Global Attractivity of a Holling-Tanner Model with Beddington-DeAngelis Functional Response: with or without Prey Refuge

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***Abstract*—A Holling-Tanner system with Beddington-DeAngelis functional response and prey refuge takes the form**

$$
\begin{aligned}
x'(t) &= x(t)(1-x(t)) - \frac{(1-m)x(t)y(t)}{a_1+b(1-m)x(t)+c_1y(t)},\\
y'(t) &= y(t)\Big[\delta - \frac{\beta y(t)}{(1-m)x(t)}\Big]
\end{aligned}
$$

is investigated in this paper, where $a_1, b, c_1, \delta,$ and β are all positive constants, m is a nonnegative constant which satisfies $0 \le m < 1$. For the system without prey refuge, i.e., $m = 0$ case, by developing the new analysis technique, we show that $c_1 \ge 2$ is enough to ensure the global attractivity of the positive equilibrium of the system, such a result seems amazing since it is independent of the parameter a_1, b, δ and β. Consequently, we can draw the conclusion that for the most of the parameters, system admits a unique globally attractive positive equilibrium. For $0 < c_1 < 2$, we also investigate the stability property of the positive equilibrium. Two examples together with their numerical simulations show the feasibility of the main results. For the system with prey refuge, we show that there exists a m^*, such that for all $m > m^*$, the system always admits a unique positive equilibrium, which means that enough large prey refuge can improve the coexistence of the species. Refuge plays important role on the persistent property of the system.

***Index Terms*—Beddington-DeAngelis functional response, Holling-Tanner, Global attractivity, Prey refuge.**

I. Introduction

Lu and Liu [1] proposed the following Holling-Tanner model with Beddington-DeAngelis functional response

$$
\begin{aligned}
\frac{dx}{dt} &= rx\big(1-\frac{x}{k}\big) - \frac{\alpha x(t)y(t)}{a+bx(t)+cy(t)},\\
\frac{dy}{dt} &= y\Big[s\big(1-h\frac{y(t)}{x(t)}\big)\Big],
\end{aligned}
\tag{1.1}
$$

where r, k, α, a, b, c, s and h are all positive constants. By nondimensionalize system (1.1) with the following scaling

$$
\begin{aligned}
&w = rt,\ \tilde{x}(\omega) = \frac{x(t)}{k},\ \tilde{y}(\omega) = \frac{\alpha y(t)}{rk},\\
&\delta = \frac{s}{r},\ \beta = \frac{sh}{\alpha},\ a_1 = \frac{a}{k},\ c_1 = \frac{cr}{\alpha}.
\end{aligned}
\tag{1.2}
$$

System (1.1) changes to the system

$$
\begin{aligned}
x'(t) &= x(t)(1-x(t)) - \frac{x(t)y(t)}{a_1+bx(t)+c_1y(t)},\\
y'(t) &= y(t)\Big[\delta - \frac{\beta y(t)}{x(t)}\Big].
\end{aligned}
\tag{1.3}
$$

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By simple computation, one could easily see that system (1.3) admits a unique positive equilibrium $E^*(x^*, y^*)$, where

$$
\begin{aligned}
x^* &= \frac{-\Delta + \sqrt{\Delta^2 + 4(b\beta + c_1\delta)a_1\beta}}{2(b\beta + c_1\delta)},\\
y^* &= \frac{\delta x^*}{\beta},
\end{aligned}
\tag{1.4}
$$

and $\Delta = a_1\beta - b\beta - c_1\delta + \delta$.

For the rest of the paper, let's set

$$M_2 \stackrel{\text{def}}{=} \frac{\delta}{\beta}, \tag{1.5}$$

$$m_1 \stackrel{\text{def}}{=} \frac{a_1 + c_1M_2 - M_2}{a_1 + c_1M_2}, \tag{1.6}$$

$$m_2 \stackrel{\text{def}}{=} \frac{\delta}{\beta}m_1. \tag{1.7}$$

Concerned with the stability property of this positive equilibrium, by constructing Lyapunov function, Lu and Liu[1] obtained the following result.

Theorem A. *If*

$$a_1 + c_1M_2 - M_2 > 0 \tag{1.8}$$

and

$$1 - \frac{b}{a_1} - \frac{1}{2a_1} - \frac{\delta}{2\beta m_1} > 0 \tag{1.9}$$

hold, then the positive equilibrium $E^*(x^*, y^*)$ is globally asymptotically stable.

At first sight, condition (1.9) is very simple, however, one could see that (1.9) requires the following three inequalities hold.

$$b < a_1,\ \ 1 < 2a_1,\ \ \delta < 2\beta m_1. \tag{1.10}$$

Now let's consider the following example.

Example 1.1.

$$
\begin{aligned}
\frac{dx}{dt} &= x\big(1-x\big) - \frac{xy}{0.4+0.3x+3y},\\
\frac{dy}{dt} &= y\big(1-\frac{y}{x}\big),\\
x(0) &> 0,\ y(0) > 0.
\end{aligned}
\tag{1.11}
$$

Here, we take $a_1 = 0.4, \delta = 2, \beta = 1, b = 1, c_1 = 3$, and so

$$
\begin{aligned}
&b = 1 > 0.4 = a_1,\ 1 > 2a_1 = 0.8,\\
&\delta = 2 > \tfrac{11}{8} = 2 \times \tfrac{11}{16} = 2\beta m_1.
\end{aligned}
\tag{1.12}
$$

That is, none of the inequalities in (1.10) holds, let along the condition (1.9), however, numeric simulation (Fig. 1)

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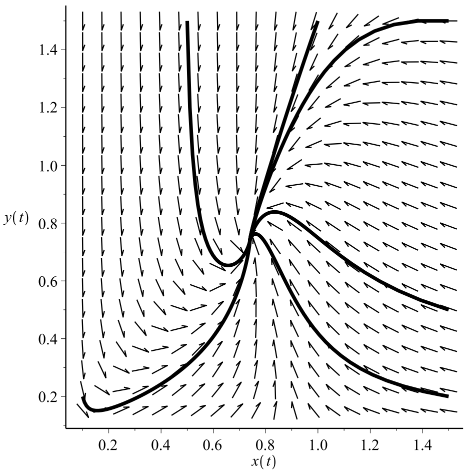

Fig. 1. Dynamic behaviors of system (1.11) with the initial condition $(x(0), y(0)) = (1, 1.5)$, $(1.5, 1.5)$, $(1.5, 0.5)$, $(0.1, 0.2)$, $(1.5, 0.2)$ and $(0.5, 1.5)$, respectively.

shows that system (1.11) admits a unique globally attractive positive equilibrium $E_1^*(x_1^*, y_1^*)$, where $x_1^* = 0.7396379247$, $y_1^* \approx 0.7396379247$.

We found that if we use the method of constructing the Lyapunov function, then (1.9) is necessary. But for the system itself, this condition may not be necessary. Motivated by the above problem, we will revisit the stability property of the positive equilibrium of system (1.3) again. By using a new method which is very different to that of [1], we will establish the following results:

Theorem 1.1. *Assume that $c_1 \geq 2$ holds, then the positive equilibrium $E^*(x^*, y^*)$ of (1.3) is globally asymptotically stable.*

Theorem 1.2. *Assume that $0 < c_1 < 2$ and (1.8) hold, assume further that*

$$a_1(a_1 + b)\beta^2 + \delta(2a_1c_1 + bc_1 - 2b)\beta + \delta^2 c_1(c_1 - 1) > 0 \tag{1.13}$$

and

$$b\beta > c_1\delta(\text{or } \ b\beta + \delta < c_1\delta) \tag{1.14}$$

holds, then the positive equilibrium $E^*(x^*, y^*)$ of (1.3) is globally asymptotically stable.

Remark 1.1. In Example 1.1, since $c_1 = 3$, condition of Theorem 1.1 holds, and consequently, the positive equilibrium of the system (1.11) is globally attractive.

Remark 1.2. Theorem 1.1 shows that for almost all of the parameters (only require $c_1 \geq 2$, and without any restriction on other parameters) of the system (1.3), two species could be coexist in a stable state, this seems very interesting, c_1 can be seen as the most important parameters in the system.

Remark 1.3. One could easily see that if a_1 large enough, b large enough (or small enough, in this case, requires $1 < c_1 < 2$), inequalities (1.13)-(1.14) hold, and it follows from Theorem 1.2 that two species could be coexist in a stable state.

On the other hand, the existence of refuges have the important effects on the coexistence of predators and prey, research on the dynamic behaviors of predator-prey system incorporating a prey refuge become a poplar topic during the last decade, see [5]-[14]. Chen and Chen[11] showed that Gause type predator prey system incorporating prey refuge on prey species could admits more than one positive equilibrium, they also gave sufficient conditions which guarantee the existence and uniqueness of limit cycle; Ma, Chen and Wu[6] considered a Lotka-Volterra predator-prey model incorporating a prey refuge and predator mutual interference, they showed that the system admits a unique positive equilibrium, which is globally asymptotically stable. Also, Chen, Chen and Xie[14] investigated the dynamic behaviors of the Leslie-Gower predator prey model with prey refuge. They showed that the system admits the unique positive equilibrium, which is globally asymptotically stable, however, the prey refuge has different influence to the final density of both prey and predator species. To the best of our knowledge, to this day, still no scholars incorporate the prey refuge to system (1.3) and studied the influence of the prey refuge. The success of Chen and Chen[11], Ma, Chen and Wu[6] and Chen, Chen and Xie[14] motivated us to propose the following system:

$$\begin{aligned} x'(t) &= x(t)(1 - x(t)) - \frac{(1-m)x(t)y(t)}{a_1 + b(1-m)x(t) + c_1y(t)}, \\ y'(t) &= y(t)\Big[\delta - \frac{\beta y(t)}{(1-m)x(t)}\Big] \end{aligned} \tag{1.15}$$

where a_1, b, c_1, δ, and β are all positive constants, m is a nonnegative constant which satisfies $0 \leq m < 1$, m describe the prey refuge, and mx is the number of prey species stay in the prey refuge.

By simple computation, one could easily see that system (1.3) admits a unique positive equilibrium $E^*(x^*, y^*)$, where

$$\begin{aligned} x_m^* &= \frac{-\Delta_m + \sqrt{\Delta_m^2 + 4(b\beta + c_1\delta)(1-m)a_1\beta}}{2(b\beta + c_1\delta)(1-m)}, \\ y_m^* &= \frac{\delta(1-m)x_m^*}{\beta}, \end{aligned} \tag{1.16}$$

and $\Delta_m = a_1\beta - b\beta(1-m) - c_1\delta(1-m) + \delta(1-m)^2$. As for as system (1.15) is concerned, one interesting problem is whether the system admits a unique positive equilibrium which is globally asymptotically stable if the refuge is enough large, which means that the coexistence of the two species. We will give an affirm answer to this problem, indeed, we have the following result:

Theorem 1.3. *Assume that*

$$m > \max\Big\{1 - \sqrt{\frac{a_1\beta}{\delta}}, 1 - c_1, 1 - \frac{a_1}{b}\Big\} \tag{1.17}$$

holds, then the positive equilibrium $E^*(x^*, y^*)$ of (1.15) is globally asymptotically stable.

The paper is arranged as follows: In Section 2, some useful Lemmas are established and then we prove Theorem 1.1 and 1.2 in Section 3, then we prove Theorem 1.3 in Section 4. In Section 5, two examples together with their numeric simulations are presented to illustrate the feasibility of Theorem 1.2. We end this paper by a briefly discussion. For more works on Leslie-Gower predator-prey model, one could refer to [1-36] and the references cited therein.

II. Lemmas

Now we state and prove several useful Lemmas.

Lemma 2.1.[4] If $a>0, b>0$ and $\dot{x} \geq x(b-ax)$, when $t \geq 0$ and $x(0)>0$, we have

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a>0, b>0$ and $\dot{x} \leq x(b-ax)$, when $t \geq 0$ and $x(0)>0$, we have

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}.$$

By using Lemma 2.1, similar to the proof of Lemma 2 and Theorem 3 in [1], we can obtain the following Lemma 2.2 and 2.3.

Lemma 2.2. Let $(x(t), y(t))$ be any positive solution of the system (1.3), then

$$\limsup_{t\to+\infty} x(t) \leq 1, \ \ \limsup_{t\to+\infty} y(t) \leq \frac{\delta}{\beta} \stackrel{\text{def}}{=} M_2.$$

Lemma 2.3. Let $(x(t), y(t))$ be any positive solution of the system (1.3), assume that $a_1+c_1M_2-M_2>0$ holds, then

$$\liminf_{t\to+\infty} x(t) \geq m_1, \ \ \liminf_{t\to+\infty} y(t) \geq m_2,$$

where m_1 and m_2 are defined by (1.6) and (1.7), respectively.

Lemma 2.4. *Assume that one of the following assumption holds*

(1) $c_1 \geq 1$*;*

(2) $0<c_1<1$, $a_1+c_1M_2-M_2>0$,

then system

$$\frac{dx}{dt} = x\Big(1-x-\frac{B}{a_1+bx+c_1B}\Big) \tag{2.1}$$

admits a unique positive equilibrium $x^*(B)$*, which is globally attractive, where* $B \in (m_2-\varepsilon, M_2+\varepsilon)$ *is some positive constant, and* $\varepsilon>0$ *is enough small such that* $-a_1+(\frac{\delta}{\beta}+\varepsilon)(1-c_1)<0$ *holds.*

Proof. The positive equilibrium of system (2.1) satisfies the equation

$$1-x-\frac{B}{a_1+bx+c_1B}=0, \tag{2.2}$$

which is equivalent to

$$bx^2+(Bc_1+a_1-b)x+B(1-c_1)-a_1=0. \tag{2.3}$$

Obviously, under the assumption of Lemma 2.4, $B(1-c_1)-a_1<0$, and so, system (2.3) has a unique positive solution

$$x^*(B) \ \ = \ \ \frac{-\Delta_1+\sqrt{\Delta_1^2-4b\big(B(1-c_1)-a_1\big)}}{2b}. \tag{2.4}$$

where $\Delta_1 = Bc_1+a_1-b$.
Set

$$F(x) = 1-x-\frac{B}{a_1+bx+c_1B},$$

since

$$F(0) = 1-\frac{B}{a_1+c_1B} \geq 1-\frac{\frac{\delta}{\beta}+\varepsilon}{a_1+c_1(\frac{\delta}{\beta}+\varepsilon)}>0$$

and $F(x^*)=0$, from the continuity of the function $F(x)$, it follows that

$$F(x)>0 \ \text{ for all } \ x \in (0, x^*)$$

and

$$F(x)<0 \ \text{ for all } \ x \in (x^*, +\infty),$$

and so apply Theorem 2.1 in [3] to system (2.1), one could see that x^* is globally stable, i. e., $\lim\limits_{t\to+\infty} x(t)=x^*$. This ends the proof of Lemma 2.4.

Lemma 2.5. *Let* $x^*(B)$ *be defined by (2.4), assume that the conditions of Lemma 2.3 and (1.13) hold, then* $x^*(B), \ B \in [m_2, M_2]$ *is a strictly decreasing function of* B*.*

Proof. Since $x^*(B)$ is the positive solution of (2.3). Let's consider the function

$$\begin{aligned} & G(x^*, B) \\ = \ & b(x^*)^2+(Bc_1+a_1-b)x^*+B(1-c_1)-a_1, \end{aligned} \tag{2.5}$$

where $x^* \in [m_1, 1], \ \ B \in (m_2, M_2]$.
It then follows from (1.13) that

$$\begin{aligned} \frac{\partial G}{\partial x^*} &= Bc_1+2bx^*+a_1-b \\ &\geq 2bm_1+m_2c_1+a_1-b \\ &= \frac{A_1\beta^2+A_2\beta+A_3}{\beta(a_1\beta+c_1\delta)}>0, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} A_1 &= a_1(a_1+b), \\ A_2 &= \big(2a_1c_1+b(c_1-2)\big)\delta, \\ A_3 &= c_1(c_1-1)\delta^2. \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \frac{\partial G}{\partial B} &= c_1x^*-c_1+1 \geq c_1m_1-c_1+1 \\ &\geq \frac{\beta a_1}{a_1\beta+c_1\delta}>0, \end{aligned} \tag{2.8}$$

Then, it follows from implicit function theorem that

$$\frac{dx^*}{dB} = -\frac{\frac{\partial G}{\partial x^*}}{\frac{\partial G}{\partial B}}<0. \tag{2.9}$$

Hence, $x^*(B)$ is the strict decreasing function of B. This ends the proof of Lemma 2.5.

Remark 2.1. (1.13) can be rewrite as follows

$$\begin{aligned} & a_1(a_1+b)\beta^2+\delta\big(2a_1c_1+b(c_1-2)\big)\beta \\ & +\delta^2c_1(c_1-1)>0. \end{aligned} \tag{2.10}$$

Obviously, if $c_1 \geq 2$, (2.10) holds, and so, the conclusion of Lemma 2.5 holds.

III. Proof of Theorem 1.1 and 1.2

Proof of Theorem 1.1. $c_1 \geq 2$ implies that (1.8) holds, and so, the conclusions of Lemma 2.3 and 2.4 hold. Also, from remark 2.1, Lemma 2.5 holds. Let $(x(t), y(t))$ be any positive solution of system (1.3), let $\varepsilon > 0$ be any positive constant enough small which satisfies

$$\delta\frac{-\Delta + \sqrt{\Delta^2 - 4b\big((\frac{\delta}{\beta}+\varepsilon)(1-c_1)-a_1\big)}}{2b} - (\beta+1)\varepsilon > 0,$$

where

$$\Delta = (\frac{\delta}{\beta}+\varepsilon)c_1 + a_1 - b.$$

It follows from Lemma 2.2 that there exists a $T > 0$ such that for all $t \geq T$,

$$x(t) < 1 + \varepsilon \stackrel{\text{def}}{=} M_1^{(1)}. \tag{3.1}$$

$$y(t) < \frac{\delta}{\beta} + \varepsilon \stackrel{\text{def}}{=} M_2^{(1)}. \tag{3.2}$$

(3.2) together with the first equation of (1.3) leads to

$$\dot{x}(t) \geq x\Big(1 - x - \frac{M_2^{(1)}}{a_1 + bx + c_1 M_2^{(1)}}\Big) \text{ for all } t \geq T. \tag{3.3}$$

Consider the auxiliary equation

$$\dot{v}(t) = v\Big(1 - v - \frac{M_2^{(1)}}{a_1 + bv + c_1 M_2^{(1)}}\Big). \tag{3.4}$$

Since $c_1 \geq 2$, according to Lemma 2.4, (3.4) admits a unique positive equilibrium

$$v_{11} = \frac{-\Delta_2 + \sqrt{\Delta_2^2 - 4b\big(M_2^{(1)}(1-c_1)-a_1\big)}}{2b}, \tag{3.5}$$

which is globally attractive, where $\Delta_2 = M_2^{(1)}c_1 + a_1 - b$. Hence, by using the differential inequality theory, there exists a $T_{11} > T$ such that

$$x(t) > v_{11} - \varepsilon \stackrel{\text{def}}{=} m_1^{(1)} > 0 \text{ for all } t \geq T_{11}. \tag{3.6}$$

(3.6) together with the second equation of (1.3) leads to

$$\frac{dy}{dt} \geq y\Big(\delta - \frac{\beta y}{m_1^{(1)}}\Big), \tag{3.7}$$

Applying Lemma 2.1 to (3.7) leads to

$$\liminf_{t\to+\infty} y(t) \geq \frac{\delta m_1^{(1)}}{\beta}. \tag{3.8}$$

That is, for above $\varepsilon > 0$, there exists a $T_{12} > T_{11}$ such that

$$y(t) > \frac{\delta m_1^{(1)}}{\beta} - \varepsilon \stackrel{\text{def}}{=} m_2^{(1)} > 0 \text{ for all } t \geq T_{12}. \tag{3.9}$$

It follows from (3.1),(3.2), (3.6) and (3.9) that for all $t \geq T_{12}$,

$$\begin{aligned} 0 < m_1^{(1)} < x(t) < M_1^{(1)}, \\ 0 < m_2^{(1)} < y(t) < M_2^{(1)}. \end{aligned} \tag{3.10}$$

(3.10) together with the first equation of (1.3) leads to

$$\dot{x}(t) \leq x\Big(1 - x - \frac{m_2^{(1)}}{a_1 + bx + c_1 m_2^{(1)}}\Big) \text{ for all } t \geq T_{12}. \tag{3.11}$$

Consider the auxiliary equation

$$\dot{v} = v\Big(1 - v - \frac{m_2^{(1)}}{a_1 + bv + c_1 m_2^{(1)}}\Big). \tag{3.12}$$

Since $c_1 \geq 2$, according to Lemma 2.3, equation (3.12) admits a unique positive equilibrium

$$v_{21} = \frac{-\Delta_3 + \sqrt{\Delta_3^2 - 4b\big(m_2^{(1)}(1-c_1)-a_1\big)}}{2b}, \tag{3.13}$$

which is globally attractive, where $\Delta_3 = m_2^{(1)}c_1 + a_1 - b$. Hence, by using the differential inequality theory, there exists a $T_{21} > T_{12}$ such that

$$x(t) < v_{21} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_1^{(2)} \text{ for all } t \geq T_{21}. \tag{3.14}$$

Since

$$\begin{aligned} & v_{21} \\ = & \frac{-\Delta_3 + \sqrt{\Delta_3^2 - 4b\big(m_2^{(1)}(1-c_1)-a_1\big)}}{2b} \\ = & \frac{-\Delta_3 + \sqrt{(m_2^{(1)}c_1 + a_1 + b)^2 - 4bm_2^{(1)}}}{2b} \\ < & \frac{-(m_2^{(1)}c_1 + a_1 - b) + \sqrt{(m_2^{(1)}c_1 + a_1 + b)^2}}{2b} \\ = & 1, \end{aligned} \tag{3.15}$$

it follows from (3.1) and (3.15) that

$$M_1^{(2)} < M_1^{(1)}. \tag{3.16}$$

From (3.16) and the second equation of (1.3), we know that for $t \geq T_{21}$,

$$\frac{dy}{dt} \leq y\Big(\delta - \frac{\beta y}{M_1^{(2)}}\Big), \tag{3.17}$$

Applying Lemma 2.1 to (3.17) leads to

$$\limsup_{t\to+\infty} y(t) \leq \frac{\delta}{\beta}M_1^{(2)}. \tag{3.18}$$

That is, for above $\varepsilon > 0$, there exists a $T_{22} > T_{21}$ such that

$$y(t) < \frac{\delta}{\beta}M_1^{(2)} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_2^{(2)} \text{ for all } t \geq T_{22}. \tag{3.19}$$

It follows from (3.2), (3.16) and (3.19) that

$$M_2^{(2)} < M_2^{(1)}. \tag{3.20}$$

Substituting (3.20) into the first equation of system (1.3), we obtain

$$\dot{x}(t) \geq x\Big(1 - x - \frac{M_2^{(2)}}{a_1 + bx + c_1 M_2^{(2)}}\Big) \text{ for all } t \geq T_{22}.$$

Similarly to the analysis of (3.3)-(3.6), there exists a $T_{23} > T_{22}$ such that

$$x(t) > v_{22} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_1^{(2)} > 0 \text{ for all } t \geq T_{23}, \tag{3.21}$$

where

$$v_{22} = \frac{-\Delta_4 + \sqrt{\Delta_4^2 - 4b\big(M_2^{(2)}(1-c_1)-a_1\big)}}{2b}. \tag{3.22}$$

here $\Delta_4 = M_2^{(2)}c_1 + a_1 - b$.
From (3.5), (3.20) and Lemma 2.5, we have

$$m_1^{(2)} > m_1^{(1)}. \tag{3.23}$$

From (3.23) and the second equation of (1.3), we know that for $t \geq T_{23}$,

$$\dot{y}(t) \geq y\Big(\delta - \frac{\beta y}{m_1^{(2)}}\Big), \tag{3.24}$$

Applying Lemma 2.1 to (3.24) leads to

$$\liminf_{t\to+\infty} y(t) \geq \frac{\delta}{\beta} m_1^{(2)}. \tag{3.25}$$

That is, for above $\varepsilon > 0$, there exists a $T_{24} > T_{23}$ such that

$$y(t) > \frac{\delta}{\beta} m_1^{(2)} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_2^{(2)} \text{ for all } t \geq T_{24}. \tag{3.26}$$

From (3.8), (3.23) and (3.26) we have

$$m_2^{(2)} > m_2^{(1)}. \tag{3.27}$$

It follows from (3.16), (3.20), (3.23) and (3.27) that for all $t \geq T_{24}$,

$$\begin{aligned} 0 < m_1^{(1)} < m_1^{(2)} < x(t) < M_1^{(2)} < M_1^{(1)}, \\ 0 < m_2^{(1)} < m_2^{(2)} < y(t) < M_2^{(2)} < M_2^{(1)}. \end{aligned} \tag{3.28}$$

Repeating the above procedure, we get four sequences $M_i^{(n)}, m_i^{(n)}, i = 1, 2, n = 1, 2, ...$ such that

$$M_1^{(n)} = v_{n1} + \frac{\varepsilon}{n}, \ \ m_1^{(n)} = v_{n2} - \frac{\varepsilon}{n}, \tag{3.29}$$

$$v_{n1} = \frac{-\Delta_{2n-1} + \sqrt{\Delta_{2n-1}^2 - 4b\big(m_2^{(n-1)}(1-c_1) - a_1\big)}}{2b}. \tag{3.30}$$

$$v_{n2} = \frac{-\Delta_{2n} + \sqrt{\Delta_{2n}^2 - 4b\big(M_2^{(n)}(1-c_1) - a_1\big)}}{2b}. \tag{3.31}$$

$$\frac{\delta}{\beta} m_1^{(n)} - \frac{\varepsilon}{n} = m_2^{(n)}, \ \ \frac{\delta}{\beta} M_1^{(n)} + \frac{\varepsilon}{n} = M_2^{(n)}, \tag{3.32}$$

where

$$\begin{aligned} \Delta_{2n-1} &= m_2^{(n-1)} c_1 + a_1 - b, \\ \Delta_{2n} &= M_2^{(n)} c_1 + a_1 - b. \end{aligned}$$

Now, we go to show that the sequences $M_i^{(n)}$ is strictly decreasing, and the sequences $m_i^{(n)}$ is strictly increasing for $i = 1, 2$ by induction. Firstly, from (3.28), we have

$$m_i^{(1)} < m_i^{(2)}, \ M_i^{(2)} < M_i^{(1)}, \ i = 1, 2. \tag{3.33}$$

Let us suppose that

$$m_i^{(n-1)} < m_i^{(n)}, \ M_i^{(n)} < M_i^{(n-1)}, \ i = 1, 2. \tag{3.34}$$

It then follows from (3.30) and Lemma 2.5 that

$$v_{n1} > v_{(n+1)1}. \tag{3.35}$$

From (3.29) we have

$$M_1^{(n)} > M_1^{(n+1)}. \tag{3.36}$$

By using (3.36), it follows from (3.32) that

$$M_2^{(n)} > M_2^{(n+1)}. \tag{3.37}$$

It then follows from (3.37), (3.31) and Lemma 2.5 that

$$v_{(n+1)2} > v_{n2}. \tag{3.38}$$

(3.38) and (3.29) show that

$$m_1^{(n+1)} > m_1^{(n)}. \tag{3.39}$$

From the relationship of $m_1^{(n)}$ and $m_2^{(n)}$, we have

$$m_2^{(n+1)} > m_2^{(n)}. \tag{3.40}$$

Therefore, we have

$$\begin{aligned} & 0 < m_1^{(1)} < m_1^{(2)} < \cdots < m_1^{(n)} < x(t) \\ < \ & M_1^{(n)} < \cdots < M_1^{(2)} < M_1^{(1)}, \\ & 0 < m_2^{(1)} < m_2^{(2)} < \cdots < m_2^{(n)} < y(t) \\ < \ & M_2^{(n)} < \cdots < M_2^{(2)} < M_2^{(1)}. \end{aligned} \tag{3.41}$$

Hence, the limits of $M_i^{(n)}$ and $m_i^{(n)}$, $i = 1, 2, n = 1, 2, ...$ exist. Denote that

$$\begin{aligned} \lim_{n\to+\infty} M_1^{(n)} &= \overline{x}, \ \lim_{n\to+\infty} m_1^{(n)} = \underline{x}, \\ \lim_{n\to+\infty} M_2^{(n)} &= \overline{y}, \ \lim_{n\to+\infty} m_2^{(n)} = \underline{y}. \end{aligned} \tag{3.42}$$

Then $\overline{x} \geq \underline{x}, \overline{y} \geq \underline{y}$. Letting $n \to +\infty$ in (3.29)-(3.33), we obtain

$$\begin{aligned} \overline{x} &= \frac{-K_1 + \sqrt{K_1^2 - 4b\big(\underline{y}(1-c_1) - a_1\big)}}{2b}. \\ \underline{x} &= \frac{-K_2 + \sqrt{K_2^2 - 4b\big(\overline{y}(1-c_1) - a_1\big)}}{2b}. \\ \frac{\delta}{\beta}\underline{x} &= \underline{y}, \ \ \frac{\delta}{\beta}\overline{x} = \overline{y}. \end{aligned} \tag{3.43}$$

where

$$K_1 = \underline{y} c_1 + a_1 - b,$$
$$K_2 = \overline{y} c_1 + a_1 - b.$$

(3.43) is equivalent to

$$\begin{aligned} & b\overline{x}^2 + (c_1\underline{y} + a_1 - b)\overline{x} + \underline{y}(1-c_1) - a_1 = 0, \\ & b\underline{x}^2 + (c_1\overline{y} + a_1 - b)\underline{x} + \overline{y}(1-c_1) - a_1 = 0, \\ & \delta\underline{x} = \beta\underline{y}, \ \ \delta\overline{x} = \beta\overline{y}. \end{aligned} \tag{3.44}$$

And so,

$$\begin{aligned} & (\overline{x} - \underline{x})\big(b(\overline{x} + \underline{x}) + a_1 - b\big) \\ & + c_1(\underline{y}\overline{x} - \overline{y}\underline{x}) + (1-c_1)(\underline{y} - \overline{y}) = 0. \end{aligned} \tag{3.45}$$

Thus,

$$\begin{aligned} & (\overline{x} - \underline{x})\big(b(\overline{x} + \underline{x}) + a_1 - b\big) \\ & + c_1\tfrac{\delta}{\beta}(\underline{x}\overline{x} - \overline{x}\underline{x}) + (1-c_1)\tfrac{\delta}{\beta}(\underline{x} - \overline{x}) = 0. \end{aligned} \tag{3.46}$$

and so,

$$(\overline{x} - \underline{x})\Big(b(\overline{x} + \underline{x}) + a_1 - b - (1-c_1)\frac{\delta}{\beta}\Big) = 0. \tag{3.47}$$

From Lemma 2.3 and $c_1 \geq 2$ we have

$$\begin{aligned} & b(\overline{x} + \underline{x}) + a_1 - b - (1-c_1)\tfrac{\delta}{\beta} \\ \geq \ & 2b\underline{x} + a_1 - b - (1-c_1)\tfrac{\delta}{\beta} \\ = \ & \frac{B_1\beta^2 + B_2\beta + B_3}{\beta(a_1\beta + c_1\delta)} \\ > \ & 0. \end{aligned}$$

where

$$B_1 = a_1(a_1+b),$$
$$B_2 = \delta(2a_1c_1+bc_1-a_1-2b),$$
$$B_3 = c_1\delta^2(c_1-1).$$

Hence, it follows from (3.47) that

$$\overline{x}=\underline{x}.$$

Also, from (3.43) we have

$$\overline{y}=\underline{y}.$$

Under the assumption of Theorem 1.1, system (1.3) admits a unique positive solution (x^*,y^*), hence $\overline{x}=\underline{x}=x^*, \overline{y}=\underline{y}=y^*$. That is to say,

$$\lim_{t\to+\infty} x(t)=x^*,\ \ \lim_{t\to+\infty} y(t)=y^*. \tag{3.48}$$

This ends the proof of the Theorem 1.1.

Proof of Theorem 1.2. Similarly to the proof of Theorem 1.1, we can finally obtain (3.45)-(3.47). Assume that $\overline{x}\neq\underline{x}$, then from (3.47) we have

$$\overline{x}=-\underline{x}+1-\frac{a_1}{b}+\frac{\delta}{b\beta}(1-c_1), \tag{3.49}$$

and

$$\underline{x}=-\overline{x}+1-\frac{a_1}{b}+\frac{\delta}{b\beta}(1-c_1). \tag{3.50}$$

Substituting (3.49) and (3.50) to (3.44) leads to

$$\begin{aligned} &D_1\overline{x}^2+D_2\overline{x}+D_3=0,\\ &D_1\underline{x}^2+D_2\underline{x}+D_3=0, \end{aligned} \tag{3.51}$$

where

$$D_1 = b\beta(b\beta-c_1\delta),$$
$$D_2 = (b\beta-c_1\delta)(a_1\beta-b\beta+c_1\delta-\delta),$$
$$D_3 = -(b\beta-c_1\delta+\delta)(a_1\beta+c_1\delta-\delta).$$

And so, $\overline{x}$ and $\underline{x}$ are the positive solution of the equation

$$D_1x^2+D_2x+D_3=0. \tag{3.52}$$

From (1.8) and (1.14) one could see that $D_1>0$ and $D_3<0$. Hence, (3.52) has a unique positive solution, this shows that $\overline{x}=\underline{x}$, the rest of the proof is similar to that of the proof of Theorem 1.1, and we omit the detail here. This ends the proof of Theorem 1.2.

IV. Proof of Theorem 1.3

Concerned with the upper and lower bound of the solutions of system (1.15), we can obtain the following Lemma 4.1 and 4.2.

Lemma 4.1. Let $(x(t),y(t))$ be any positive solution of the system (1.15), then

$$\limsup_{t\to+\infty} x(t)\le 1\stackrel{\text{def}}{=}N_1,\ \ \limsup_{t\to+\infty} y(t)\le\frac{\delta(1-m)}{\beta}\stackrel{\text{def}}{=}N_2.$$

Lemma 4.2. Let $(x(t),y(t))$ be any positive solution of the system (1.15), assume that $a_1+c_1N_2-N_2(1-m)>0$ holds, then

$$\liminf_{t\to+\infty} x(t)\ge n_1,\ \ \liminf_{t\to+\infty} y(t)\ge n_2,$$

where

$$n_1\stackrel{\text{def}}{=}\frac{a_1+c_1N_2-N_2(1-m)}{a_1+c_1N_2}, \tag{4.1}$$

$$n_2\stackrel{\text{def}}{=}\frac{\delta(1-m)}{\beta}n_1. \tag{4.2}$$

Remark 4.1 If

$$m>1-\sqrt{\frac{a_1\beta}{\delta}}, \tag{4.3}$$

then the inequality $a_1+c_1N_2-N_2(1-m)>0$ always holds, that is, if the prey refuge is enough large, then the inequality $a_1+c_1N_2-N_2(1-m)>0$ holds.

Lemma 4.3. *Assume that*

$$m>1-c_1, \tag{4.4}$$

then system

$$\frac{dx}{dt}=x\Big(1-x-\frac{B(1-m)}{a_1+b(1-m)x+c_1B}\Big) \tag{4.5}$$

admits a unique positive equilibrium $x_m^*(B)$, which is globally attractive, where $B\in(n_2-\varepsilon,N_2+\varepsilon)$ is some positive constant, and $\varepsilon>0$ is enough small such that $\varepsilon<m_2$ holds.

Proof. The positive equilibrium of system (4.5) satisfies the equation

$$1-x-\frac{B(1-m)}{a_1+b(1-m)x+c_1B}=0, \tag{4.6}$$

which is equivalent to

$$b(1-m)x^2+(Bc_1+a_1-b(1-m))x+B(1-c_1-m)-a_1=0. \tag{4.7}$$

Obviously, under the assumption of Lemma 4.3, $B(1-c_1-m)-a_1<0$, and so, system (4.7) has a unique positive solution

$$x_m^*(B) = \frac{-\Gamma_1+\sqrt{\Gamma_1^2-4b(1-m)\big(B(1-c_1-m)-a_1\big)}}{2b(1-m)}. \tag{4.8}$$

where $\Gamma_1=Bc_1+a_1-b(1-m)$.
Similarly to the proof of Lemma 2.4, we could show that $\lim\limits_{t\to+\infty} x(t)=x_m^*(B)$. This ends the proof of Lemma 4.3.

Lemma 4.4. *Let $x_m^*(B)$ be defined by (4.8), assume that the conditions of Lemma 4.3 and (4.4) hold, assume further that*

$$m>1-\frac{a_1}{b} \tag{4.9}$$

holds, then $x_m^*(B),\ B\in[n_2,N_2]$ is a strictly decreasing function of B.

Proof. Since $x^*(B)$ is the positive solution of (4.7). Let's consider the function

$$\begin{aligned} &G(x^*,B)\\ =\ &b(1-m)(x_m^*)^2+(Bc_1+a_1-b(1-m))x_m^*\\ &+B(1-c_1-m)-a_1=0, \end{aligned} \tag{4.10}$$

where $x_m^* \in [n_1, 1]$, $B \in (n_2, N_2]$.
It then follows from (4.9) that

$$\begin{aligned}\frac{\partial G}{\partial x_m^*} &= Bc_1 + 2b(1-m)x_m^* + a_1 + b(1-m) \\ &\geq a_1 - b(1-m) > 0,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\frac{\partial G}{\partial B} &= c_1 x_m^* - c_1 + 1 \geq c_1 n_1 - c_1 + 1 \\ &= \frac{\beta a_1(1-m)}{a_1\beta + c_1\delta(1-m)} > 0,\end{aligned} \tag{4.12}$$

Then, it follows from implicit function theorem that

$$\frac{dx_m^*}{dB} = -\frac{\frac{\partial G}{\partial x_m^*}}{\frac{\partial G}{\partial B}} < 0. \tag{4.13}$$

Hence, $x_m^*(B)$ is the strict decreasing function of B. This ends the proof of Lemma 4.4.

Proof of Theorem 1.3. It follows from Lemma 4.1 that there exists a $T > 0$ such that for all $t \geq T$,

$$x(t) < 1 + \varepsilon \overset{\text{def}}{=} M_1^{(1)}. \tag{4.14}$$

$$y(t) < \frac{\delta(1-m)}{\beta} + \varepsilon \overset{\text{def}}{=} M_2^{(1)}. \tag{4.15}$$

(4.15) together with the first equation of (1.15) leads to

$$\dot{x}(t) \geq x\Big(1 - x - \frac{(1-m)M_2^{(1)}}{a_1 + b(1-m)x + c_1 M_2^{(1)}}\Big). \tag{4.16}$$

for all $t \geq T$.

Consider the auxiliary equation

$$\dot{v}(t) = v\Big(1 - v - \frac{(1-m)M_2^{(1)}}{a_1 + b(1-m)v + c_1 M_2^{(1)}}\Big). \tag{4.17}$$

According to Lemma 4.4, (4.17) admits a unique positive equilibrium

$$v_{11} = \frac{-\Delta_2 + \sqrt{\Delta_2^2 - 4b(1-m)\big(M_2^{(1)}(1-c_1-m) - a_1\big)}}{2b(1-m)},$$

which is globally attractive, where $\Delta_2 = M_2^{(1)}c_1 + a_1 - b(1-m)$. Hence, by using the differential inequality theory, there exists a $T_{11} > T$ such that

$$x(t) > v_{11} - \varepsilon \overset{\text{def}}{=} m_1^{(1)} > 0 \ \text{ for all } \ t \geq T_{11}. \tag{4.18}$$

(4.18) together with the second equation of (1.15) leads to

$$\frac{dy}{dt} \geq y\Big(\delta - \frac{\beta y}{(1-m)m_1^{(1)}}\Big), \tag{4.19}$$

Applying Lemma 2.1 to (4.19) leads to

$$\liminf_{t\to+\infty} y(t) \geq \frac{\delta(1-m)m_1^{(1)}}{\beta}. \tag{4.20}$$

That is, for above $\varepsilon > 0$, there exists a $T_{12} > T_{11}$ such that

$$y(t) > \frac{\delta(1-m)m_1^{(1)}}{\beta} - \varepsilon \overset{\text{def}}{=} m_2^{(1)} > 0 \ \text{ for all } \ t \geq T_{12}. \tag{4.21}$$

Repeating the above procedure, we get four sequences $M_i^{(n)}, m_i^{(n)}, i = 1, 2, n = 1, 2, ...$ such that

$$M_1^{(n)} = v_{n1} + \frac{\varepsilon}{n}, \ \ m_1^{(n)} = v_{n2} - \frac{\varepsilon}{n}, \tag{4.22}$$

$$v_{n1} = \frac{-\Delta_{2n-1} + \sqrt{\Delta_{2n-1}^2 - 4b(1-m)L_1}}{2b(1-m)}. \tag{4.23}$$

$$v_{n2} = \frac{-\Delta_{2n} + \sqrt{\Delta_{2n}^2 - 4b(1-mL_2}}{2b(1-m)}. \tag{4.24}$$

$$\frac{\delta}{\beta}m_1^{(n)} - \frac{\varepsilon}{n} = m_2^{(n)}, \ \ \frac{\delta}{\beta}M_1^{(n)} + \frac{\varepsilon}{n} = M_2^{(n)}, \tag{4.25}$$

where

$$\begin{aligned}\Delta_{2n-1} &= m_2^{(n-1)}c_1 + a_1 - b(1-m), \\ \Delta_{2n} &= M_2^{(n)}c_1 + a_1 - b(1-m), \\ L_1 &= \big(m_2^{(n-1)}(1-c_1-m) - a_1\big), \\ L_2 &= \big(M_2^{(n)}(1-c_1-m) - a_1\big).\end{aligned}$$

Similar to the analysis of (3.33)-(3.41), we can show that the sequences $M_i^{(n)}$ is strictly decreasing, and the sequences $m_i^{(n)}$ is strictly increasing for $i = 1, 2$. Hence, the limits of $M_i^{(n)}$ and $m_i^{(n)}$, $i = 1, 2, n = 1, 2, ...$ exist. Denote that

$$\begin{aligned}\lim_{n\to+\infty} M_1^{(n)} &= \overline{x}, \ \lim_{n\to+\infty} m_1^{(n)} = \underline{x}, \\ \lim_{n\to+\infty} M_2^{(n)} &= \overline{y}, \ \lim_{n\to+\infty} m_2^{(n)} = \underline{y}.\end{aligned} \tag{4.26}$$

Similarly to the analysis of (3.42)-(3.48), one could show that under the assumption of Theorem 1.3,

$$\lim_{t\to+\infty} x(t) = x_m^*, \ \lim_{t\to+\infty} y(t) = y_m^*. \tag{4.27}$$

This ends the proof of the Theorem 1.3.

V. Numeric simulations

In section I, we gave an example to show the feasibility of the Theorem 1.1, now let's consider the following two examples which illustrate the feasibility of the Theorem 1.2.

Example 5.1 Now let us consider the following system

$$\begin{aligned}\frac{dx}{dt} &= x\big(1-x\big) - \frac{xy}{4 + 2x + \frac{1}{2}y}, \\ \frac{dy}{dt} &= y\big(1 - \frac{y}{x}\big), \\ x(0) &> 0, \ y(0) > 0.\end{aligned} \tag{5.1}$$

Here, we take $a_1 = 4, \delta = \beta = 1$, $b = 2, c_1 = \frac{1}{2}$, and so, by simple computation, we have

$$\begin{aligned}&a_1(a_1+b)\beta^2 + \delta(2a_1c_1 + bc_1 - 2b)\beta + \delta^2 c_1(c_1 - 1) \\ &= \frac{99}{4} > 0,\end{aligned} \tag{5.2}$$

$$b\beta = 2 > \frac{1}{2} = c_1\delta \tag{5.3}$$

and

$$a_1\beta + c_1\delta = \frac{9}{2} > 1 = \delta \tag{5.4}$$

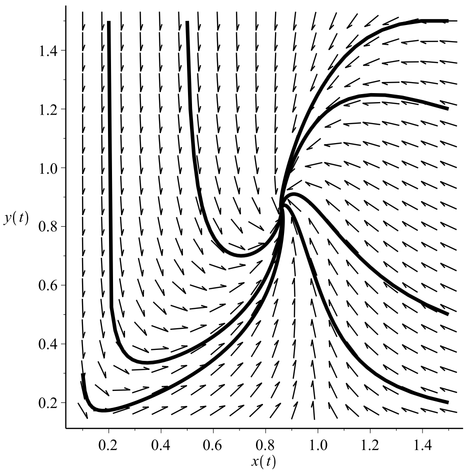

Fig. 2. Dynamic behaviors of system (4.1) with the initial condition $(x(0), y(0)) = (0.2, 1.5)$, $(1.5, 1.5)$, $(1.5, 0.5)$, $(0.1, 0.3)$, $(1.5, 0.2)$ and $(0.5, 1.5)$, respectively.

hold, then the positive equilibrium $E_2^*(x_2^*, y_2^*)$ of (4.1) is globally asymptotically stable. Numeric simulation (Fig. 2) shows that system (4.1) admits a unique globally attractive positive equilibrium $E_2^*(x_2^*, y_2^*)$, where $x_2^* = y_2^* \approx 0.8601470509$.

Example 5.2 Now let us consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{xy}{4 + \frac{1}{10}x + \frac{3}{2}y}, \\ \frac{dy}{dt} &= y(1 - \frac{y}{x}), \\ x(0) &> 0,\ y(0) > 0. \end{aligned} \tag{5.5}$$

Here, we take $a_1 = 4, \delta = \beta = 1,\ b = \frac{1}{10}, c_1 = \frac{3}{2}$, and so, by simple computation, we have

$$2a_1c_1 + bc_1 - 2b = 12 - \frac{1}{20} > 0, c_1 - 1 = \frac{1}{2} > 0, \tag{5.6}$$

that is, (1.13) holds. Also,

$$b\beta + \delta = \frac{1}{10} < \frac{3}{2} = c_1\delta, \tag{5.7}$$

then the positive equilibrium $E_3^*(x_3^*, y_3^*)$ of (5.5) is globally asymptotically stable, where $x_3^* = y_3^* \approx 0.8424688318$. Since the numeric simulation is similar to Fig. 2, we omit the detail here.

VI. Discussion

In this paper, we revisit the Holling-Tanner system with Beddington-DeAngelis functional response, which was proposed by Lu and Liu[1]. By developing some new analysis technique and using the new method, two set of sufficient conditions which ensure the global attractivity of the positive equilibrium are obtained.

It is nature to conjecture that system (1.3) admits complex dynamic behaviors, since the system contains five parameters. However, Theorem 1.1 shows that if $c_1 \geq 2$, then the rest of the parameters have no influence on the dynamic behaviors of the system, and the system always admits a unique positive equilibrium, which is globally attractive. Theorem 1.2 shows that for the case $0 < c_1 < 2$, if a_1 enough large, b enough large or enough small, then the system also has a positive equilibrium which is globally attractive.

To summarize: System (1.3) admits a very simple dynamic behaviors for most of the parameters.

On the other hand, we incorporate the prey refuge to system (1.3), this leads to the system (1.15), Theorem 1.3 shows that if the prey refuge is enough large, then two species could be coexist in a stable state.

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