Some Efficient Algorithms Based on Elliptical Arc Artificial Boundary Condition for Anisotropic Problems

Yajun Chen, and Qikui Du

***Abstract*—In this paper, some exact nonlocal boundary conditions are derived on an elliptical arc artificial boundary, and they are applied to solving the exterior anisotropic problems in concave angle domains. Based on the above artificial boundary conditions, the Schwarz alternating algorithm is presented. The convergence of this algorithm is examined. Finally, some numerical examples are given to show the effectiveness of our methods..**

***Index Terms*—Schwarz alternating algorithm, elliptical arc artificial boundary condition, anisotropic problem**

I. INTRODUCTION

MANY scientific and engineering computing problems can be modeled by boundary value problems of partial differential equations in unbounded domains. There is a variety of numerical methods to solve such problems. One of the commonly used techniques is the method of artificial boundary conditions [1]-[9]. The method may be summarized as follows: (i) Introduce an artificial boundary $\mathcal{B}$, which divides the original unbounded domain into two non-overlapping subdomains: a bounded computational domain Ω and an infinite residual domain D. (ii) By analyzing the problem in the infinite residual domain D, obtain a relation on the artificial boundary $\mathcal{B}$ involving the unknown function and its derivatives. (iii) Using the relation as a boundary condition on $\mathcal{B}$, to obtain a well-posed problem in the bounded computational domain Ω. (iv) Solve the problem in the bounded computational domain Ω be the standard finite element methods or some other numerical methods.

Based on artificial boundary conditions, the overlapping and non-overlapping domain decomposition methods can be viewed as effective ways to solve problems in unbounded domains. These techniques have been used to solve many linear or nonlinear problems [10]-[17]. Recently, the authors used a new elliptical arc artificial boundary and some iteration methods to solve Poisson problems and anisotropic problems [18]-[22]. In this paper, we derive an exact elliptical arc artificial boundary condition for anisotropic problems in an unbounded domain with a concave angle, and apply the methods in [10] to solving the above problems.

Let Ω be an exterior concave angle domain with angle ω, and $0 < \omega < 2\pi$. The boundary of domain Ω is decomposed into three disjoint parts: Γ, Γ_0 and Γ_ω (see Fig. 1), i.e. $\partial\Omega = \overline{\Gamma \cup \Gamma_0 \cup \Gamma_\omega}$, $\Gamma_0 \cap \Gamma_\omega = \emptyset$, $\Gamma \cap \Gamma_0 = \emptyset$, $\Gamma \cap \Gamma_\omega = \emptyset$. The boundary Γ is a simple smooth curve part, Γ_0 and Γ_ω are two half lines.

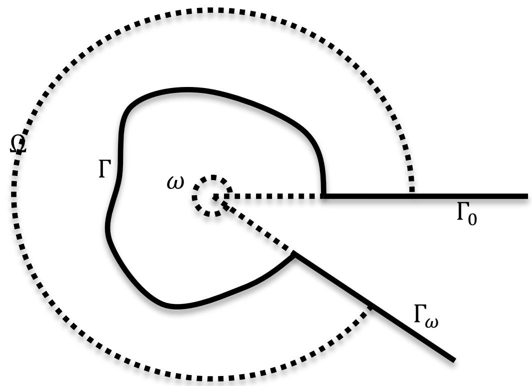

Fig. 1. The illustration of domain Ω

We consider the following anisotropic problem:

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) = f, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\omega, \\ \mathcal{A}\nabla u \cdot n = g, \quad \text{on } \Gamma, \\ u \text{ is vanish at infinity}, \end{cases} \tag{1}$$

and

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) = f, \quad \text{in } \Omega, \\ \mathcal{A}\nabla u \cdot n = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\omega, \\ u = h, \quad \text{on } \Gamma, \\ u \text{ is bounded at infinity}, \end{cases} \tag{2}$$

where $\mathcal{A} = \begin{pmatrix} k^2 & 0 \\ 0 & 1 \end{pmatrix}$, k is a constant and $0 < k < 1$, u is the unknown function, $f \in L^2(\Omega)$ and $g, h \in L^2(\Gamma)$ are given functions, $\mathrm{supp}(f)$ is compact.

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Yajun Chen is with the School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023 and the Department of Mathematics, Shanghai Maritime University, Shanghai 200136, China. (E-mail: chenyajun@shmtu.edu.cn).

Qikui Du is with the School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China.

The outline of the paper is as follows. In Section 2, we derive an exact elliptical arc artificial boundary condition for the above anisotropic problem. In Section 3, we construct a Schwarz alternating method. In Section 4, we give the convergence of the method, and analyze the convergence rate for a typical domain. Finally, in Section 5 we present some numerical results to show its accuracy and the effectiveness of our methods

II. The Exact Artificial Boundary Condition

Let f_0 denote the half distance between the two foci of an ellipse, we introduce an elliptic system of co-ordinates (μ,φ) such that the artificial boundary $\mathcal{B}$ coincides with the elliptical arc $\{(\mu,\varphi)|\mu=\mu_R,\ 0<\varphi<\omega\}$, where

$$f_0=\frac{\sqrt{1-k^2}}{k}R,\ \mu_R=\ln\frac{1+k}{\sqrt{1-k^2}}.$$

Thus, the Cartesian co-ordinates (ξ,η) are related to the elliptic co-ordinates (μ,φ), that is $\xi=f_0\cosh\mu\cos\varphi$, $\eta=f_0\sinh\mu\sin\varphi$. The domain exterior to $\mathcal{B}$, namely the $\{(\mu,\varphi)|\mu>\mu_R,\ 0<\varphi<\omega\}$ is denoted by D. Let $\mathcal{B}$ be a circle arc with radius R at the origin, enclosing Γ and $\mathrm{supp}(f)$. We first introduce the following transformation $x=k\xi$, $y=\eta$, then the anisotropic problem (1) become the following Poisson problem:

$$\begin{cases}-\Delta u=f, & \text{in } \tilde{\Omega},\\ u=0, & \text{on } \tilde{\Gamma}_0\cup\tilde{\Gamma}_\omega,\\ \dfrac{\partial u}{\partial n}=\tilde{g}, & \text{on } \tilde{\Gamma},\end{cases}\tag{3}$$

where $\tilde{g}=\frac{R}{k\sqrt{J}}g, J=\frac{R^2}{k^2}(\sin^2\varphi+k^2\cos^2\varphi)$. The artificial boundary is an elliptical arc

$$\tilde{\mathcal{B}}=\{(\xi,\eta)|k^2\xi^2+\eta^2=R^2,(\xi,\eta)\in\tilde{\Omega}\},$$

and the exterior domain to $\tilde{\mathcal{B}}$ is

$$\tilde{D}=\{(\xi,\eta)|k^2\xi^2+\eta^2>R^2,\ (\xi,\eta)\in\tilde{\Omega}\}.$$

Assume that $f=0$ in the domain $\tilde{D}$, then problem (3) confines in $\tilde{D}$ is

$$\begin{cases}-\Delta u=0, & \text{in } \tilde{D},\\ u=0, & \text{on } \tilde{\Gamma}_0\cup\tilde{\Gamma}_\omega,\\ u \text{ is vanish at infinite.}\end{cases}\tag{4}$$

By separation of variables, we know that the solution of problem (4) has the form

$$u(\mu,\varphi)=\sum_{n=1}^{+\infty}b_n e^{(\mu_R-\mu)\frac{n\pi}{\omega}}\sin\frac{n\pi\varphi}{\omega},\tag{5}$$

where

$$b_n=\frac{2}{\omega}\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\phi}{\omega}d\phi,\ n=1,2,\cdots.\tag{6}$$

Thus (6) can be written as

$$\begin{aligned}&u(\mu,\varphi)\\&=\frac{2}{\omega}\sum_{n=1}^{+\infty}e^{(\mu_R-\mu)\frac{n\pi}{\omega}}\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}d\phi\\&\triangleq H(\mu_R,\mu,\varphi).\end{aligned}\tag{7}$$

We differentiate (7) with respect to μ and set $\mu=\mu_R$ to obtain

$$\frac{\partial u}{\partial\mu}|_{\tilde{\mathcal{B}}}=-\frac{2\pi}{\omega^2}\sum_{n=1}^{+\infty}n\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}d\phi.\tag{8}$$

Since $\frac{\partial u}{\partial n}|_{\tilde{\mathcal{B}}}=-\frac{1}{\sqrt{J}}\frac{\partial u}{\partial\mu}|_{\tilde{\mathcal{B}}}$, we obtain the exact artificial boundary condition on $\tilde{\mathcal{B}}$:

$$\begin{aligned}\frac{\partial u}{\partial n}|_{\tilde{\mathcal{B}}}&=\frac{2\pi}{\omega^2\sqrt{J}}\sum_{n=1}^{+\infty}n\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}d\phi\\&\triangleq\mathcal{K}u(\mu_R,\varphi).\end{aligned}\tag{9}$$

III. Schwarz Alternating Method

We introduce two elliptical arc artificial boundaries $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ with the same foci, $\tilde{\Gamma}_i=\{(\mu,\varphi)|\mu>\mu_i,\ 0<\varphi<\omega\}$, $i=1,2$, which enclose $\tilde{\Gamma}$ such that $\mathrm{dist}(\tilde{\Gamma},\tilde{\Gamma}_2)>0$ and $\mu_1>\mu_2>0$. Then $\tilde{\Omega}$ is divided into two overlapping subdomains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ (see Fig. 2). Let $\tilde{\Omega}_1$ be the bounded domain among $\tilde{\Gamma}$, $\tilde{\Gamma}_0$, $\tilde{\Gamma}_\omega$ and $\tilde{\Gamma}_1$, and $\tilde{\Omega}_2$ be the unbounded domain outside $\tilde{\Gamma}_2$, $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_\omega$. Let $u_i^{(l)}=u^{(l)}|_{\tilde{\Omega}_i}$, $i=1,2$.

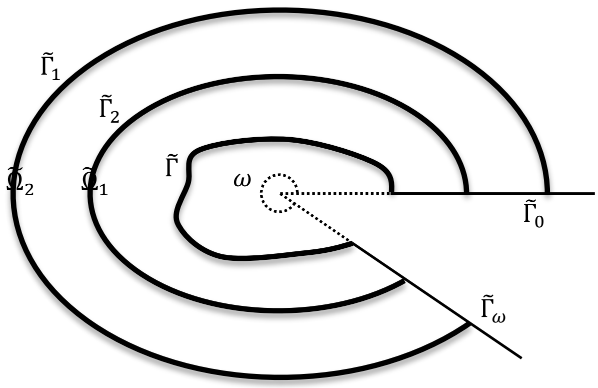

Fig.2: The illustration of domain $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$

The Schwarz alternating method is given by

$$\begin{cases}-\Delta u_1^{(l)}=f, & \text{in } \tilde{\Omega}_1,\\ u_1^{(l)}=0, & \text{on } \tilde{\Gamma}_0\cup\tilde{\Gamma}_\omega,\\ \dfrac{\partial u_1^{(l)}}{\partial n}=\tilde{g}, & \text{on } \tilde{\Gamma},\\ u_1^{(l)}=u_2^{(l-1)}, & \text{on } \tilde{\Gamma}_1,\end{cases}\quad l=1,2,\cdots,\tag{10}$$

and

$$\begin{cases} -\Delta u_2^{(l)} = f, \quad \text{in } \tilde{\Omega}_2, \\ u_2^{(l)} = 0, \quad \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ u_2^{(l)} = u_1^{(l)}, \quad \text{on } \tilde{\Gamma}_2, \\ u_2^{(l)} \text{ is vanish at infinity,} \end{cases} \quad l = 1,2,\cdots. \tag{11}$$

For problem (2), we can also construct the following Schwarz alternating method:

$$\begin{cases} -\Delta u_1^{(l)} = f, \quad \text{in } \tilde{\Omega}_1, \\ \frac{\partial u_1^{(l)}}{\partial n} = 0, \quad \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ u_1^{(l)} = \tilde{h}, \quad \text{on } \tilde{\Gamma}, \\ u_1^{(l)} = u_2^{(l-1)}, \quad \text{on } \tilde{\Gamma}_1, \end{cases} \quad l = 1,2,\cdots, \tag{12}$$

and

$$\begin{cases} -\Delta u_2^{(l)} = f, \quad \text{in } \tilde{\Omega}_2, \\ \frac{\partial u_2^{(l)}}{\partial n} = 0, \quad \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ u_2^{(l)} = u_1^{(l)}, \quad \text{on } \tilde{\Gamma}_2, \\ u_2^{(l)} \text{ is bounded at infinity,} \end{cases} \quad l = 1,2,\cdots. \tag{13}$$

Taking some initial value of function u_0 on boundary $\tilde{\Gamma}_1$, e.g. $u|_{\tilde{\Gamma}_1} = 0$. Combining it with the given boundary condition on $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \cup \tilde{\Gamma}$, we can solve the interior boundary value problem in domain $\tilde{\Omega}_1$, get the value of solution $u_1|_{\tilde{\Gamma}_2}$ on $\tilde{\Gamma}_2$, and then solve the exterior boundary value problem in domain $\tilde{\Omega}_2$, get the value of solution $u_2|_{\tilde{\Gamma}_1}$ on $\tilde{\Gamma}_1$, and then solve the problem in $\tilde{\Omega}_1$ again, ..., and so on.

In the following sections, we just consider the convergence and convergence rate of problem (1), we can obtain corresponding result of problem (2) in the same way.

IV. Convergence of the Method

The solution of problems (3) is in space

$$V = \{v \in W_0^1(\tilde{\Omega}) | v = 0, \text{ on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega\},$$

where

$$W_0^1(\Omega) = \{v | \frac{v}{\sqrt{x^2+y^2+1}\ln(x^2+y^2+2)}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\tilde{\Omega})\}.$$

Functions $u_1^{(l)} \in H^1(\tilde{\Omega}_1)$ and $u_2^{(l)} \in W_0^1(\tilde{\Omega}_2)$ can be extended to functions in V. Let

$$V_1 = \{v \in H^1(\tilde{\Omega}_1) | v = 0, \text{ on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \cup \tilde{\Gamma}_1\},$$

$$V_2 = \{v \in W_0^1(\tilde{\Omega}_2) | v = 0, \text{ on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \cup \tilde{\Gamma}_2\}.$$

Then

$$u_1^{(l)} - u_2^{(l-1)} \in V_1, \; u_2^{(l)} - u_1^{(l)} \in V_2.$$

We can look upon V_1 and V_2 as the subspaces of V. Define the bilinear form as follows

$$a(u,v) = \int_\Omega \nabla u \nabla v dx.$$

From this, the inner product $a(u,v)$ and the norm $\|v\|_1$ in V can be defined. Then (10) and (11) are equivalent to variational problems

$$\begin{cases} \text{Find } u_1^{(l)} \in V_1 + u_2^{(l-1)}, \quad \text{such that} \\ a(u_1^{(l)} - u, v_1) = 0, \quad \forall\, v_1 \in V_1, \end{cases} \tag{14}$$

and

$$\begin{cases} \text{Find } u_2^{(l)} \in V_2 + u_1^{(l)}, \quad \text{such that} \\ a(u_2^{(l)} - u, v_2) = 0, \quad \forall\, v_2 \in V_2. \end{cases} \tag{15}$$

Let $P_{V_i}: V \to V_i, i = 1,2$ denote the orthogonal projectors under the inner product $a(\cdot,\cdot)$. We have

$$\begin{cases} u_1^{(l)} - u_2^{(l-1)} = P_{V_1}(u - u_2^{(l-1)}), \\ u_2^{(l)} - u_1^{(l)} = P_{V_2}(u - u_1^{(l)}), \end{cases} \quad l = 1,2,\cdots \tag{16}$$

or equivalently

$$\begin{cases} u - u_1^{(l)} = P_{V_1^\perp}(u - u_2^{(l-1)}), \\ u - u_2^{(l)} = P_{V_2^\perp}(u - u_1^{(l)}), \end{cases} \quad l = 1,2,\cdots \tag{17}$$

where $V_i^\perp, i = 1,2$ are the orthogonal complementary spaces of V_i in V. Let

$$e_i^{(l)} = u - u_i^{(l)}, \; i = 1,2,$$

be errors. Then (17) is

$$\begin{cases} e_1^{(l)} = P_{V_1^\perp} e_2^{(l-1)}, \\ e_2^{(l)} = P_{V_2^\perp} e_1^{(l)}, \end{cases} \quad l = 1,2,\cdots \tag{18}$$

Therefore

$$\begin{cases} e_1^{(l+1)} = P_{V_1^\perp} P_{V_2^\perp} e_2^{(l)}, \quad l = 1,2,\cdots \\ e_2^{(l+1)} = P_{V_2^\perp} P_{V_1^\perp} e_1^{(l)}, \quad l = 0,1,\cdots \end{cases}$$

This implies that, if $\{e_i^{(l)}\}, i = 1,2$, are convergent, then their limits are in $V_1^\perp \cap V_2^\perp$. Similar to the proofs given in [10] we can show the following results

Theorem 1. $\lim_{l\to\infty} \left\| e_i^{(l)} \right\|_1 = 0$ for $i = 1,2$.

Theorem 2. There exists a constant $\delta, 0 \leq \delta < 1$, such that

$$\|e_1^{(l)}\|_1 \leq \delta^{l-1}\|e_1^{(1)}\|_1, \quad \|e_2^{(l)}\|_1 \leq \delta^l\|e_2^{(0)}\|_1.$$

Theorems 1 and 2 show that the Schwarz alternating method converges geometrically, and the contraction factor is δ. We find it is quite difficult to analyze the rate of convergence δ for general unbounded domain $\tilde{\Omega}$. However, it is possible to find δ when $\tilde{\Gamma}$ is an elliptical arc.

For simplicity, we let $\tilde{\Gamma}, \tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be elliptical arcs with the same foci, $\tilde{\Gamma} = \{(\mu,\varphi)|\mu = \mu_0,\ 0 < \varphi < \omega\}$, $\tilde{\Gamma}_i = \{(u, \varphi)|\ \mu > \mu_i,\ 0 < \varphi < \omega\}$, $i = 1,2$, and $\mu_1 > \mu_2 > \mu_0$. Let

$$e_2^{(0)}(\mu_1,\varphi) = \sum_{n=1}^{+\infty} b_n \sin\frac{n\pi\varphi}{\omega}, \tag{19}$$

is given on the artificial boundary $\tilde{\Gamma}_1$ and

$$\frac{\partial e_1^{(1)}}{\partial\mu} = 0, \text{ on } \tilde{\Gamma}. \tag{20}$$

And let

$$e_1^{(1)}(\mu,\varphi) = \sum_{n=1}^{+\infty}(A_n e^{\frac{n\pi\mu}{\omega}} + B_n e^{-\frac{n\pi\mu}{\omega}}) \sin\frac{n\pi\varphi}{\omega}, \text{ in } \tilde{\Omega}_1.$$

From (18) and (19) we have

$$A_n = \frac{b_n e^{-\frac{n\pi\mu_0}{\omega}}}{e^{\frac{n\pi}{\omega}(\mu_1-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_1)}}, \quad B_n = \frac{b_n e^{\frac{n\pi\mu_0}{\omega}}}{e^{\frac{n\pi}{\omega}(\mu_1-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_1)}}.$$

Hence

$$e_1^{(1)}(\mu,\varphi) = \sum_{n=1}^{+\infty} \frac{e^{\frac{n\pi}{\omega}(\mu-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu)}}{e^{\frac{n\pi}{\omega}(\mu_1-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_1)}} b_n \sin\frac{n\pi\varphi}{\omega}.$$

Therefore

$$e_1^{(1)}(\mu_2,\varphi) = \sum_{n=1}^{+\infty} \frac{e^{\frac{n\pi}{\omega}(\mu_2-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_2)}}{e^{\frac{n\pi}{\omega}(\mu_1-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_1)}} b_n \sin\frac{n\pi\varphi}{\omega}.$$

Using (7), we can obtain the value of function on $\tilde{\Gamma}_1$:

$$e_2^{(1)}(\mu_2,\varphi) = \sum_{n=1}^{+\infty} e^{\frac{n\pi}{\omega}(\mu_2-\mu_1)} \frac{e^{\frac{n\pi}{\omega}(\mu_2-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_2)}}{e^{\frac{n\pi}{\omega}(\mu_1-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_1)}} b_n \sin\frac{n\pi\varphi}{\omega}.$$

So

$$\begin{aligned}
&\|e_2^{(1)}\|^2_{\frac{1}{2},\tilde{\Gamma}_1} \\
&= \sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}}|e^{\frac{n\pi}{\omega}(\mu_2-\mu_1)} \frac{e^{\frac{n\pi}{\omega}(\mu_2-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_2)}}{e^{\frac{n\pi}{\omega}(\mu_1-\mu_0)}+e^{\frac{n\pi}{\omega}(\mu_0-\mu_1)}} b_n| \\
&\leq \sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}}|e^{\frac{n\pi}{\omega}(\mu_2-\mu_1)} \\
&\leq e^{\frac{2\pi}{\omega}(\mu_2-\mu_1)} \sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}}|b_n|^2 \\
&\leq e^{\frac{2\pi}{\omega}(\mu_2-\mu_1)} \|e_2^{(0)}\|^2_{\frac{1}{2},\tilde{\Gamma}_1}.
\end{aligned}$$

Similarly, we can obtain

$$\|e_1^{(2)}\|^2_{\frac{1}{2},\tilde{\Gamma}_2} \leq e^{\frac{2\pi}{\omega}(\mu_2-\mu_1)}\|e_1^{(1)}\|^2_{\frac{1}{2},\tilde{\Gamma}_2}.$$

Using mathematics induction, we have

$$\|e_2^{(l)}\|^2_{\frac{1}{2},\tilde{\Gamma}_1} \leq e^{\frac{2l\pi}{\omega}(\mu_2-\mu_1)}\|e_2^{(0)}\|^2_{\frac{1}{2},\tilde{\Gamma}_1},\ l = 1,2,\cdots,$$

$$\|e_1^{(l+1)}\|^2_{\frac{1}{2},\tilde{\Gamma}_2} \leq e^{\frac{2l\pi}{\omega}(\mu_2-\mu_1)}\|e_1^{(1)}\|^2_{\frac{1}{2},\tilde{\Gamma}_2},\ l = 1,2,\cdots.$$

Therefore, we have

Theorem 3. Let $\tilde{\Gamma}, \tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be elliptical arcs with the same foci, $\tilde{\Gamma} = \{(\mu,\varphi)|\mu = \mu_0,\ 0 < \varphi < \omega\}$, $\tilde{\Gamma}_i = \{(u, \varphi)|\ \mu > \mu_i,\ 0 < \varphi < \omega\}$, $i = 1,2$, and $\mu_1 > \mu_2 > \mu_0$. If we apply the Schwarz alternating method (10) and (11) to problem (3), then

$$\|e_2^{(l)}\|_{\frac{1}{2},\tilde{\Gamma}_1} \leq \delta^l\|e_2^{(0)}\|_{\frac{1}{2},\tilde{\Gamma}_1},\ l = 1,2,\cdots,$$

$$\|e_1^{(l+1)}\|_{\frac{1}{2},\tilde{\Gamma}_2} \leq \delta^l\|e_1^{(1)}\|_{\frac{1}{2},\tilde{\Gamma}_2},\ l = 1,2,\cdots,$$

where $\delta = e^{\frac{\pi}{\omega}(\mu_2-\mu_1)}$.

Finally, using the trace theorem we have

$$\|e_2^{(l)}\|_{\frac{1}{2},\tilde{\Gamma}_1} \leq C\delta^l,\ l = 1,2,\cdots,$$

$$\|e_1^{(l+1)}\|_{\frac{1}{2},\tilde{\Gamma}_2} \leq C\delta^l,\ l = 1,2,\cdots.$$

The smaller the $\mu_2 - \mu_1$ is, the faster the convergence is.

V. Numerical Examples

In this section, we give a numerical example to show the effectiveness of Schwarz alternating method. The finite element method with liner elements is used in the computation.

Example 1. We consider problem (1), where $\Omega = \{(r,\theta)|r > 2,\ 0 < \theta < 2\pi\}$, $\Gamma = \{(r,\theta)|r = 2,\ 0 < \theta <$

$2\pi\}$, $\Gamma_0 = \{(r,\theta)|r > 2,\ \theta = 0\}$, and $\Gamma_\omega = \{(r,\theta)|r > 2,\ \theta = 2\pi\}$. By using coordinate transformation $x = k\xi$, $y = \eta$, we turn the original problem into the problem as the following

$$\begin{cases} -\Delta u = f, & \text{in } \tilde{\Omega}, \\ u = 0, & \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ \frac{\partial u}{\partial n} = \tilde{g}, & \text{on } \tilde{\Gamma}, \end{cases} \tag{21}$$

where $\tilde{\Omega} = \{(\mu,\varphi)|\mu > \mu_0,\ 0 < \varphi < 2\pi\}$, $\tilde{\Gamma} = \{(\mu,\varphi)|\mu = \mu_0,\ 0 < \varphi < 2\pi\}$, $\tilde{\Gamma}_0 = \{(\mu,\varphi)|\mu = \mu_0,\ \varphi = 0\}$, $\tilde{\Gamma}_\omega = \{(\mu,\varphi)|\mu = \mu_0,\ \varphi = 2\pi\}$, $f_0 = \frac{2\sqrt{1-k^2}}{k}$, and $\mu_0 = \ln\frac{1+k}{\sqrt{1-k^2}}$. Let $u(x,y) = \frac{k^2 y}{x^2+k^2y^2}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$.

u_{1h} is the finite element solution in $\tilde{\Omega}_1$, e and e_h denote the maximal error of all node functions in $\tilde{\Omega}_1$, respectively, i.e.,

$$e(l) = \sup_{P_i \in \tilde{\Omega}_1} |u(P_i) - u_{1h}^l(P_i)|,$$

$$e_h(l) = \sup_{P_i \in \tilde{\Omega}_1} |u_{1h}^{l+1}(P_i) - u_{1h}^l(P_i)|.$$

$q_h(l)$ is the approximation of the convergence rate, i.e.,

$$q_h(l) = \frac{e_h(l-1)}{e_h(l)}.$$

Let $\tilde{\Gamma}_i = \{(\mu,\varphi)|\mu = \mu_0 + t_i,\ 0 < \varphi < 2\pi\}$, $i = 1,2$ be the artificial boundaries, and $t_1 = 1$. Fig. 3 shows the mesh h of subdomain $\tilde{\Omega}_1$, Table 1 shows the convergence rate for different anisotropic coefficient k (Mesh $h/4$, $t_2 = 0.25$). Table 2 shows the relation between convergence rate and mesh ($k = 0.5$, $t_2 = 0.25$). Table 3 shows the relation between convergence rate and overlapping degree ($k = 0.5$, Mesh $h/4$). Fig. 4 shows $L^\infty(\tilde{\Omega}_1)$ errors for different mesh. Fig. 5 shows $L^\infty(\tilde{\Omega}_1)$ errors for different overlapping degree.

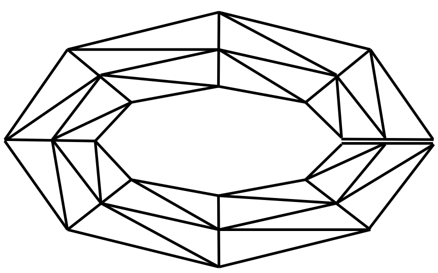

Fig. 3: Mesh h of domain $\tilde{\Omega}_1$.

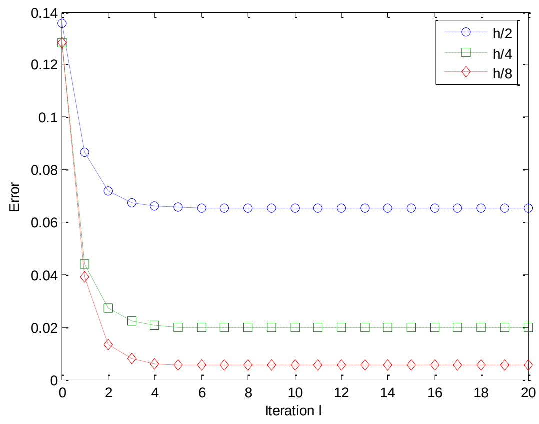

Fig. 4: $L^\infty(\tilde{\Omega}_1)$ errors for different mesh.

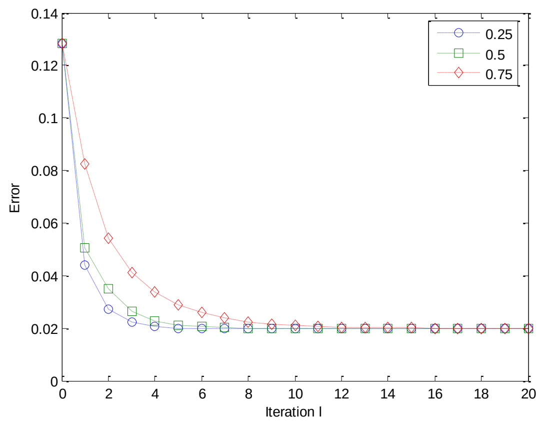

Fig. 5: $L^\infty(\tilde{\Omega}_1)$ errors for different overlapping degree.

TABLE 1: THE CONVERGENCE RATE FOR DIFFERENT ANISOTROPIC COEFFICIENT K (MESH $H/4$, $T_2 = 0.25$).

k	l	Numbers of iteration and corresponding values					
		0	1	2	3	4	5
0.8	$e(l)$	0.166	0.054	0.024	0.017	0.015	0.014
	$e_h(l)$		0.112	0.034	0.011	0.003	0.001
	$q_h(l)$			3.257	3.194	3.192	3.192
0.5	$e(l)$	0.128	0.044	0.027	0.022	0.021	0.020
	$e_h(l)$		0.088	0.026	0.008	0.003	0.001
	$q_h(l)$			3.399	3.207	3.198	3.197
0.2	$e(l)$	0.097	0.070	0.062	0.059	0.059	0.058
	$e_h(l)$		0.047	0.013	0.004	0.001	0.000
	$q_h(l)$			3.629	3.229	3.212	3.210

TABLE 2: THE RELATION BETWEEN CONVERGENCE RATE AND MESH ($K = 0.5$, $T_2 = 0.25$).

M	l	Numbers of iteration and corresponding values					
		0	1	2	3	4	5
$h/2$	$e(l)$	0.136	0.087	0.072	0.067	0.066	0.066
	$e_h(l)$		0.083	0.024	0.007	0.002	0.001
	$q_h(l)$			3.475	3.293	3.279	3.267
$h/4$	$e(l)$	0.128	0.044	0.027	0.022	0.021	0.020
	$e_h(l)$		0.088	0.026	0.008	0.003	0.001
	$q_h(l)$			3.399	3.207	3.198	3.197
$h/8$	$e(l)$	0.128	0.039	0.013	0.008	0.006	0.006
	$e_h(l)$		0.089	0.026	0.008	0.003	0.001
	$q_h(l)$			3.380	3.184	3.175	3.175

TABLE 3: THE RELATION BETWEEN CONVERGENCE RATE AND OVERLAPPING DEGREE ($\kappa = 0.5$, MESH $h/4$).

t_2	l	Numbers of iteration and corresponding values					
		0	1	2	3	4	5
0.25	$e(l)$	0.128	0.044	0.027	0.022	0.021	0.020
	$e_h(l)$		0.088	0.026	0.008	0.003	0.001
	$q_h(l)$			3.399	3.207	3.198	3.197
0.5	$e(l)$	0.128	0.050	0.035	0.027	0.023	0.021
	$e_h(l)$		0.072	0.030	0.013	0.006	0.003
	$q_h(l)$			2.426	2.288	2.272	2.270
0.75	$e(l)$	0.128	0.082	0.054	0.041	0.034	0.029
	$e_h(l)$		0.046	0.028	0.018	0.012	0.008
	$q_h(l)$			1.638	1.570	1.547	1.539

The numerical results show that the Schwarz alternating method is feasible and convergent quickly. Its convergence rate is related to the degree of overlapping of subdomains. The higher the overlapping degree of the two subdomains is, the faster the convergence is. Moreover, the convergence rate is nearly not affected by finite element mesh.

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