Analytical Investigation of the MHD Jeffery-Hamel Flow Through Convergent and Divergent Channel by New Scheme

Abdul-Sattar J. A. Al-Saif, Abeer Majeed Jasim

***Abstract*—In this article, the problem of an incompressible viscous fluid between non parallel plane walls as known Jeffery-Hamel flow has been studied by using a new scheme. This new scheme depends on integrating n^{th} order of differential equation with known data which was performed using Taylor series expansion. The resulting solution is a new analytical approximate solution that represents a power series with physical parameters. The comparison between the present result from anew scheme with a numerical result and other methods was investigated. Interestingly the results confirm the applicability, efficiency, validity and converge of the new scheme.**

***Index Terms*—Jeffery-Hamel flow, Magnetohydrodynamics, power series, analytical-approximate solution, convergence analysis.**

I. INTRODUCTION

Jeffery-Hamel flow of an incompressible viscous fluid between non-parallel walls was firstly introduced by Jeffery and Hamel [1], [2]. The study of Jeffery-Hamel flow has been expanded to involve the effects of an external magnetic field on an electrical conducting fluid [3]. The Jeffery-Hamel flow is one of the most applicable types of the flow in fluid mechanics[4]. This type of flow possess diverse applications particularly in chemical, mechanical and bio-mechanical engineering. The Jeffery- Hamel flow can be described using the similar solution of the Navier-Stokes equations in the special case of two-dimensional flow. It was reported that the presence of a magnetohydrodynamics (MHD) field can affect on the Jeffery- Hamel flow [5], [6]. The theoretical study of MHD channel has offered many applications in the design of cooling systems, liquid metals accelerators, pumps and flow meters[7], [8], [9]. The magnetic field acts as a control parameter beside the flow Reynolds number and the angle of the walls. The mechanic problems of Jeffery-Hamel flow are considered as the most scientific inherently nonlinear problems. However there is a limited number of these problems, most of them don't have exact solutions. Therefore, several methods have been investigated to solve these problems. Esmaeilpour and Ganji [10] employed optimal homotopy asymptotic method (OHAM) to solve these problems. The adomian decomposition method (ADM) was used by many authors to solve the ordinary differential equation (ODE) which revealed the problems of this flow. Cherruault et al. [11], [12] also reported the convergence of the ADM. An advantage of this method is to provide an analytical approximation solution for a wide class of nonlinear equations without use of linearization, perturbation and discretization methods. Jin and Liu [13] modified ADM for solving a kind of evolution equation. Ganji et al [14] used ADM to obtain an analytical approximate solution of nonlinear differential equation governing Jeffery- Hamel flow with high magnetic field. Dib et al [15] modified adomian decomposition method to solve the MHD Jeffery- Hamel flow. In this work we design new scheme to obtain an analytical-approximate solution of the MHD Jeffery-Hamel problem at an external magnetic field and to make a comparison with ADM [14], the Runge-Kutta fourth order method and the Duan-Rach approach[15]. This method is fundamentally based on the integration of differential equation as well as using Taylor series. This solution from this new scheme was used to study the effects of physical parameters and angle varies on the problem. The present solution which obtained from the new scheme is an infinite power series for appropriate initial approximation. Consequently, the success of this new scheme for solving the highly nonlinear problem will be considered as a useful tool for solution of a nonlinear problem in science and engineering.

A. J. A. Al-Saif & A. M. Jasim Department of Mathematics, College of education for pure Science, University of Basrah, Basrah, Iraq. Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq. Email address: sattaralsaif@yahoo.com & abeer.jassem@yahoo.com

II. MATHEMATICAL FORMULATION

It was considered the steady two-dimensional flow of an incompressible conductive viscous fluid between two rigid plane walls that meet at an angle 2α as shown in Figure (1). The velocity is purely radial and depends on r and θ. We can define the continuity equation and Navies -Stokes in polar coordinates as follows:

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru(r,\theta)) = 0, \tag{1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\sigma B_0^2}{\rho r^2}u(r,\theta) - \mu[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}] = 0, \tag{2}$$

(Advance online publication: 12 August 2019)

$$\frac{1}{\rho r}\frac{\partial p}{\partial \theta} - \frac{2\mu}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0, \tag{3}$$

where p is the fluid pressure, B_0 the electromagnetic induction, σ the conductively of the fluid, ρ the fluid density and μ is the coefficient of kinematic viscosity.

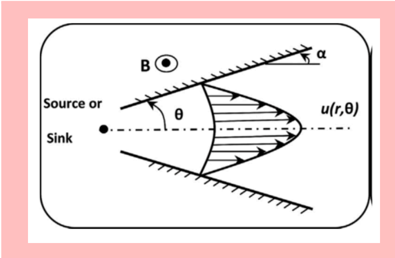

Fig. 1. The geometry of the MHD Jeffery-Hamel flow.

The Equation (1) can be deduced to yield equation

$$g(\theta) = ru(r,\theta), \tag{4}$$

with the use of dimensionless parameters award the Equation (5) can be written

$$f(\eta) = \frac{g(\theta)}{g_{max}}, \quad \eta = \frac{\theta}{\alpha}, \tag{5}$$

A third-order ordinary differential equation for the normal function profile $f(\eta)$ after removing from the equations (2) and (3)

$$f'''(\eta) + 2\alpha Re f(\eta) f'(\eta) + (4 - Ha)\alpha^2 f'(\eta) = 0, \tag{6}$$

when $Ha = 0$ Equation (6) become

$$f'''(\eta) + 2\alpha Re f(\eta) f'(\eta) + 4\alpha^2 f'(\eta) = 0, \tag{7}$$

since a symmetric geometry, the boundary conditions become

$$f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0. \tag{8}$$

The Reynolds number is

$$Re = \frac{U_{max} r \alpha}{\mu} \begin{pmatrix} divergent: \alpha > 0, U_{max} > 0 \\ convergent: \alpha < 0, U_{max} < 0 \end{pmatrix}, \tag{9}$$

the Hartmann number is

$$Ha = \sqrt{\frac{\sigma B_0^2}{\rho \mu}}. \tag{10}$$

III. Description of the New Scheme

This section described how can obtain a new scheme to calculate the coefficients of the power series solution which result from solving nonlinear ordinary differential equations and find an analytical-approximate solution. These coefficients are important bases to construct the solution formula, therefore they should be computed by differential ways. To illustrate the computations and operations of these coefficients in order to derive the new scheme, the detailed new outlook can be summarized in the following steps:

Step 1: The non-linear differential equation can be considered as follows:

$$H(f(\eta), f'(\eta), f''(\eta), ..., f^{(n-1)}(\eta), f^{(n)}(\eta)) = 0, \tag{11}$$

integration of Equation (11) with respect to η on $[0, \eta]$ yields

$$f(\eta) = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} + ...+$$
$$f^{(n-1)}(0)\frac{\eta^{n-1}}{(n-1)!} + L^{-1}G[f(\eta)], \tag{12}$$

where,

$$G[f(\eta)] = H(f(\eta), f'(\eta), f''(\eta), ..., f^{(n-1)}(\eta)),$$
$$L^{-1} = \int_0^\eta \int_0^\eta ... \int_0^\eta (d\eta)^n, \tag{13}$$

Step 2 : Taylor series expansion of the function $G[f(\eta)]$ about $\eta = \eta_0$ can be taken as follows

$$G[f(\eta)] = \sum_{n=0}^{\infty} \frac{(\Delta\eta)^n}{n!} \frac{d^n G(f_0(\eta))}{d\eta^n}, \tag{14}$$

The Equation (14) can be written in the following

$$G[f(\eta)] = \frac{(\Delta\eta)^0}{0!} G[f_0(\eta)] + \frac{(\Delta\eta)^1}{1!} G'[f_0(\eta)] +$$
$$\frac{(\Delta\eta)^2}{2!} G''[f_0(\eta)] + \frac{(\Delta\eta)^3}{3!} G'''[f_0(\eta)] + ..., \tag{15}$$

Now, it was assumed that $\Delta\eta = Max\{\eta, \eta_0\}$ and the substitution Equation (15) in Equation (12) gives

$$f(\eta) = f_0 + f_1 + f_2 + f_3 + f_4 + ..., \tag{16}$$

where,

$$f_0 = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} ... + f^{(n-1)}(0)\frac{\eta^{(n-1)}}{(n-1)!},$$
$$f_1 = L^{-1}\frac{(Max\{\eta, \eta_0\})^0}{0!} G[f_0(\eta)],$$
$$f_2 = L^{-1}\frac{(Max\{\eta, \eta_0\})^1}{1!} G'[f_0(\eta)],$$
$$f_3 = L^{-1}\frac{(Max\{\eta, \eta_0\})^2}{2!} G''[f_0(\eta)],$$
$$f_4 = L^{-1}\frac{(Max\{\eta, \eta_0\})^3}{3!} G'''[f_0(\eta)], ... \tag{17}$$

Step 3 : The computation of the derivatives G with respect to η was indicated as a crucial part of the proposed method. Calculation can be started from $G[f(\eta)], G'[f(\eta)], G''[f(\eta)], G'''[f(\eta)], ...$

$$G[f(\eta)] = H(f(\eta), f'(\eta), f''(\eta), ..., f^{(n-1)}(\eta)), \tag{18}$$

$$G'[f(\eta)] = \frac{dG[f(\eta)]}{d\eta} = G_f.f_\eta + G_{f'}.(f_\eta)'$$
$$+ ... + G_{f^{(n-1)}}.(f_\eta)^{(n-1)}, \tag{19}$$

$$G''[f(\eta)] = \frac{d^2G[f(\eta)]}{d\eta^2} = G_{ff}.(f_\eta)^2 + G_{ff'}.(f_\eta)^{'} f_\eta$$
$$+G_{ff''}.f_\eta(f_\eta)'' + ... + G_{ff^{(n-1)}}.(f_\eta)(f_\eta)^{(n-1)} + G_f$$
$$.f_{\eta\eta} + G_{f'f}.(f_\eta)'.f_\eta + G_{f'f'}.(f_\eta)^{'2} + ... + G_{f'f^{(n-1)}}$$
$$.(f_\eta)'(f_\eta)^{(n-1)} + G_{f'}.(f_{\eta\eta})' + G_{f''f}.(f_\eta)''.f_\eta + G_{f''f'}.$$
$$(f_\eta)^{'}(f_\eta)'' + G_{f''f''}.(f_\eta)^{''2} + G_{f''f'}.(f_\eta)^{'}(f_\eta)''+$$
$$... + G_{f''f^{(n-1)}}.(f_\eta)''(f_\eta)^{(n-1)} + G_{f''}.(f_{\eta\eta})'' + ...+$$
$$G_{f^{(n-1)}f}.(f_\eta)^{(n-1)}.f_\eta + G_{f^{(n-1)}f'}(f_\eta)^{(n-1)}.(f_\eta)' + ...+$$
$$G_{f^{(n-1)}f^{(n-1)}}.(f_\eta)^{(n-1)2} + G_{f^{(n-1)}}.(f_{\eta\eta})^{(n-1)}, \quad (20)$$

$$G'''[f(\eta)] = \frac{d^3G[f(\eta)]}{d\eta^3} = G_{fff}.(f_\eta)^3 + G_{fff'}.(f_\eta)^2$$
$$(f_\eta)' + ... + G_{fff^{(n-1)}}.(f_\eta)^2.(f_\eta)^{(n-1)} + G_{ff}.2(f_\eta).f_{\eta\eta}$$
$$+G_{ff'f}.(f_\eta)'(f_\eta)^2 + G_{ff'f'}.(f_\eta)^{'2}(f_\eta) + ... + G_{ff'f^{(n-1)}}$$
$$.(f_\eta)'(f_\eta).(f_\eta)^{(n-1)} + G_{ff'}.[(f_{\eta\eta})'.f_\eta + (f_\eta)'.f_{\eta\eta}]+$$
$$.(f_\eta)^{''}(f_\eta)^2 + G_{ff''f'}.(f_\eta)^{''}(f_\eta).(f_\eta)' + ...+$$
$$.(f_\eta)^{''}(f_\eta).(f_\eta)^{(n-1)} + G_{ff''}.[f_{\eta\eta}.(f_\eta)'' + + ...+$$
$$G_{ff^{(n-1)f}}.(f_\eta)^2.(f_\eta)^{(n-1)} + G_{ff^{(n-1)f'}}.(f_\eta).(f_\eta)'.$$
$$+ ... + G_{ff^{(n-1)f^{(n-1)}}}.(f_\eta).(f_\eta)^{(n-1)2} + G_{ff^{(n-1)}}.[(f_{\eta\eta})$$
$$.(f_\eta)^{(n-1)} + (f_\eta)(f_{\eta\eta})^{(n-1)}] + G_{ff}.f_{\eta\eta}.(f_\eta) + G_{ff'}$$
$$+.. + G_{ff^{(n-1)}}.f_{\eta\eta}.(f_z)^{(n-1)} + G_\eta.f_{\eta\eta\eta} + G_{f'ff}.(f_\eta)$$
$$G_{f'f'f}.(f_\eta)^{'2}(f_\eta) + ... + G_{f'ff^{(n-1)}}.(f_\eta)'(f_\eta).$$
$$+G_{f'f}.[(f_{\eta\eta})'.f_\eta + (f_\eta)'.f_{\eta\eta}] + G_{f'f'f}.(f_\eta)^{'2}.f_\eta+$$
$$.(f_\eta)^{'3} + ... + G_{f'f'f^{(n-1)}}.(f_\eta)^{'2}.(f_\eta)^{(n-1)} + G_{f'f'}.2(f_\eta)'$$
$$.(f_{\eta\eta})' + ... + G_{f^{(n-1)}f^{(n-1)}f}.(f_\eta)^{(n-1)2}.f_\eta+$$
$$(f_\eta)^{(n-1)2}.(f_\eta)' + ... + G_{f^{(n-1)}f^{(n-1)}f^{(n-1)}}.(f_\eta)^{(n-1)3}$$
$$.(f_{\eta\eta})^{(n-1)}.f_\eta + G_{f^{(n-1)}f'}.(f_{\eta\eta})^{(n-1)}.(f_\eta)' + ...+$$
$$f^{(n-1)}.(f_{\eta\eta})^{(n-1)}.(f_\eta)^{(n-1)} + G_{f^{(n-1)}}.(f_{\eta\eta\eta})^{(n-1)}. \quad (21)$$

$$\vdots$$

The calculations are more complicated in the second and third derivatives due to the product rules. Consequently, the systematic structure on calculation is important due to the assumption that the operator G and the solution f are analytic functions as well as the mixed derivatives are equivalent. It was noted that the derivatives function f was unknown, therefore the following hypothesis can be illustrated

$$f_\eta = f_1 = L^{-1}\frac{(Max\{\eta,\eta_0\})^0}{0!}G[f_0(\eta)],$$
$$f_{\eta\eta} = f_2 = L^{-1}\frac{(Max\{\eta,\eta_0\})^1}{1!}G'[f_0(\eta)],$$
$$f_{\eta\eta\eta} = f_3 = L^{-1}\frac{(Max\{\eta,\eta_0\})^2}{2!}G''[f_0(\eta)],$$
$$f_{\eta\eta\eta\eta} = f_4 = L^{-1}1\frac{(Max\{\eta,\eta_0\})^3}{3!}G'''[f_0(\eta)], ... \quad (22)$$

Therefore Equations (18)- (21) are evaluated by

$$G[f_0(\eta)] = H(f_0(\eta), f_0'(\eta), ..., f_0^{(n-1)}(\eta)), \quad (23)$$
$$G'[f_0(\eta)] = G_{f_0}.f_1 + G_{f_0'}.(f_1)' + ... + G_{f_0^{(n-1)}}.(f_1)^{(n-1)}, \quad (24)$$
$$G''[f_0(\eta)] = G_{f_0f_0}.(f_1)^2 + G_{f_0f_0'}.(f_1)^{'}f_1 + G_{f_0f_0''}.f_1$$
$$(f_1)'' + ... + G_{f_0f_0^{(n-1)}}.(f_1)(f_1)^{(n-1)} + G_{f_0}.f_2 + G_{f_0'f_0}$$
$$.(f_1)'.f(_1) + G_{f_0'f_0'}.(f_1)^{'2} + ... + G_{f_0'f_0^{(n-1)}}.(f_1)'$$
$$(f_1)^{(n-1)} + G_{f_0'}.(f_2)' + G_{f_0''f_0}.(f_1)''.f_1 + G_{f_0''f_0'}.$$
$$(f_1)^{'}(f_1)'' + G_{f_0''f_0''}.(f_1)^{''2} + ... + G_{f_0''f_0^{(n-1)}}.$$
$$(f_1)''(f_1)^{(n-1)} + G_{f_0''}.(f_2)'' + G_{f^{(n-1)}f}.(f_1)^{(n-1)}.f_1$$
$$+G_{f_0^{(n-1)}f_0'}.(f_1)^{(n-1)}.(f_1)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}}$$
$$.(f_1)^{(n-1)2} + ... + G_{f_0^{(n-1)}}.(f_2)^{(n-1)}, \quad (25)$$

$$G'''[f_0(\eta)] = G_{f_0f_0f_0}.(f_1)^3 + G_{f_0f_0f_0'}.(f_1)^2(f_1)'+$$
$$... + G_{f_0f_0f_0^{(n-1)}}.(f_1)^2.(f_1)^{(n-1)} + G_{f_0f_0}.2(f_1).f_2$$
$$+G_{f_0f_0'f_0}.(f_1)'(f_1)^2 + G_{f_0f_0'f_0'}.(f_1)^{'2}(f_1) + ...+$$
$$G_{f_0f_0'f_0^{(n-1)}}(f_1)^2 + G_{f_0f_0''f_0'}.(f_1)^{''}.(f_1)'(f_1)$$
$$.(f_1)^{(n-1)} + G_{f_0f_0'}.[(f_2)'.f_z + (f_1)'.f_2] + G_{f_0f_0''f_0}.$$
$$(f_1)^{''}(f_1).(f_1)' + ... + G_{f_0f_0''f_0^{(n-1)}}.(f_1)^{''}f_1).(f((_1)^{(n-1)}$$
$$+G_{f_0f_0''}.[f_2.(f_1)'' + f_1.(f_2)''] + ... + G_{f_0f_0^{(n-1)}f_0}.(f_1)^2$$
$$.(f_1)^{(n-1)} + G_{f_0f_0^{(n-1)}f_0'}.(f_1).(f_1)'.(f_1)^{(n-1)} + ...+$$
$$G_{f_0f_0^{(n-1)}f_0^{(n-1)}}.(f_1).(f_1)^{2(n-1)} + G_{f_0f_0^{(n-1)}}$$
$$.[(f_2).(f_2)^{(n-1)} + (f_1)(f_2)^{(n-1)}] + G_{f_0f_0}.f_2.(f_1) + G_{f_0f_0'}$$
$$.f_2.(f_1)' + .. + G_{f_0f_0^{(n-1)}}.f_2.(f_1)^{(n-1)} + G_{f_0}.f_3 + G_{f_0'f_0f_0}$$
$$.(f_1)'(f_1)^2 + G_{f_0'f_0'f_0}.(f_1)^{'2}(f_1) + ... + G_{f_0'f_0f_0^{(n-1)}}.(f_1)'$$
$$(f_1).(f_1)^{(n-1)} + G_{f_0'f_0}.[(f_2)'.f_1 + (f_1)'.f_1] + G_{f_0'f_0'f_0}.$$
$$(f_1)^{'2}.f_1 + G_{f_0'f_0'f_0'}.(f_1)^{'3} + ... + G_{f_0'f_0'f_0^{(n-1)}}.(f_1)^{'2}.$$
$$(f_1)^{(n-1)} + G_{f_0'f_0'}.2(f_1)'.(f_2)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}f_0}.(f_1)^{(n-1)2}$$
$$.f_1 + G_{f_0^{(n-1)}f_0^{(n-1)}f_0'}.(f_1)^{(n-1)2}.(f_1)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}f_0^{(n-1)}}$$
$$.(f_1)^{(n-1)3} + G_{f_0^{(n-1)}f_0^{(n-1)}}.2.(f_1)^{(n-1)}.(f_2)^{(n-1)} + G_{f_0^{(n-1)}f_0}$$
$$.(f_2^{(n-1)}.f_1 + G_{f_0^{(n-1)}f_0'}.(f_2)^{(n-1)}.(f_1)' + ... + G_{f_0^{(n-1)}f_0^{(n-1)}}$$

$$.(f_2)^{(n-1)}.(f_1)^{(n-1)} + G_{f_0^{(n-1)}}.(f_3)^{(n-1)}, \quad (26)$$

$$\vdots$$

Step 4 : Substitution of Equations(23)-(26) in Equation (16) offers the required analytical-approximate solution for the Equation (11).

IV. Application of the new scheme to Jeffery-Hamel flow

The new scheme described which in the previous section can be used as a powerful solver for the non-linear differential equations of Jeffery-Hamel flow (6)-(8) and to find a new analytical-approximate solution. From **step(1)**

$$f(\eta) = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} + L^{-1}[-2\alpha Re\, f(\eta)f'(\eta) - (4 - Ha)\alpha^2 f'(\eta)], \quad (27)$$

the Equation (27) can be rewritten as follows

$$f(\eta) = B_1 + B_2\eta + B_3\frac{\eta^2}{2!} + L^{-1}G[f(\eta)], \quad (28)$$

where,

$$B_1 = f(0), \quad B_2 = f'(0), \quad B_3 = f''(0),$$

$$G[f] = -2\alpha Re f(\eta)f'(\eta) - (4 - Ha)\alpha^2 f'(\eta),$$

$$and \quad L^{-1}(.) = \int_0^\eta \int_0^\eta \int_0^\eta (d\eta)^3. \quad (29)$$

From the boundary conditions the Equation (28) becomes

$$f(\eta) = 1 + B_3\frac{\eta^2}{2!} + L^{-1}[G[f(\eta)], \quad (30)$$

From **step(2)** it was supposed that $\Delta\eta = Max\{1, 0\} = 1$, we get

$$f_0 = 1 + B_3\frac{\eta^2}{2!}, \quad f_1 = L^{-1}G[f_0(\eta)],$$

$$f_2 = L^{-1}G'[f_0(\eta)], \quad f_3 = L^{-1}G''[f_0(\eta)], ... \quad (31)$$

From **step(3)** the following equations become

$$G[f(\eta)]) = -2\alpha Re f(\eta)f'(\eta) - (4 - Ha)\alpha^2 f'(\eta), \quad (32)$$

$$G'[f(\eta)] = \frac{dG[f(\eta)]}{d\eta} = G_f.f_\eta + G_{f'}.(f_\eta)', \quad (33)$$

$$G''[f(\eta)] = \frac{d^2G[f(\eta)]}{d\eta^2} = G_{ff}.(f_\eta)^2 + 2.G_{ff'}. + G_{f'f'}.(f_\eta)^{'2} + G_f.f_{\eta\eta} + G_{f'}.(f_{\eta\eta})' \quad (34)$$

$$G'''[f(\eta)] = \frac{d^3G[f(\eta)]}{d\eta^3} = G_{fff}(f_\eta)^3 + 3.G_{fff'}(f_\eta)^2. + 3.G_{f'f}.(f_{\eta\eta})'(f_\eta) + 3.G_{f'f'}.(f_{\eta\eta})'(f_\eta)' + G_{f'}.(f_{\eta\eta\eta})' \quad (35)$$

$$\vdots$$

It was observed that the derivatives of f with respect to η as given in (22) can be computed by Equations (32)-(35) as followed

$$G[f_0(\eta)] = -2\alpha Re f_0(\eta)f_0'(\eta) - (4 - Ha)\alpha^2 f_0'(\eta), \quad (36)$$

$$G'[f_0(\eta)] = G_{f_0}.f_1 + G_{f_0'}.(f_1)', \quad (37)$$

$$G''[f_0(\eta)] = G_{f_0f_0}.(f_1)^2 + 2.G_{f_0f_0'}.f_1(f_1)' + G_{f_0'f_0'}.(f_1)^{'2} + G_{f_0}.f_2 + G_{f_0'}.(f_2)', \quad (38)$$

$$G'''[f_0(\eta)] = G_{f_0f_0f_0}(f_1)^3 + 3.G_{f_0f_0f_0'}(f_1)^2.(f_1)' + 3.G_{f_0f_0'f_0'}.(f_1)(f_1)^{'2} + G_{f_0'f_0'f_0'}.(f_1)^{'3} + 3.G_{f_0f_0}.f_2.f_1\ 3.G_{f_0f_0'}.f_2.(f_1)' + G_{f_0}.f_3 + 3.G_{f_0'f_0}.(f_2)'(f_1) + 3.G_{f_0'f_0'}.(f_2)'(f_1)' + G_{f_0'}.(f_3)' \quad (39)$$

$$\vdots$$

Now, the extraction of the first derivatives of G are needed to be as followed:

$$G_{f_0} = -2\alpha Re f_0'(\eta), \quad G_{f_0f_0} = 0, \quad G_{f_0f_0'} = -2\alpha Re,$$
$$G_{f_0f_0f_0} = G_{f_0f_0'f_0} = G_{f_0f_0'f_0'} = G_{f_0f_0f_0'} = 0,$$
$$G_{f_0'} = -2\alpha Re f_0(\eta) - (4 - Ha)\alpha^2, \quad G_{f_0'f_0} = -2\alpha Re,$$
$$G_{f_0'f_0'} = 0, G_{f_0'f_0f_0} = G_{f_0'f_0'f_0} = G_{f_0'f_0f_0'} = G_{f_0'f_0'f_0'} = 0, \quad (40)$$

from Equation (31) and using Equations (36)-(40) the below equations can be obtained

$$f_0 = 1 + \frac{1}{2}B_3\eta^2, \quad (41)$$

$$f_1 = -\frac{1}{120}\alpha Re B_3^2\eta^6 - (\frac{1}{12}\alpha Re B_3 + \frac{1}{6}\alpha^2 B_3 - \frac{1}{24}\alpha^2 Ha B_3)\eta^4, \quad (42)$$

$$f_2 = \frac{1}{10800}\alpha^2 Re^2 B_3^3\eta^{10} + (\frac{1}{280}\alpha^3 Re B_3^2 - \frac{1}{1120}\alpha^3 Re Ha B_3^2 + \frac{1}{560}\alpha^2 Re^2 B_3^2)\eta^8 + (\frac{1}{180}\alpha^2 Re^2 B_3 + \frac{1}{45}\alpha^3 Re B_3 - \frac{1}{180}\alpha^3 Re Ha B + \frac{1}{45}\alpha^4 B_3 + \frac{1}{90}\alpha^4 Re Ha B_3 + \frac{1}{720}\alpha^4 Re Ha^2 B_3)\eta^6, \quad (43)$$

$$f_3 = -\frac{1}{1572480}\alpha^3 Re^3 B_3^4\eta^{14} - (\frac{359}{9979200}\alpha^4 Re^2$$

$$B_3^3 - \frac{359}{39916800}\alpha^4 Re^2 HaB_3^3 + \frac{359}{19958400}$$

$$\alpha^3 Re^3 B_3^3)\eta^{12} + (\frac{29}{226800}\alpha^4 Re^2 HaB_3^2 + \frac{29}{113400}$$

$$\alpha^5 ReHaB_3^2 - \frac{29}{907200}\alpha^5 ReHa^2 B_3^2 - \frac{29}{56700}$$

$$\alpha^4 Re^2 B_3^2 \alpha^3 Re^3 B_3^2 - \frac{29}{56700}\alpha^5 ReB_3^2)\eta^{10}$$

$$-(\frac{1}{10080}\alpha^3 Re^3 B_3 + \frac{1}{1680}\alpha^4 Re^2 B_3 - \frac{1}{6720}$$

$$-\frac{1}{1680}\alpha^5 ReHaB_3 + \frac{1}{13440}\alpha^5 ReHa^2 B_3 + \frac{1}{1260}$$

$$\alpha^6 B_3 \alpha^4 Re^2 HaB_3 + \frac{1}{840}\alpha^5 ReB_3 - \frac{1}{1680}$$

$$\alpha^6 HaB_3 + \frac{1}{6720}\alpha^6 Ha^2 B_3 - \frac{1}{80640}\alpha^6 Ha^3 B_3)\eta^8, \tag{44}$$

$$\vdots$$

From **step(4)** the substitution of Equations(41)-(44) in Equation (16), the analytical-approximate solution becomes

$$f(\eta) = 1 + \frac{1}{2}B_3\eta^2 - (\frac{1}{12}\alpha ReB_3 + \frac{1}{6}\alpha^2 B_3 - \frac{1}{24}$$

$$\alpha^2 HaB_3)\eta^4 + (\frac{1}{180}\alpha^2 Re^2 B_3 - \frac{1}{120}\alpha ReB_3^2 +$$

$$\frac{1}{45}\alpha^3 ReB_3 - \frac{1}{180}\alpha^3 ReHaB_3 + \frac{1}{45}\alpha^4 B_3$$

$$+\frac{1}{90}\alpha^4 ReHaB_3 + \frac{1}{720}\alpha^4 + ReHa^2 B_3)\eta^6$$

$$+(\frac{1}{280}\alpha^3 ReB_3^2 - \frac{1}{1120}\alpha^3 ReHaB_3^2 + \frac{1}{560}$$

$$\alpha^2 Re^2 B_3^2 - (\frac{1}{10080}\alpha^3 Re^3 B_3 + \frac{1}{1680}\alpha^4 Re^2 B_3$$

$$-\frac{1}{6720}\alpha^4 Re^2 HaB_3 + \frac{1}{840}\alpha^5 ReB_3 - \frac{1}{1680}$$

$$\alpha^5 ReHaB_3 + \frac{1}{13440}\alpha^5 ReHa^2 B_3 + \frac{1}{1260}$$

$$\alpha^6 B_3 - \frac{1}{1680}\alpha^6 HaB_3 + \frac{1}{6720}\alpha^6 Ha^2 B_3$$

$$-\frac{1}{80640}\alpha^6 Ha^3 B_3)\eta^8 + \frac{1}{10800}\alpha^2 Re^2 B_3^3 \eta^{10} + ..., \tag{45}$$

V. Convergence analysis

The analysis of convergence for the analytical-approximate solution (45) that was resulted from the application of new power series scheme for solving the Jeffery-Hamel flow problem has been extensively investigated.

Definition 5.1. H is supposed as Banach space, R the real numbers and $G[F]$ is a nonlinear operator which is defined $G[F]\colon H \longrightarrow R$. Consequently, the solutions that generated from the new scheme can be written as

$$F_{n+1} = G[F_n], \quad F_n = \sum_{k=0}^{n} f_k, \quad n = 0,1,2,3,... \tag{46}$$

where G[F] satisfies Lipchitz condition such that for $\gamma > 0$, $\gamma \in R$, as follows

$$\| G[F_n] - G[F_{n-1}] \leqslant \gamma \| F_n - F_{n-1} \|, \tag{47}$$

Theorem 5.1. The series of the analytical-approximate solution $f(\eta) = \sum_{k=0}^{\infty} f_k(\eta)$ was generated by new scheme converge :

$$\| F_n - F_m \| \longrightarrow 0, \quad m \longrightarrow \infty \quad for \quad 0 \leqslant \gamma < 1, \tag{48}$$

Proof. From the above definition the next equation should be written as

$$\| F_n - F_m \| = \| \sum_{k=0}^{n} f_k - \sum_{k=0}^{m} f_k \|,$$

$$= \| [f_0 + L^{-1}\sum_{k=1}^{n}\frac{\Delta^{(k-1)}}{(k-1)!}\frac{d^{(k-1)}G[f_0(\eta)]}{d\eta^{(k-1)}}] -$$

$$[f_0 + L^{-1}\sum_{k=1}^{m}\frac{\Delta^{(k-1)}}{(k-1)!}\frac{d^{(k-1)}G[f_0(\eta)]}{d\eta^{(k-1)}}] \|,$$

$$= \| L^{-1}G[\sum_{k=0}^{n-1} f_k] - L^{-1}G[\sum_{k=0}^{m-1} f_k] \|, (since\ F_n = G[F_{n-1}])$$

$$\leqslant |Ł^{-1}| \| G[\sum_{k=0}^{n-1} f_k] - G[\sum_{k=0}^{m-1} f_k] \|,$$

$$\leqslant |L^{-1}| \| G[F_{n-1}] - G[F_{m-1}] \|$$

$$\leqslant \gamma \| F_{n-1} - F_{m-1} \|, \tag{49}$$

since G[F] satisfies Lipchitz condition. Let $n = m+1$, then

$$\| F_{m+1} - F_m \| \leqslant \gamma \| F_m - F_{m-1} \|, \tag{50}$$

hence,

$$\| F_m - F_{m-1} \| \leqslant \gamma \| F_{m-1} - F_{m-2} \| \leqslant ... \leqslant \gamma^{m-1} \| F_1 - F_0 \|, \tag{51}$$

from Equation (51) we get

$$\| F_2 - F_1 \| \leqslant \gamma \| F_1 - F_0 \|,$$

Triangle inequality was used as follows

$$\| F_n - F_m \| = \| F_n - F_{n-1} - F_{n-2} - ... - F_{m+1} - F_m \|,$$

$$\leqslant \| F_n + F_{n-1} \| + ... + \| F_{m+1} - F_m \|,$$
$$\leqslant [\gamma^{n-1} + \gamma^{n-2} + ... + \gamma^m] \| F_1 - F_0 \|,$$
$$= \gamma^m [\gamma^{n-m-1} + \gamma^{n-m-2} + ... + 1] \| F_1 - F_0 \|,$$
$$\leqslant \frac{\gamma^m}{1-\gamma} \| F_1 - F_0 \|,$$

as $m \longrightarrow \infty$, leads to $\| F_n - F_m \| \longrightarrow 0$, then F_n is a Cauchy sequence in Banach space H.□

Theorem 5.2. The convergence of the analytical-approximate solution $\sum_{k=0}^{\infty} a_{k0} \frac{\eta^{4k}}{(4k)!} + \sum_{k=0}^{\infty} a_{k1} \frac{\eta^{4k+2}}{(4k+2)!}$ which generated by the new scheme can be verified when

$$\exists 0 \leqslant \gamma < 1, \| F_{n+1} - F_n \| \longrightarrow 0, \quad asn \longrightarrow \infty, \tag{52}$$

Proof. Let denote to the series solution using F_k where k is the nth term of solution (45), the results become

$$F_0 = f_0 = a_{00}\eta^0 + a_{01}\frac{\eta^2}{2!},$$
$$F_1 = f_0 + f_1 = a_{00}\eta^0 + a_{01}\frac{\eta^2}{2!} + a_{10}\frac{\eta^4}{4!} + a_{11}\frac{\eta^6}{6!},$$
$$F_2 = f_0 + f_1 + f_2 = a_{00}\eta^0 + a_{01}\frac{\eta^2}{2!} + ... + a_{20}\frac{\eta^8}{8!} + a_{21}\frac{\eta^{10}}{10!},$$
$$= a_{00}\eta^0 + a_{01}\frac{\eta^2}{2!} + ... + a_{21}\frac{\eta^{10}}{10!} + a_{30}\frac{\eta^{12}}{12!} + a_{31}\frac{\eta^{14}}{14!},$$
$$\vdots$$
$$F_n = f_0 + f_1 + f_2 + f_3 + f_4 + f_5 ... + f_{n-1} + f_n,$$
$$= a_{00}\eta^0 + a_{01}\frac{\eta^2}{2!} + ... + a_{n0}\frac{\eta^{4n}}{(4n)!} + a_{n1}\frac{\eta^{4n+2}}{(4n+2)!},$$
$$\| F_{n+1} - F_n \| = \| \sum_{k=0}^{n+1} (a_{k0}\frac{z^{4k}}{(4k)!} + a_{k1}\frac{z^{4k+2}}{(4k+2)!})$$
$$- \sum_{k=0}^{n} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4k+2)!}) \|,$$
$$\leqslant \gamma \| \sum_{k=0}^{n} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4(k+2)!})$$
$$- \sum_{k=0}^{n-1} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4k+2)!}) \|$$
$$\leqslant \gamma^2 \| \sum_{k=0}^{n-1} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4k+2)!})$$
$$- \sum_{k=0}^{n-2} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4k+2)!}) \|,$$
$$\vdots$$
$$\leqslant \gamma^n \| \sum_{k=0}^{1} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4(k+2)!})$$
$$- \sum_{k=0}^{0} (a_{k0}\frac{\eta^{4k}}{(4k)!} + a_{k1}\frac{\eta^{4k+2}}{(4k+2)!}) \|,$$
$$= \gamma^n \| a_{00}\eta^0 + a_{01}\frac{\eta^2}{2!} + a_{10}\frac{\eta^4}{4!} + a_{11}\frac{\eta^6}{6!} - \eta^0 - \frac{\eta^2}{2!} \|$$
$$= \gamma^n \| F_1 - F_0 \|, \tag{53}$$

as $n \longrightarrow \infty$, then $\| F_{n+1} - F_n \| \longrightarrow 0$ for $0 \leqslant \gamma < 1$.□

In practice, the theorems 5.1 and 5.2 propose to compute the value of γ as described in the following definition

Defention 5.2 For $k = 1, 2, 3, ...$

$$\gamma^k = \begin{cases} \frac{\|F_{k+1} - F_k\|}{\|F_1 - F_0\|} = \frac{\|f_{k+1}\|}{\|f_1\|}, & \| f_1 \| \neq 0, \\ 0, & \| f_1 \| = 0, \end{cases} \tag{54}$$

Now, the definition 5.2 can be applied on Jeffery-Hamel flow to find convergence for examples as below:

if $Re = 1, \alpha = 3°, Ha = 100, B_3 = -1.970428543$ the value of γ becomes

$$\| F_2 - F_1 \|_2 \leqslant \gamma \| F_1 - F_0 \|_2 \Longrightarrow \gamma = 0.0056732847 < 1,$$
$$\| F_3 - F_2 \|_2 \leqslant \gamma^2 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^2 = 0.000014627 < 1,$$
$$\| F_4 - F_3 \|_2 \leqslant \gamma^3 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^3 = 3.286145 * 10^{-8} < 1,$$
$$\vdots$$
$$\| F_2 - F_1 \|_{+\infty} \leqslant \gamma \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma = 0.005275191 < 1,$$
$$\| F_3 - F_2 \|_{+\infty} \leqslant \gamma^2 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^2 = 0.00001252 < 1,$$
$$\| F_4 - F_3 \|_{+\infty} \leqslant \gamma^3 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^3 = 3.06462 * 10^{-8} < 1,$$
$$\vdots$$

Also, if $Re = 10, \alpha = -5°, Ha = 120, B_3 = -1.6563099$ the results become

$$\| F_2 - F_1 \|_2 \leqslant \gamma \| F_1 - F_0 \|_2 \Longrightarrow \gamma = 0.087564650 < 1,$$
$$\| F_3 - F_2 \|_2 \leqslant \gamma^2 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^2 = 0.003569926 < 1,$$
$$\| F_4 - F_3 \|_2 \leqslant \gamma^3 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^3 = 0.000124651 < 1,$$
$$\vdots$$
$$\| F_2 - F_1 \|_{+\infty} \leqslant \gamma \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma = 0.087564650 < 1,$$
$$\| F_3 - F_2 \|_{+\infty} \leqslant \gamma^2 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^2 = 0.00291155 < 1,$$
$$\| F_4 - F_3 \|_{+\infty} \leqslant \gamma^3 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^3 = 0.000118 < 1,$$
$$\vdots$$

When compensation of $Re = 50, \alpha = 5°, Ha = 1000, B_3 = -1.916858326$ therefor the results become

$$\| F_2 - F_1 \|_2 \leqslant \gamma \| F_1 - F_0 \|_2 \Longrightarrow \gamma = 0.12877333 < 1,$$
$$\| F_3 - F_2 \|_2 \leqslant \gamma^2 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^2 = 0.0104528 < 1,$$

$$\| F_4 - F_3 \|_2 \leqslant \gamma^3 \| F_1 - F_0 \|_2 \Longrightarrow \gamma^3 = 0.00056938 < 1,$$

$$\vdots$$

$$\| F_2 - F_1 \|_{+\infty} \leqslant \gamma \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma = 0.1226718 < 1,$$

$$\| F_3 - F_2 \|_{+\infty} \leqslant \gamma^2 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^2 = 0.0103297 < 1,$$

$$\| F_4 - F_3 \|_{+\infty} \leqslant \gamma^3 \| F_1 - F_0 \|_{+\infty} \Longrightarrow \gamma^3 = 0.0005162 < 1,$$

$$\vdots$$

Then $\sum_{k=0}^{\infty} f_k(\eta)$ converges to the solution $f(\eta)$ when $0 \leqslant \gamma^k < 1,\ k = 1, 2, ...$

VI. Results and discussions

The objective of the present study is to apply the new scheme and to obtain an analytical solution of the MHD Jeffery-Hamel problem. The influence of magnetic field (Hartmann number) and Reynold number in Jeffery-Hamel flow was discussed. Tables (I) and (II) elucidate that the new scheme allows to evaluate the coefficient B_3 which can be extracted from the boundary condition $f(1) = 0$. It can also be observed that the values of B_3 is convergent and being fixed in the fourth approximation. The Tables (III) and (IV) explain the comparison of the analytical solutions between the new scheme with ADM [14] and DRA[15] respectively. From this comparison, it was realized that the solutions are well matched. In another comparison between the new scheme and Runge-kutta fourth order scheme as indicated in Tables (V) and (VI), the resulting solution and numerical solution have prefect agreement. The most important effects of the present work were displayed as follows:

- **The effect of Hartmann number.**
  Figure (2)shows the magnetic field effect on the velocity profiles for convergent and divergent channels for fixed Reynolds numbers respectively. The results prove that an increasing in the velocity with increasing Hartmann numbers as no backflow. The fluid velocity becomes flat and thickness of the boundary layer decreased for all Hartmann numbers.
- **The effect of Reynolds number.**
  Figure (3) indicates the effect of an increasing Reynold numbers on the fluid velocity for fixed Hartmann numbers. The case of converging channels comprises exclusion of backflow while the case of divergent channels is possible for large Reynolds numbers.
- **The effect of angle α.**
  Finally, Figures (4 - 7) show the influence of varied angles for fixed values of Hartmann number and Reynolds number. Figure (4) represents the velocity profile which becomes a flat with backflow when $Re = 40$ and Hartmann number is small $Ha = 0$ in the divergent channel. In addition, the situation of the convergent channel is similar except that no backflow. The backflow can be seen in Figure (5) in both cases (divergent and convergent channels) when $Re = 40$ and the Hartmann number is high $Ha = 1000$ with observation of the velocity profile is a flat and thickness of boundary layer decreases in the divergent channel. Figure (6) shows the backflow and the opening angle α in the divergent channel when the $Re = 50$ and $Ha = 0$ therefor, there is no backflow in convergent channel case. Figure (7) displays the effect of large Hartmann number $Ha = 1000$ and the curves which represent the increase of the velocity profile with the increase the angle α have no backflow in the convergent and divergent channels.

Finally, the results confirm that the fluid velocity increases with an increasing the Hartman numbers. Moreover the resulting solutions from the new scheme are more convergent than the resulting solutions from the numerical method when the Reynolds numbers are small and Hartmann number is high.

TABLE I
Convergence of the values $B_3 = f''(0)$ when $Re = 1, Ha = 100$.

	$(\alpha = 3°)$	$(\alpha = -3°)$
Approximation	B_3	B_3
1 term	-1.970624254	-1.94375349
2 term	-1.970429116	-1.94316613
3 term	-1.970428544	-1.93165334
4 term	-1.970428543	-1.93165345
5 term	-1.970428543	-1.93165345
6 term	-1.970428543	-1.93165345
7 term	-1.970428543	-1.93165345
8 term	-1.970428543	-1.93165345

TABLE II
Convergence of the values $B_3 = f''(0)$ when $Re = 10, Ha = 120$.

	$(\alpha = 5°)$	$(\alpha = -5°)$
Approximation	B_3	B_3
1 term	-2.08658567	-1.6742382
2 term	-2.08684806	-1.6566137
3 term	-2.08679124	-1.6562876
4 term	-2.08679148	-1.6563098
5 term	-2.08679148	-1.6563098
6 term	-2.08679148	-1.6563098
7 term	-2.08679148	-1.6563098
8 term	-2.08679148	-1.6563098

VII. Conclusion

In this article the magnetohydrodynamic Jeffery-Hamel flow in a diverging and converging channel was analytically studied using the new scheme. The convergence of the results is explicitly introduced. Graphical results and tables were presented to investigate the influence of physical parameters on the velocity. An analytical solution was obtained using the new scheme and compared with the numerical results. The resulting solution confirms that the new scheme converges with the numerical solution was successfully applied to solve a variety of nonlinear boundary value problems.

TABLE III
COMPARISON BETWEEN ADM[14], NEW SCHEME AND $R-K4$ SCHEME FOR THE ANALYTICAL SOLUTION $f(\eta)$.

	$Re=25, Ha=0, \alpha=5^\circ$		
η	ADM[14]	present result	$(R-K4)$
0.0	1.000000	1.000000	1.000000
0.1	0.986637	0.986677	0.986677
0.2	0.947129	0.947285	0.947286
0.3	0.883178	0.883478	0.883482
0.4	0.797347	0.797799	0.797867
0.5	0.692820	0.693338	0.693398
0.6	0.573015	0.573616	0.573655
0.7	0.441265	0.441806	0.441896
0.8	0.300475	0.300867	0.300574
0.9	0.152914	0.153097	0.153441
1.0	0.000000	0.000000	0.000000

TABLE IV
COMPARISON BETWEEN DRA [15], NEW SCHEME AND RK-4 SCHEME FOR THE ANALYTICAL SOLUTION $f(\eta)$.

	$Re=50, Ha=1000, \alpha=5^\circ$		
η	DRA[15]	present results	$(R-K4)$
0.00	1.000000	1.000000	1.000000
0.05	0.997665	0.997604	0.997604
0.10	0.990427	0.990425	0.990425
0.15	0.978486	0.978480	0.978480
0.20	0.961810	0.961800	0.961800
0.25	0.940436	0.940422	0.940422
0.30	0.914404	0.914383	0.914383
0.35	0.883748	0.883716	0.883716
0.40	0.848474	0.848440	0.848440
0.45	0.808593	0.808553	0.808553
0.50	0.764064	0.764018	0.764018
0.55	0.714805	0.714755	0.714755
0.60	0.660677	0.660623	0.660623
0.65	0.601462	0.601406	0.601406
0.70	0.536852	0.536796	0.536797
0.75	0.466421	0.466368	0.466370
0.80	0.389651	0.389555	0.389559
0.85	0.305652	0.305613	0.305623
0.90	0.213611	0.213583	0.213603
0.95	0.112250	0.112234	0.112272
1.00	0.000000	0.000000	0.000000

ACKNOWLEDGMENTS

The authors gratefully acknowledge Dr. Ahmed Majeed Jassem and Assis. Prof. Mahdi Mohsin Mohammed for performing the proofreading English

REFERENCES

[1] GB. Jeffery, The two-dimensional steady motion of a viscous fluid, Philos Mag, 29, 455-65, 1915.

[2] G. Hamel, Spiralformige bewegungen zaher flussigkeiten, Math Verein, 25, 34-60, 1916.

[3] Wl. Axford, The magnetohydrodynamics Jeffery-Hamel problem for a weakly conducting fluid. QJ Mech Appl. Math., 14, 335-51, 1961.

[4] Q. Esmaili, A. Ramiar, E. Alizadeh, DD. Ganji, An approximation of the analytical solution of the Jeffery-Hamel flow by decomposition method, Phys Letter A, 372:3434-9, 2008.

[5] S.Motsa, P.Sibanda, F. Awad, S. Shateyi, A new spectral-homotopy analysis method for the MHD JefferyHamel problem,Comput Fluids, 39, 121925.

[6] S.Moghimi,G. Domairry, S. Soleimani, E. Ghasem, G. Bararnia, Application of homotopy analysis method to solve MHD JefferyHamel flows in non-parallel walls, Adv Eng Software, 42, 10813, 2011.

[7] J. Cha, Y.Ahn, H. Kim, Flow measurement with an electromagnetic flowmeter in two-phase bubbly and slug flow regimes, Flow Measure Instrum, 12, 32939, 2002.

TABLE V
COMPARISON BETWEEN THE NEW SCHEME AND RK-4 SCHEME FOR $f(\eta)$ WHEN $Re=1, Ha=100$.

	$\alpha=-3^\circ$		$\alpha=3^\circ$	
η	Present result	$RK-4$	Present result	$RK-4$
0.00	1.000000	1.000000	1.000000	1.000000
0.05	0.997571	0.997571	0.997537	0.997537
0.10	0.990281	0.990281	0.990147	0.990147
0.15	0.978124	0.978124	0.977826	0.977826
0.20	0.961089	0.961089	0.960571	0.960571
0.25	0.939160	0.939160	0.938372	0.938372
0.30	0.912317	0.912317	0.911224	0.911224
0.35	0.880537	0.880537	0.879113	0.879113
0.40	0.843790	0.843790	0.842026	0.842026
0.45	0.802045	0.802045	0.799947	0.799947
0.50	0.755264	0.755264	0.752857	0.752857
0.55	0.703409	0.703409	0.700735	0.700735
0.60	0.703408	0.703408	0.643556	0.643556
0.65	0.646433	0.646433	0.581294	0.581294
0.70	0.584292	0.584292	0.513916	0.513916
0.75	0.444302	0.444302	0.441389	0.441389
0.80	0.366343	0.366343	0.363674	0.363674
0.85	0.282996	0.282996	0.280728	0.280728
0.90	0.194199	0.194199	0.192502	0.192502
0.95	0.099889	0.099889	0.098946	0.098946
1.00	0.000000	0.000000	0.000000	0.000000

TABLE VI
COMPARISON BETWEEN THE NEW SCHEME AND RK-4 SCHEME FOR $f(\eta)$ WHEN $Re=10, Ha=120$.

	$\alpha=-5^\circ$		$\alpha=5^\circ$	
η	Present result	$RK-4$	Present result	$RK-4$
0.00	1.000000	1.000000	1.000000	1.000000
0.05	0.997928	0.997928	0.997392	0.997392
0.10	0.991700	0.991700	0.989574	0.989574
0.15	0.981275	0.981275	0.976561	0.976561
0.20	0.966584	0.966584	0.958382	0.958382
0.25	0.947533	0.947533	0.935072	0.935072
0.30	0.924001	0.924001	0.906677	0.906677
0.35	0.895839	0.895839	0.873247	0.873247
0.40	0.862874	0.862874	0.834838	0.834838
0.45	0.824907	0.824907	0.791501	0.791501
0.50	0.781715	0.781715	0.743318	0.743318
0.55	0.733051	0.733051	0.690318	0.690318
0.60	0.678647	0.678644	0.632559	0.632559
0.65	0.618219	0.618219	0.570078	0.570078
0.70	0.551461	0.551447	0.502903	0.502903
0.75	0.478058	0.478058	0.431041	0.431041
0.80	0.397686	0.397647	0.354481	0.354483
0.85	0.310015	0.310015	0.273183	0.273183
0.90	0.214718	0.214627	0.187080	0.187080
0.95	0.111471	0.111471	0.096067	0.096067
1.00	0.000000	0.000000	0.000000	0.000000

[8] J. Mossimo, Some nonlinear problems involving a free boundary plasma physics, J Differ Eq, 34, 11438, 1979.

[9] R.Nijsing, W.Eifler, A computational analysis of transient heat transfer in fuel rod bundles with single phase liquid metal cooling, Nucl Eng Des, 62, 3968, 1980.

[10] Esmaeilpour M, Ganji DD. Solution of the JefferyHamel flow problem by optimal homotopy asymptotic method, Comput Math Appl, 59, 11, 3405-3411, June 2010.

[11] Y. Cherruault, Convergence of Adomians method,Kybernetes, 18, 318, 1989.

[12] Y. Cherruault, Adomian G. Decomposition methods: a new proof of convergence. Mathl. Comput. Modelling 18, 12, 103-106, 1993.

[13] C. Jin , M. Liu, A new modification of Adomian decomposition method for solving a kind of evolution equation, Appl Math Comput, 169, 953962, 2005.

[14] D. Ganji, M. Sheikholeslami, R. Ashorynejad, Analytical approximate solution of nonlinear differential equation gov-

erning JefferyHamel flow with high magnetic field by adomian decomposition method, ISRN Mathematical Analysis, Article ID 937830, 16, 2011.
[15] A.Dib, A.Haiahem, B.Bou-said, An analytical solution of the MHD Jeffery- Hamel flow by the modified Adomian decomposition method, Computers & Fluids, 102, 111-115, 2014.
[16] Adnan, U. Khanb, N. Ahmedb, S. Tauseef Mohyud-Din, Thermo-diffusion and diffusion-thermo effects on flow of second grade fluid between two inclined plane walls, Journal of Molecular Liquids, 224, 10741082, 2016.
[17] A.A. Joneidi, G. Domairry, M. Babaelahi, Three analytical methods applied to JefferyHamel flow,Commun Nonlinear Sci Numer Simulat 15, 34233434, 2010.
[18] M. B. Gerdroodbary, M. R. Takami, and D. D. Ganji, Investigation of thermal radiation on traditional Jeffery-Hamel flow to stretchable convergent/divergent channels,Case Studies in Thermal Engineering 6, 2839, 2015.
[19] N. Freidoonimehr and M. M. Rashidi, Dual solutions for MHD Jeffery-Hamel nano-fluid flow in non-parallel walls using predictor homotopy analysis method, J. Appl. Fluid Mech., 8, 911919, 2015.
[20] U. Khan, N. Ahmed, S. Tauseef, y. Mohyud-Din, Soret and Dufour effects on flow in converging and diverging channels with chemical reaction, Aerospace Science and Technology, 49, 135143, 2016.
[21] A. Ara, N. Alam Khan,F. Naz, M. Asif Zahoor Raja and Q. Rubbab, Numerical simulation for Jeffery-Hamel flow and heat transfer of micropolar fluid based on differential evolution algorithm, AIP Advances, 8, 015201-17, 2018.
[22] F. Yu, S. Xiao, M. Zhu, and Z, Wang," A Novel Method to Improve the Accuracy of Torque Estimation for Robotic Joint with Harmonic Drive Transmission," Engineering Letters, vol. 26, no. 4, pp 455-460, 2019.

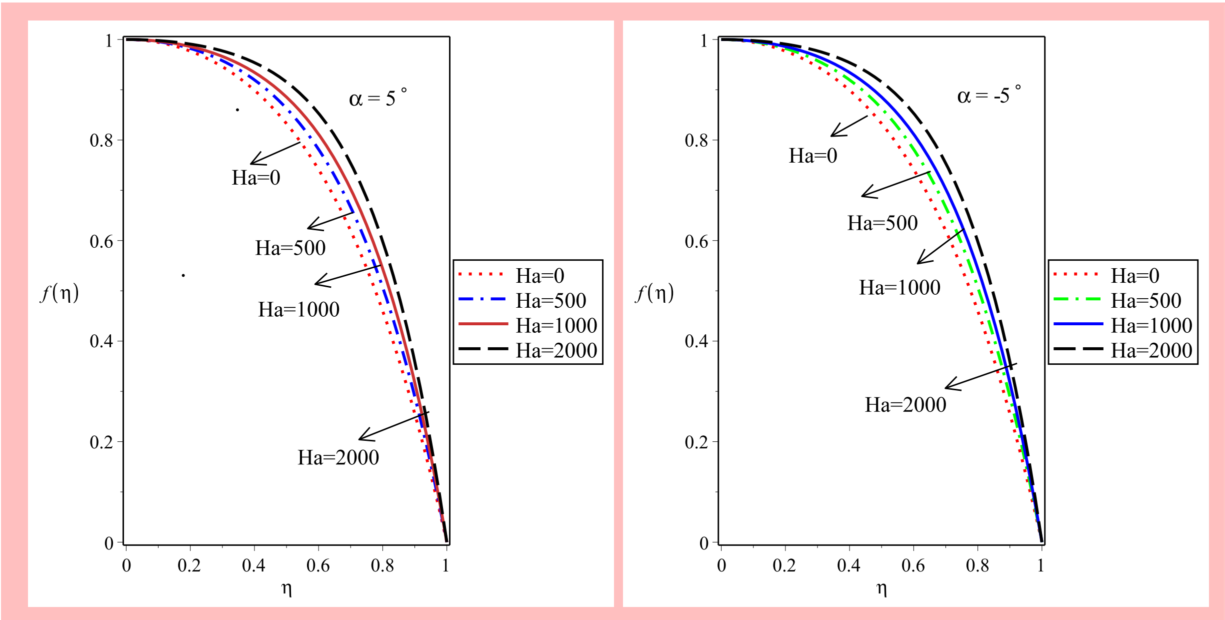

Fig. 2. The velocity profile $f(\eta)$ for the value $Re = 50$ when Ha is varied

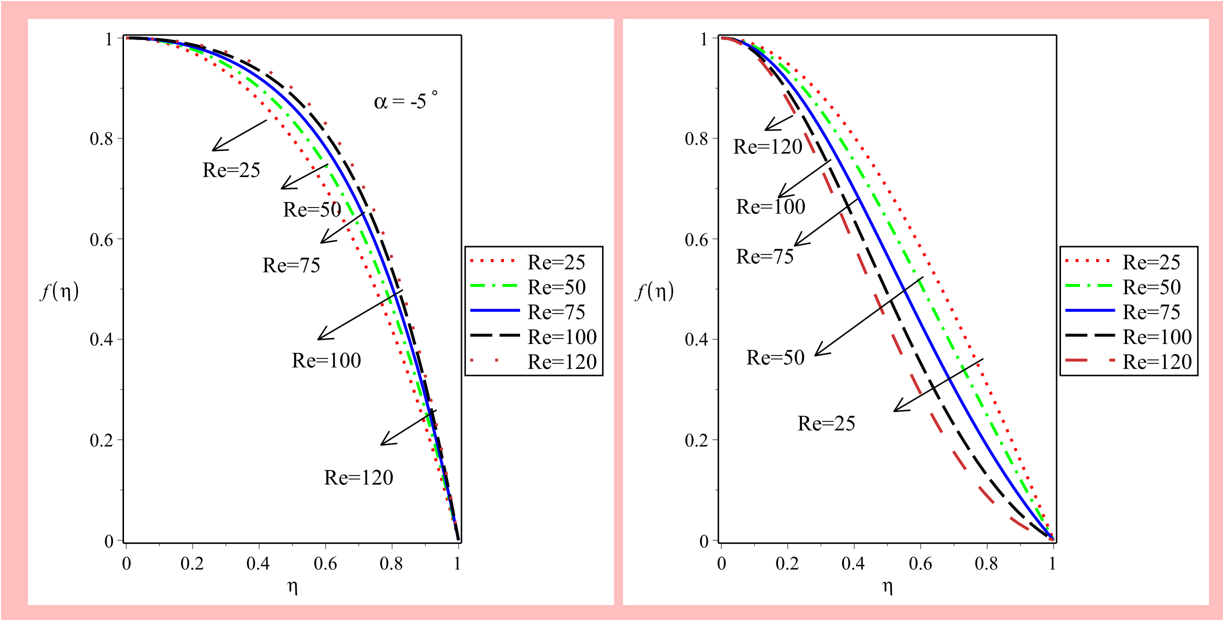

Fig. 3. The velocity profile $f(\eta)$ for the value $Ha = 50$ when Re is varied

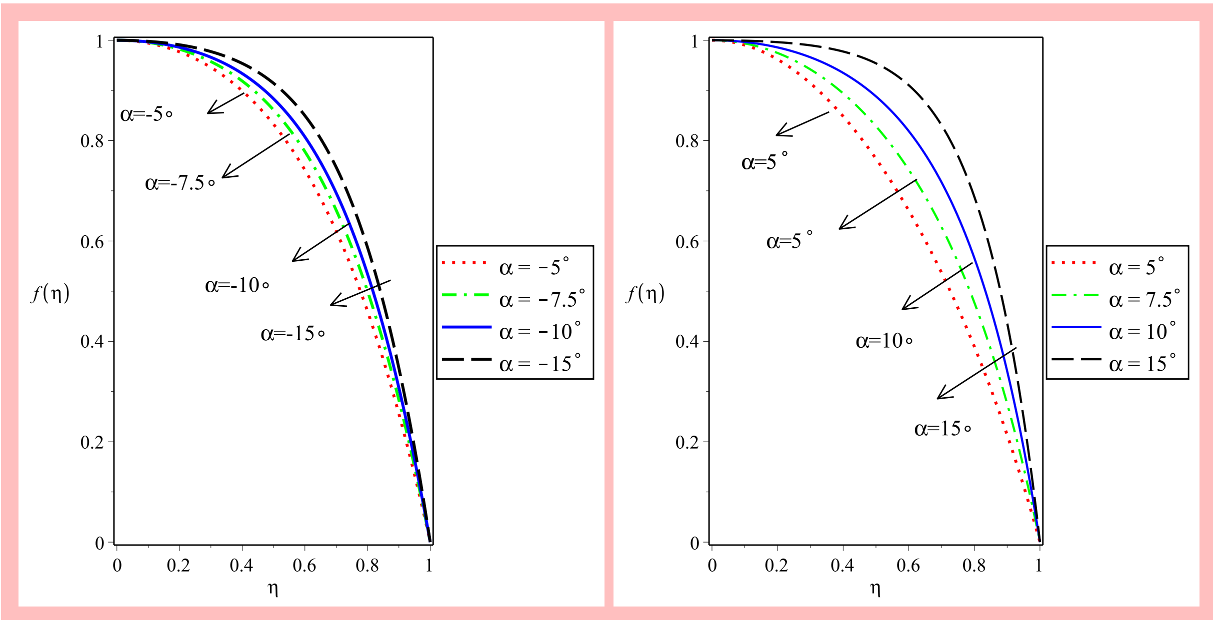

Fig. 4. The velocity profile $f(\eta)$ for the value $Re = 40, Ha = 0$ when the angle α is varied

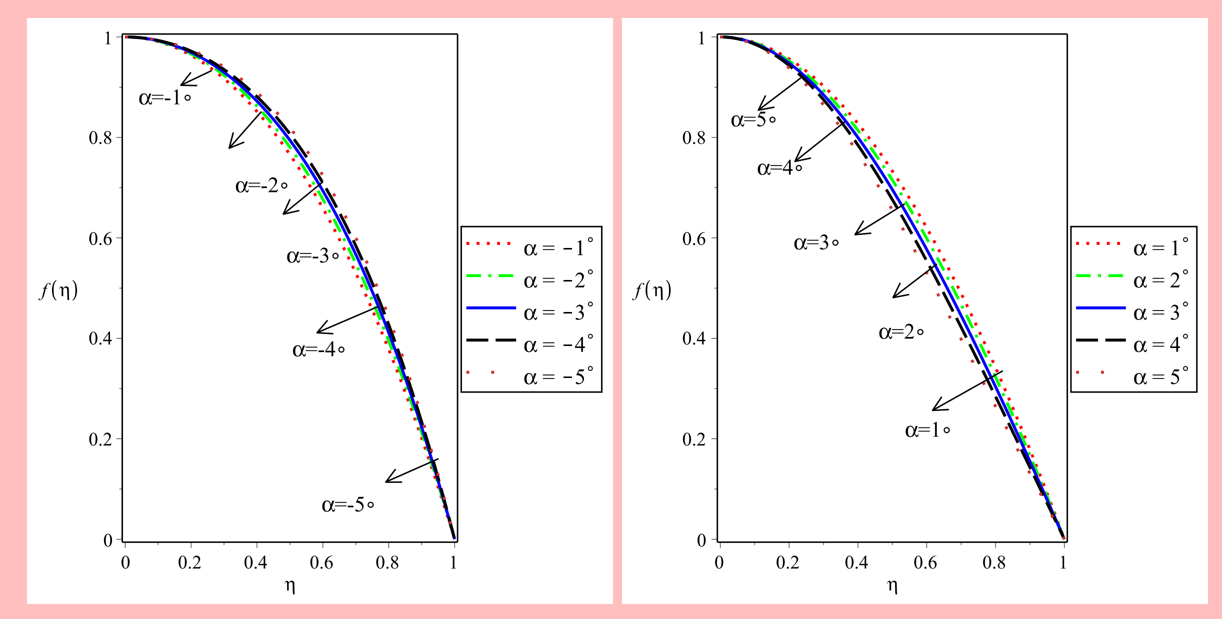

Fig. 5. The velocity profile $f(\eta)$ for the value $Re = 40, Ha = 1000$ when the angle α is varied

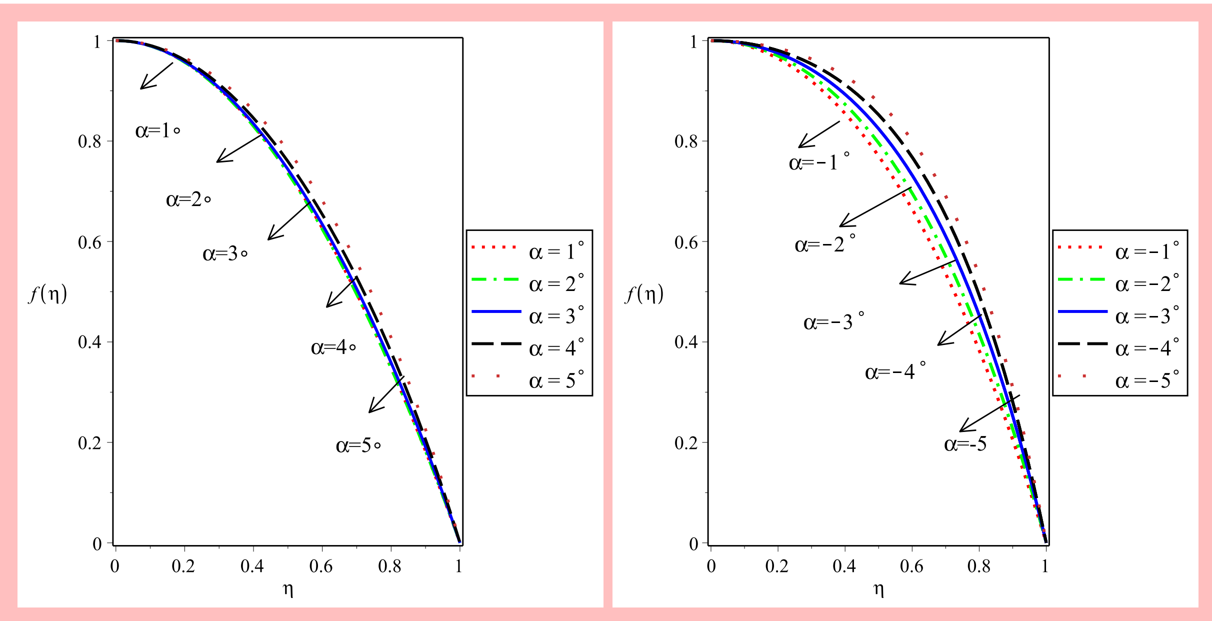

Fig. 6. The velocity profile $f(\eta)$ for the value $Re = 50, Ha = 0$ when the angle α is varied

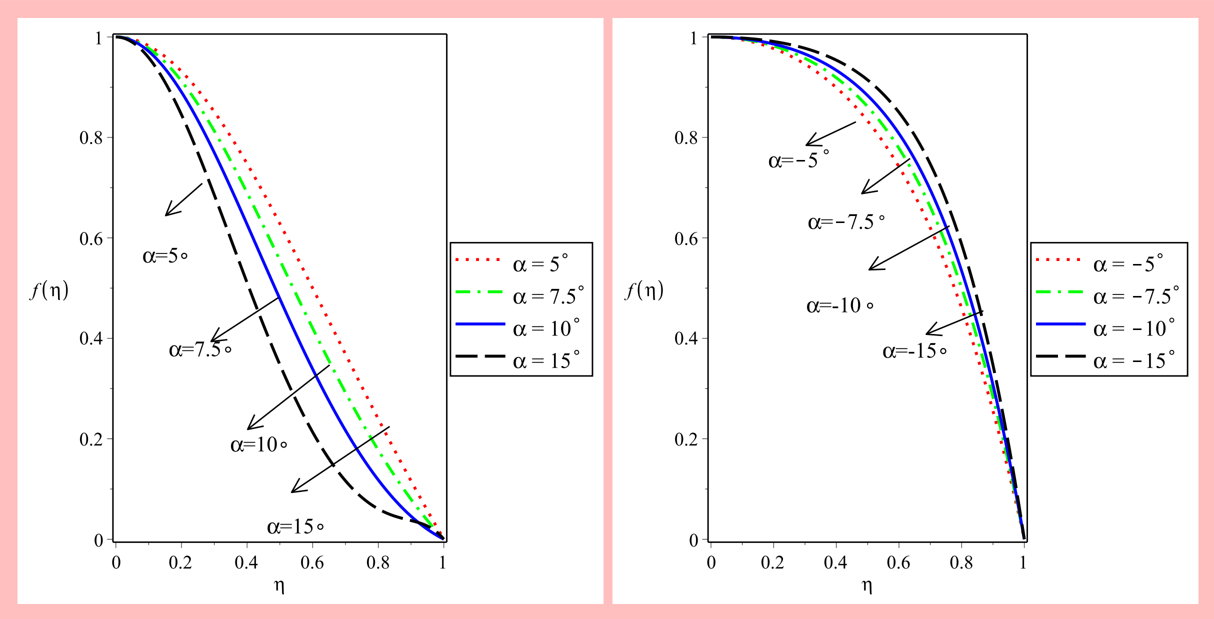

Fig. 7. The velocity profile $f(\eta)$ for the value $Re = 50, Ha = 1000$ when the angle α is varied