1) in more detail, we need to investigate the behavior of Eq. (2.11).

III. Stability and Boundedness of solutions of Eq. (1.1)

First of all, we need to determine the identity of equilibrium points of Eq. (2.11), where these equilibrium points are also the critical points of Eq. (1.1).

Theorem 1. *Eq.* (2.11) *has unique equilibrium point* $\bar{x}$ *satisfying* $0 < \bar{x} < e^{\frac{r}{a_1 + a_2}}$.

Proof: From (2.6), we have $x(t) > 0$, that is, $x(n) > 0$.

From Eq. (2.11), using $x(n+1) - x(n) = 0$, and $m_1 \neq 0$, we get

$$(e^{m_1} - 1) - a_0 x(n) \frac{e^{m_1} - 1}{m_1} = 0,$$

that is

$$x(n) = \frac{1}{a_0}(r - a_1 \ln(x(n)) - a_2 \ln(x(n-1))).$$

Utilizing $x(n) - x(n-1) = 0$ again, we get the positive equilibrium point $\bar{x}$ of Eq. (2.11) satisfying

$$\bar{x} = \frac{1}{a_0}(r - (a_1 + a_2)\ln(\bar{x})),$$

or,

$$(a_1 + a_2)\ln(\bar{x}) = r - a_0 \bar{x}. \quad (3.1)$$

Clearly, the exact expression of the solution $\bar{x}$ of Eq. (3.1) is difficult to obtain, but it exists and satisfies $0 < \bar{x} < e^{\frac{r}{a_1 + a_2}}$.

Indeed, let

$$\begin{cases} y = (a_1 + a_2)\ln x, \\ y = r - a_0 x. \end{cases} \quad (3.2)$$

Obviously, if (x, y) is a solution of the system (3.2), then x is a equilibrium point of Eq. (2.11). On the other hand, the logarithmic curve $y = (a_1 + a_2)\ln x$ and beeline $y = r - a_0 x$ always intersect in $(0, e^{\frac{r}{a_1 + a_2}})$.

So there is a unique solution of system (3.2), and its abscissa is the equilibrium point $\bar{x}$ of Eq. (2.11). ■

Let $m = a_0 \bar{x} = r - (a_1 + a_2)\ln \bar{x}$, from Eq. (2.11), we get the characteristic equation of the form:

$$\lambda^2 - \left(-\frac{a_1}{m} + \frac{1}{e^m} + \frac{a_1}{m e^m}\right)\lambda - a_2\left(\frac{1}{m e^m} - \frac{1}{m}\right) = 0. \quad (3.3)$$

Clearly, $m > 0$, and the local stability of equilibrium point $\bar{x}$ depends on the roots of the characteristic of Eq. (3.3). To analyse the local stability of the positive equilibrium point, we use the linearized stability as follows

Theorem 2. *If* $2a_2 - a_1 > 0$, *then the positive equilibrium point of Eq.* (2.11) *is locally asymptotically stable if and only if*

$$\frac{1}{m} - \frac{1}{m e^m} < \frac{1}{a_2}.$$

Proof: From Lemma 2.1 (1) and (2), the positive equilibrium point of Eq. (2.11) is locally asymptotically stable if and only if

$$|-\frac{a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m}|<1-a_2(\frac{1}{me^m}-\frac{1}{m})<2. \quad (3.4)$$

From (3.4), two cases can be considered here which are:

(a) $\quad |-\frac{a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m}|<1-a_2(\frac{1}{me^m}-\frac{1}{m})$,
(b) $\quad 0<1-a_2(\frac{1}{me^m}-\frac{1}{m})<2$.

From (b), we obtain

$$\frac{1}{m}-\frac{1}{me^m}<\frac{1}{a_2}, \quad \text{that is,} \quad me^m>a_2(e^m-1). \quad (3.5)$$

From (a), we get

$$\pm\big(-\frac{a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m}\big)<1-a_2(\frac{1}{me^m}-\frac{1}{m}). \quad (3.6)$$

According to $2a_2-a_1>0$, $m>0$ and (3.5), we have $e^m(m+a_2-a_1)>a_2-a_1-m$, hence, (3.6) holds.

Therefore, we get that the positive equilibrium point of Eq. (2.11) which is locally asymptotically stable if and only if (3.5) holds. This completes the proof of Theorem 2. ■

Theorem 3. *If $2a_2-a_1>0$, then the positive equilibrium point $\bar{x}$ of Eq. (2.11) is unstable and called a repeller point if and only if*

$$\frac{1}{m}-\frac{1}{me^m}>\frac{1}{a_2}.$$

Proof: From Lemma 1, the positive equilibrium point of Eq. (2.11) is a repeller if an only if

$$|a_2(\frac{1}{me^m}-\frac{1}{m})|>1,$$

$$|\frac{-a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m}|<|1-a_2(\frac{1}{me^m}-\frac{1}{m})|.$$

Since $|a_2(\frac{1}{me^m}-\frac{1}{m})|>1$, we have

$$\frac{1}{m}-\frac{1}{me^m}>\frac{1}{a_2}. \quad (3.7)$$

By the inequality $|\frac{-a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m}|<|1-a_2(\frac{1}{me^m}-\frac{1}{m})|$, we have

(c) $\quad \frac{-a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m}<1-a_2(\frac{1}{me^m}-\frac{1}{m})$,
(d) $\quad -(\frac{-a_1}{m}+\frac{1}{e^m}+\frac{a_1}{me^m})<1-a_2(\frac{1}{me^m}-\frac{1}{m})$.

That is easy to prove (c) holds.

From (d), we get

$$\bigg(\frac{1}{m}-\frac{1}{me^m}\bigg)(a_1-a_2)<\frac{1}{e^m}+1. \quad (3.8)$$

If $a_1-a_2\leq 0$, then the inequality (3.8) holds.

If $a_1-a_2>0$, from (3.7) and $2a_2-a_1>0$, we get

$$\frac{1}{e^m}+1>\frac{a_1-a_2}{a_2}=\frac{a_1}{a_2}-1 \quad (3.9)$$

Distinctly, the inequality (3.9) holds. This completes the proof of Theorem 3. ■

We can proof that there is no saddle point for Eq.(2.11).

Theorem 4. *Every positive solution x of Eq. (2.11) is bounded with the bound $(0,\frac{r}{a_0(1-e^{-r})})$.*

Proof: Let $\{x(n)\}_{n=-1}^{\infty}$ be a positive solution of Eq. (2.11).

If $0<x(n)\leqslant 1$,for $n=-1,0,1,2,\ldots$, then the theorem has been proved.

Following, we prove it from two aspects.

(i) If $x(n)>1$ $(n=0,1,2,\ldots)$ and $m_1=r-a_1\ln(x(n))-a_2\ln(x(n-1))>0$. Since $\ln(x(k))>1>0(k\in N^+)$, thus, $0<m_1<r$ and $e^{-r}<e^{-m_1}<1$.

So we obtain

$$x(n+1)=\frac{x(n)m_1}{a_0x(n)+[m_1-a_0x(n)]e^{-m_1}}<\frac{r}{a_0(1-e^{-r})}.$$

Because $x(n+1)=\frac{m_1}{a_0+[\frac{m_1}{x(n)}-a_0]e^{-m_1}}<\frac{m_1}{a_0-a_0e^{-m_1}}$, and the function $y=\frac{x}{a_0-a_0e^{-x}}(0<x<r)$ are monotone increasing.

(ii) If $x(n)>1$ $(n=0,1,2,\ldots)$, and $m_1=r-a_1\ln(x(n))-a_2\ln(x(n-1))<0$, then $e^{-m_1}>1$.

Because

$$x(n+1)=\frac{-m_1}{[\frac{-m_1}{x(n)}+a_0]e^{-m_1}-a_0}<\frac{-m_1}{a_0e^{-m_1}-a_0}. (3.10)$$

Because the function $y=\frac{x}{a_0-a_0e^{-x}}(x<0)$ is monotone increasing, thus

$$y<\lim_{x\to 0}\frac{-x}{a_0e^{-x}-a_0}=\frac{1}{a_0}.$$

But $\frac{1}{a_0}<\frac{r}{a_0(1-e^{-r})}(r>0)$ hold. This completes the proof. ■

IV. Semicycle analysis

We believe that a semicycle analysis of the solutions of a scalar difference equation is a powerful tool for a detailed understanding of the entire character of solutions and often leads to straightforward proofs of their long term behavior.

we now give the definitions of positive and negative semicycle of a solutions of Eq. (2.1) relative to an the equilibrium point $\bar{x}$.

By [17], a positive semicycle of a solutions $\{x_n\}$ of Eq. (2.1) consists of a "string" of terms $\{x_l,x_{l+1},\ldots,x_m\}$, is all greater than or equal to the equilibrium point $\bar{x}$, with $l\geqslant -1$ and $m\leqslant\infty$ and such that

$$\text{either } l=-1, \text{ or, } l>-1 \text{ and } x_{l-1}<\bar{x}$$

and

$$\text{either } m=\infty, \text{ or, } m<\infty \text{ and } x_{m+1}<\bar{x}.$$

A negative semicycle of a solutions $\{x_n\}$ of Eq. (2.1) consists of a "string" of terms $\{x_l,x_{l+1},\ldots,x_m\}$, all less than the equilibrium point $\bar{x}$, with $l\geqslant -1$ and $m\leqslant\infty$ and such that

$$\text{either } l=-1, \text{ or, } l>-1 \text{ and } x_{l-1}\geqslant\bar{x}$$

and

$$\text{either } m=\infty, \text{ or, } m<\infty \text{ and } x_{m+1}\geqslant\bar{x}.$$

In the following, we shall apply the lemma 2 to analyse in detail the conditions of semicycle and damped oscillation of every oscillatory solution of Eq. (2.11)

Theorem 5. *Suppose $a_1\geqslant 1$, and $x^{a_1}y^{a_2}>e^{r+1}$, then $f(x,y)$ is decreasing in both arguments, where*

$$f(x,y)=\frac{x(r-a_1\ln x-a_2\ln y)}{a_0x+(r-a_1\ln x-a_2\ln y-a_0x)p_{xy}}, \quad (4.1)$$

where $p_{xy} = e^{-(r-a_1\ln x-a_2\ln y)}$, $f \in C[(0,\infty)\times(0,\infty),(0,\infty)]$.

Proof: We set $t = r - a_1\ln x - a_2\ln y$, then $f(x,y) = \frac{tx}{a_0x+(t-a_0x)e^{-t}}$. The first derivative of (4.1) with respect to x and y is respectively

$$\frac{\partial f}{\partial x} = \frac{A_1}{B} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{A_2}{B},$$

where

$$B = [a_0x + (t - a_0x)e^{-t}]^2,$$

$$A_1 = -a_0a_1x + [(1-a_1)t^2 + a_0a_1xt + a_0a_1x]e^{-t}]$$

and

$$A_2 = \frac{a_2x}{y}[-a_0x + (-t^2 + a_0xt + a_0x)e^{-t}].$$

After that, we shall proof the rationality of the assumption of theorem-self.

Let

$$g_1(t) = (1-a_1)t^2 + a_0a_1xt + a_0a_1x,$$

$$g_2(t) = -t^2 + a_0xt + a_0x.$$

If

$$a_1 > 1 \quad \text{and} \quad x^{a_1}y^{a_2} > e^{r+1}, \tag{4.2}$$

then, $t+1 = (r - a_1\ln x - a_2\ln y)+1 < 0$, we have $g_1(t) < 0$ and $g_2(t) < 0$, therefore, $A_1 < 0$ and $A_2 < 0$, then $f(x,y)$ is decreasing in both arguments.

From $g_1(t)$, since $1-a_1 < 0$, thus, $\Delta_1 = [a_0a_1x]^2 - 4(1-a_1)a_0a_1x > 0$ holds.

By $g_2(t)$, $\Delta_2 = [a_0x]^2 + 4a_0x > 0$ holds.

Since $\frac{a_0a_1x\pm\sqrt{\Delta_1}}{2(a_1-1)}$ are the two roots of equation $(1-a_1)t^2 + a_0a_1xt + a_0a_1x = 0$ with respect to t, and $\frac{a_0x\pm\sqrt{\Delta_2}}{2}$ are the two roots of equation $-t^2 + a_0xt + a_0x = 0$ with respect to t.

Because $\sqrt{\Delta_2} < 2 + a_0x$ holds,

$$2(a_1-1)\sqrt{\Delta_2} < 2(a_1-1)(2+a_0x),$$

that is,

$$[a_0a_1x]^2 - 4(1-a_1)a_0a_1x$$

$$> (a_0x)^2 + (a_1-1)^2[(a_0x)^2 + 4a_0x] + 2a_0x(a_1-1)\sqrt{\Delta_2}.$$

We can find

$$\sqrt{\Delta_1} > a_0x + (a_1-1)\sqrt{\Delta_2}, \tag{4.3}$$

then

$$\frac{a_0a_1x - \sqrt{\Delta_1}}{2(a_1-1)} < \frac{a_0x - \sqrt{\Delta_2}}{2}. \tag{4.4}$$

Because (4.3) holds, hence, $\sqrt{\Delta_1} > -a_0x + (a_1-1)\sqrt{\Delta_2}$ holds.

We can gain

$$\frac{a_0a_1x + \sqrt{\Delta_1}}{2(a_1-1)} > \frac{a_0x + \sqrt{\Delta_2}}{2}. \tag{4.5}$$

By (4.4) and (4.5), we have, when $t < \frac{a_0a_1x-\sqrt{\Delta_1}}{2(a_1-1)}$, $g_1(t) < 0$ and $g_2(t) < 0$ hold. Here $t > \frac{a_0a_1x+\sqrt{\Delta_1}}{2(a_1-1)}$ doesn't hold for $t < 0$.

Clearly, $a_0a_1x - \sqrt{\Delta_1} < 0$.

If $t < \frac{a_0a_1x-\sqrt{\Delta_1}}{2(a_1-1)}$, then

$$\sqrt{\Delta_1} < a_0a_1x - 2t(a_1-1),$$

and then

$$a_0a_1x(t+1) < t^2(a_1-1). \tag{4.6}$$

Distinctly, when $t+1 < 0$, the inequality (4.6) holds.

If $a_1 = 1$, then $g_2(t) < g_1(t) = a_0a_1x(t+1) < 0$ always holds for $t+1 < 0$.

If $1 - a_1 > 0$.

Because $\sqrt{\Delta_2} < 2 + a_0x$ holds, thus

$$2(1-a_1)\sqrt{\Delta_2} < 2(1-a_1)(2+a_0x),$$

that is,

$$[a_0a_1x]^2 - 4(1-a_1)a_0a_1x$$

$$< (a_0x)^2 + (1-a_1)^2[(a_0x)^2 + 4a_0x] - 2a_0x(1-a_1)\sqrt{\Delta_2}.$$

We can find

$$\sqrt{\Delta_1} < (1-a_1)\sqrt{\Delta_2} - a_0x, \tag{4.7}$$

then

$$\frac{-a_0a_1x - \sqrt{\Delta_1}}{2(1-a_1)} > \frac{a_0x - \sqrt{\Delta_2}}{2}. \tag{4.8}$$

Because (4.7) holds, hence, $\sqrt{\Delta_1} < a_0x + (1-a_1)\sqrt{\Delta_2}$ holds.

We can gain

$$\frac{-a_0a_1x + \sqrt{\Delta_1}}{2(1-a_1)} < \frac{a_0x + \sqrt{\Delta_2}}{2}. \tag{4.9}$$

By (4.8) and (4.9), when

$$t \in (\frac{-a_0a_1x - \sqrt{\Delta_1}}{2(1-a_1)}, \frac{-a_0a_1x + \sqrt{\Delta_1}}{2(1-a_1)})$$

$$\subset (\frac{a_0x - \sqrt{\Delta_2}}{2}, \frac{a_0x + \sqrt{\Delta_2}}{2}),$$

we have $g_1(t) < 0$ and $g_2(t) > 0$, and then $A_2 > 0$. therefore $f(x,y)$ is nondecreasing with respect to y.

Therefore, it completes to proof the rationality of the assumption of theorem-self, and completes the proof. ■

Theorem 6. *Let* $\{x(n)_{n=-1}^{\infty}\}$ *be a positive solution of Eq.* (2.11)*. The following statements are true.*

(I) *Let* $r - a_1\ln(x(n)) - a_2\ln(x(n-1)) < 0$*, then the sequence* $\{x(n)_{n=-1}^{\infty}\}$ *is monotone decreasing.*

(II) *Let* $r - a_1\ln(x(n)) - a_2\ln(x(n-1)) > a_0x(n)$*, then* $\{x(n)_{n=-1}^{\infty}\}$ *is monotone increasing.*

Proof: Let $m_1 = r - a_1\ln(x(n)) - a_2\ln(x(n-1))$.

(I) From Eq. (2.11) we can obtain

$$x(n+1) - x(n) = \frac{x(n)(m_1 - a_0x(n))(1 - e^{-m_1})}{a_0x(n) + [m_1 - a_0x(n)]e^{-m_1}}.$$

Since $m_1 = r - a_1\ln(x(n)) - a_2\ln(x(n-1)) < 0$, we have

$$m_1 - a_0x(n) < 0 \quad \text{and} \quad 1 - e^{-m_1} < 0.$$

Furthermore, from $e^{-m_1} > 1$, we get

$$a_0x(n) + [m_1 - a_0x(n)]e^{-m_1} < m_1 < 0, \tag{4.10}$$

which gives $x(n+1) - x(n) < 0$ for $n = 0, 1, 2, \ldots$.

(II) The proof is similar with (I) and will be omitted. ■

It can be shown that in Theorem 4. the sequence $\{x(n)_{n=-1}^{\infty}\}$ looks as follows:

If $r-a_1\ln(x(n))-a_2\ln(x(n-1))<0$, then for $x(n)<\bar{x}$, one can write

$$\ldots<x(n+2)<x(n+1)<x(n)<\bar{x},$$

and by assuming that $r-a_1\ln(x(n))-a_2\ln(x(n-1))>a_0x(n)$, then for $x(n)>\bar{x}$, we have

$$\ldots>x(n+2)>x(n+1)>x(n)>\bar{x}.$$

Theorem 7. *Let $\{x(n)_{n=-1}^{\infty}\}$ be a positive solution of Eq.* (2.11)*, and $x(n)\in(1,e^{\frac{r}{a_1+a_2}})$. Assume that*

$$\ln(\frac{x(2n-1)}{x(2n)})$$
$$>r-a_1\ln(x(2n))-a_2\ln(x(2n-1))>a_0x(2n),$$

if $x(2n)<\bar{x}<x(2n-1)$, then the solution of Eq. (2.11) *has damped oscillations.*

Proof: Set $M=r-a_1\ln(x(2n))-a_2\ln(x(2n-1))$ and $M>a_0x(2n)>0$.

Simplifying Eq. (2.11), we can write

$$x(2n+1)=\frac{x(2n)M}{a_0x(2n)+[M-a_0x(2n)]e^{-M}}.$$

Now, we consider the three cases as follows:

Case (1), we discuss

$$x(2n+1)-x(2n)=\frac{x(2n)(M-a_0x(2n))(1-e^{-M})}{a_0x(2n)+[M-a_0x(2n)]e^{-M}}.$$

Clearly, $x(2n+1)-x(2n)>0$ holds, which gives us that $x(2n+1)>x(2n)$, $n=1,2,\ldots$.

Likewise it can be shown that

$$x(2n+3)>x(2n+2),x(2n+5)>x(2n+4),\ldots.$$

Case (2), we debate

$$x(2n+1)-x(2n-1)$$
$$=\frac{x(2n)M}{a_0x(2n)+[M-a_0x(2n)]e^{-M}}-x(2n-1)$$
$$=\frac{M(x(2n)-x(2n-1))-a_0x(2n)x(2n-1)(1-e^{-M})}{a_0x(2n)+[M-a_0x(2n)]e^{-M}}.$$

Since $\ln(\frac{x(2n-1)}{x(2n)})>M>a_0x(2n)$, thus $x(2n+1)-x(2n-1)<0$, and then

$$x(2n-1)>x(2n+1)>x(2n+3)>x(2n+5)>\ldots.$$

Case (3), we study $x(2n+2)-x(2n)=\frac{C}{D}>0$.

Let $M_1=r-a_1\ln(x(2n+1))-a_2\ln(x(2n))$, then

$$x(2n+2)=\frac{x(2n+1)M_1}{a_0x(2n+1)+[M_1-a_0x(2n+1)]e^{-M_1}},$$

and then

$$D=a_0x(2n+1)+[M_1-a_0x(2n+1)]e^{-M_1},$$
$$C=x(2n+1)M_1-x(2n)D.$$

First, we proof $D>0$ by the conditions of theorem-self.

Since $x(2n+1)>x(2n)$ and $x(n)\in(0,e^{\frac{r}{a_1+a_2}})$, hence

$$M_1>r-(a_1+a_2)\ln(x(2n+1))$$
$$>r-(a_1+a_2)\frac{r}{(a_1+a_2)}=0$$

and

$$D=a_0x(2n+1)(1-e^{-M_1})+M_1e^{-M_1},\ D<M_1. \quad (4.11)$$

Next, we proof $C>0$.

By utilizing $x(2n)<x(2n+1)$ and (4.11), $C>0$ holds.

So we obtain $x(2n+2)>x(2n)$, iterating this result, we get that for $n=1,2,3,\ldots$,

$$\cdots>x(2n+4)>x(2n+2)>x(2n).$$

This completes the proof. ■

V. Global Behavior

In this section, by using a suitable Lyapunov function, we investigate the global asymptotic stability of the positive equilibrium point of the difference equation

$$x(n+1)=\frac{x(n)e^{m_1}}{1+a_0x(n)\frac{e^{m_1-1}}{m_1}},\quad n=1,2,\ldots,$$

where $m_1=r-a_1\ln(x(n))-a_2\ln(x(n-1))$.

Theorem 8. *Let the conditions of Theorem* 2.2 *hold. Furthermore assume that $m_1<0$. If $x(n)>2\bar{x}$, then the positive equilibrium point $\bar{x}$ of Eq.* (2.11) *is globally asymptotically stable.*

Proof: We consider a Lyapunov function $V(n)$ defined by

$$V(x(n))=[x(n)-\bar{x}]^2,\quad n=1,2,\ldots.$$

By Theorem 1, we can know the unique equilibrium point $\bar{x}$ of Eq. (2.11) satisfy

$$\bar{x}\in(0,e^{\frac{r}{a_1+a_2}}).$$

First, we calculate the upper left derivative of $V(n)$ along (4.1),

$$\begin{aligned}\Delta V(x(n)) &= V(x(n+1))-V(x(n))\\ &=[x(n+1)-\bar{x}]^2-[x(n)-\bar{x}]^2\\ &=[x(n+1)-x(n)][x(n+1)+x(n)-2\bar{x}].\end{aligned}$$

Applying the Theorem $6(I)$, we have $x(n+1)-x(n)<0$.

Because $m_1<0$, let $E=a_0x(n)+[m_1-a_0x(n)]e^{-m_1}$, from (3.10), $E<m_1<0$,

$$\begin{aligned}x(n+1)+x(n)-2\bar{x} &= \frac{x(n)m_1}{E}+x(n)-2\bar{x}\\ &> \frac{x(n)m_1}{E}>0.\end{aligned}$$

This implies that $\Delta V(n)<0$, which is also the condition for the global asymptotic stability of the positive equilibrium point of Eq. (2.11). ■

VI. Some examples

Example 5.1 From Theorem 2.2, by determining the parameters in Eq. (2.11) as $r=a_0=a_2=2$, $a_1=1$.

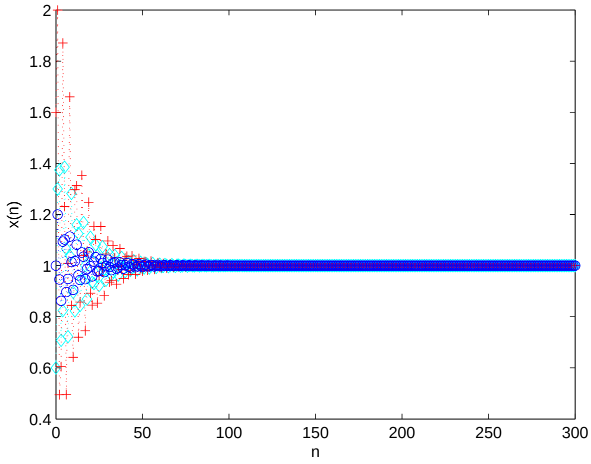

Fig. 1 Red "+" denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 0.6$ and $x(0) = 1.3$. Green " $\diamond$ " denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 1.6$ and $x(0) = 2$. Blue " $\circ$ " denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 1$ and $x(0) = 1.2$. When $r = a_0 = a_2 = 2$, $a_1 = 1$, the solution of Eq. (2.11) is locally asymptotically stable.

By easy calculation, $\bar{x} = 1$, $m = 2$. Furthermore, we have $\frac{1}{m} - \frac{1}{me^m} < \frac{1}{a_2}$. The graph of the first 300 iterations of Eq. (2.11) are given in figure 1. It can be seen that under the conditions given in Theorem 2.2, the solution of Eq. (2.11) is locally asymptotically stable.

Example 5.2 In Eq. (2.11), taking the values like $r = a_0 = 0.5$, $a_1 = 1$, $a_2 = 2$. By simple calculation, $\bar{x} = 1$, $m = 0.5$. Furthermore, we have $\frac{1}{m} - \frac{1}{me^m} > \frac{1}{a_2}$. The graph of the first 500 iterations of Eq. (2.11) is given in figure 2. It can be seen that under the conditions given in Theorem 2.3, the positive equilibrium point of Eq. (2.11) is a repeller.

Example 5.3 In Eq. (2.11), by taking the values of $r = a_0 = 1$, $a_1 = a_2 = 1.5$. By simple calculation, $\bar{x} = 1$, $m = 1$. Furthermore, when $x(n) > 2\bar{x}$, we have $m_1 = r - a_1 ln(x(n)) - a_2 ln(x(n-1)) < 1 - 3ln2 < 0$. The graph of the first 400 iterations of Eq. (2.11) is given in figure 3. It can be seen that under the conditions given in Theorem 4.1, the positive equilibrium point of Eq. (2.11) is global asymptotically stable.

VII. Conclusion

Throughout this paper, we can know that it is difficult to obtain the exact expression of the equilibrium point $\bar{x}$ of Eq. (2.11) about a logarithmic population model, but we have discussed locally asymptotically stable of the equilibrium point $\bar{x}$ of Eq. (2.11) and obtained that $\bar{x}$ is existent and unique, as well as satisfies $0 < \bar{x} < e^{\frac{r}{a_1+a_2}}$. Also, we get that the local behavior of the positive solution of the logarithmic population model depends on the conditions of the coefficients and proves that every positive solution tends to the equilibrium point $\bar{x}$ as $t \to +\infty$. In addition, we give a

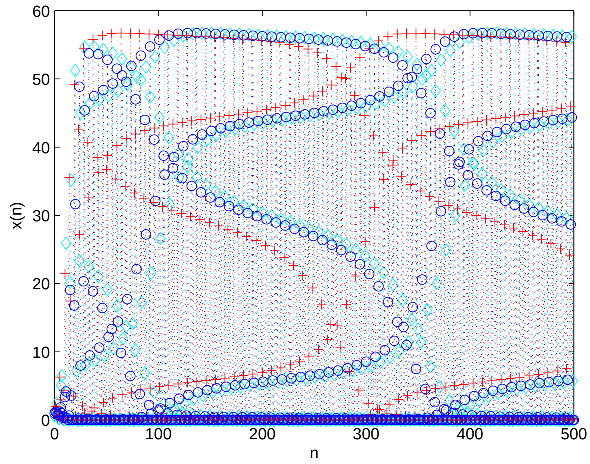

Fig. 2 Red "+" denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 0.6$ and $x(0) = 1.3$. Green " $\diamond$ " denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 1.6$ and $x(0) = 2$. Blue " $\circ$ " denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 1$ and $x(0) = 1.2$. When $r = a_0 = 0.5$, $a_1 = 1$, $a_2 = 2$, the positive equilibrium point of Eq. (2.11) is a repeller.

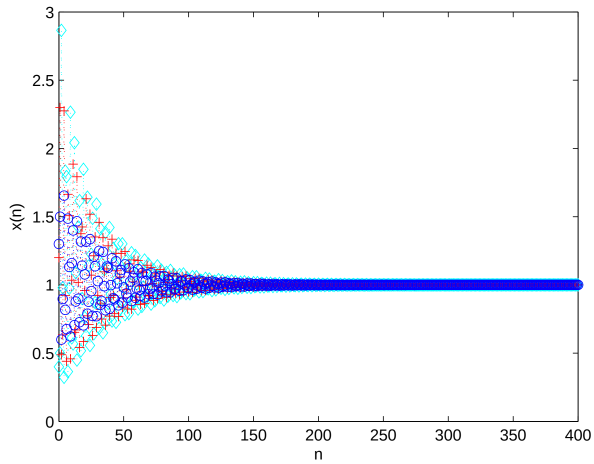

Fig. 3 Red "+" denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 0.4$ and $x(0) = 0.5$. Green " $\diamond$ " denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 1.2$ and $x(0) = 2.3$. Blue " $\circ$ " denotes the track of the graph of the iteration solution of x(n) for $x(-1) = 1.3$ and $x(0) = 1.5$. When $r = a_0 = 1$, $a_1 = a_2 = 1.5$, the positive equilibrium point of Eq. (2.11) is global asymptotically stable.

detailed description and conditions of semicycle and damped oscillation of the model.

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