Abstract—In this paper, some exact nonlocal boundary conditions are derived on an elliptical arc artificial boundary, and they are applied to solving the exterior anisotropic problems in concave angle domains. Based on the above artificial boundary conditions, the Schwarz alternating algorithm is presented. The convergence of this algorithm is examined. Finally, some numerical examples are given to show the effectiveness of our methods.

Index Terms—Schwarz alternating algorithm, elliptical arc artificial boundary condition, anisotropic problem

I. INTRODUCTION

ANY scientific and engineering computing problems can be modeled by boundary value problems of partial differential equations in unbounded domains. There is a variety of numerical methods to solve such problems. One of the commonly used techniques is the method of artificial boundary conditions [1]-[9]. The method may be summarized as follows: (i) Introduce an artificial boundary \( \mathcal{B} \), which divides the original unbounded domain into two non-overlapping subdomains: a bounded computational domain \( \Omega \) and an infinite residual domain \( D \). (ii) By analyzing the problem in the infinite residual domain \( D \), obtain a relation on the artificial boundary \( \mathcal{B} \) involving the unknown function and its derivatives. (iii) Using the relation as a boundary condition on \( \mathcal{B} \), to obtain a well-posed problem in the bounded computational domain \( \Omega \). (iv) Solve the problem in the bounded computational domain \( \Omega \) be the standard finite element methods or some other numerical methods.

Based on artificial boundary conditions, the overlapping and non-overlapping domain decomposition methods can be viewed as effective ways to solve problems in unbounded domains. These techniques have been used to solve many linear or nonlinear problems [10]-[17]. Recently, the authors used a new elliptical arc artificial boundary and some iteration methods to solve Poisson problems and anisotropic problems [18]-[22]. In this paper, we derive an exact elliptical arc artificial boundary condition for anisotropic problems in an unbounded domain with a concave angle, and apply the methods in [10] to solving the above problems.

Let \( \Omega \) be an exterior concave angle domain with angle \( \omega \), and \( 0 < \omega < 2\pi \). The boundary of domain \( \Omega \) is decomposed into three disjoint parts: \( \Gamma, \Gamma_0 \) and \( \Gamma_\omega \) (see Fig. 1), i.e., \( \partial \Omega = \Gamma \cup \Gamma_0 \cup \Gamma_\omega \), \( \Gamma_0 \cap \Gamma_\omega = \emptyset \), \( \Gamma \cap \Gamma_0 = \emptyset \), \( \Gamma \cap \Gamma_\omega = \emptyset \). The boundary \( \Gamma \) is a simple smooth curve part, \( \Gamma_0 \) and \( \Gamma_\omega \) are two half lines.

\[ \begin{align*}
-\nabla \cdot (A \nabla u) &= f, & \text{in } \Omega, \\
u &= 0, & \text{on } \Gamma_0 \cup \Gamma_\omega, \\
A \nabla u \cdot n &= g, & \text{on } \Gamma, \\
u &= 0 & \text{at infinity},
\end{align*} \]

\[ \begin{align*}
-\nabla \cdot (A \nabla u) &= f, & \text{in } \Omega, \\
A \nabla u \cdot n &= 0, & \text{on } \Gamma_0 \cup \Gamma, \\
u &= h, & \text{on } \Gamma, \\
u &= 0 & \text{at infinity},
\end{align*} \]

where \( A = \begin{pmatrix} k^2 & 0 \\ 0 & 1 \end{pmatrix} \), \( k \) is a constant and \( 0 < k < 1 \), \( u \) is the unknown function, \( f \in L^2(\Omega) \) and \( g, h \in L^2(\Gamma) \) are given functions, \( \text{supp}(f) \) is compact.

Fig. 1. The illustration of domain \( \Omega \)

We consider the following anisotropic problem:

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Manuscript received April 8, 2019; revised August 5, 2019. This work was supported by the National Natural Science Foundation of China (Grant No. 11371198).

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The outline of the paper is as follows. In Section 2, we derive an exact elliptical arc artificial boundary condition for the above anisotropic problem. In Section 3, we construct a Schwarz alternating method. In Section 4, we give the convergence of the method, and analyze the convergence rate for a typical domain. Finally, in Section 5 we present some numerical results to show its accuracy and the effectiveness of our methods.

II. THE EXACT ARTIFICIAL BOUNDARY CONDITION

Let \( f_0 \) denote the half distance between the two foci of an ellipse, we introduce an elliptic system of co-ordinates \((\mu, \varphi)\) such that the artificial boundary \( \mathcal{B} \) coincides with the elliptical arc \( (\mu, \varphi) | \mu = \mu_R, 0 < \varphi < \omega \), where

\[
f_0 = \frac{\sqrt{1-k^2}}{k} R, \quad \mu_R = \ln \frac{1+k}{\sqrt{1-k^2}}.
\]

Thus, the Cartesian co-ordinates \((x, y)\) are related to the elliptic co-ordinates \((\mu, \varphi)\), that is \( x = f_0 \cos \mu \cos \varphi, \quad \eta = f_0 \sin \mu \sin \varphi \). The domain exterior to \( \mathcal{B} \) is denoted by \( \mathcal{B}^* = (\mu, \varphi) | \mu > \mu_R, 0 < \varphi < \omega \). We first introduce the following transformation \( x = k \xi, \ y = \eta \), then the anisotropic problem (1) become the following Poisson problem:

\[
\begin{cases}
-\Delta u = f, & \text{in } \Omega, \\
u_0 = 0, & \text{on } \mathcal{B}^* \\
\frac{\partial u}{\partial n} = g, & \text{on } \mathcal{B},
\end{cases}
\]

(3)

where \( g = \frac{R}{k^2} f_0 = \frac{R^2}{k^2} \left( \sin^2 \varphi + k^2 \cos^2 \varphi \right) \). The artificial boundary is an elliptical arc \( \mathcal{B} = ((\xi, \eta) | k^2 \xi^2 + \eta^2 = R^2, (\xi, \eta) \in \Omega) \), and the exterior domain to \( \mathcal{B} \) is \( \mathcal{B}^* = ((\xi, \eta) | k^2 \xi^2 + \eta^2 > R^2, (\xi, \eta) \in \Omega) \).

Assume that \( f = 0 \) in the domain \( \mathcal{B}^* \), then problem (3) confines in \( \mathcal{B} \) is

\[
\begin{cases}
-\Delta u = 0, & \text{in } \mathcal{D}, \\
u_0 = 0, & \text{on } \mathcal{B}^* \cup \mathcal{B}^*, \\
u \text{ is vanish at infinite.}
\end{cases}
\]

(4)

By separation of variables, we know that the solution of problem (4) has the form

\[
u(\mu, \varphi) = \sum_{n=1}^{+\infty} b_n e^{(\mu_R-\mu)\frac{n\pi}{\omega}} \sin \frac{n\pi \varphi}{\omega},
\]

(5)

where

\[
b_n = \frac{2}{\omega} \int_0^\omega u(\mu_R, \varphi) \sin \frac{n\pi \varphi}{\omega} d\varphi, \quad n = 1, 2, \cdots.
\]

(6)

Thus (6) can be written as

\[

u(\mu, \varphi) = \frac{2}{\omega} \sum_{n=1}^{+\infty} \frac{e^{(\mu_R-\mu)\frac{n\pi}{\omega}}}{\omega} \int_0^\omega u(\mu_R, \varphi) \sin \frac{n\pi \varphi}{\omega} d\varphi \sin \frac{n\pi \varphi}{\omega} d\varphi
\]

(7)

\[

\Delta u = -H(\mu_R, \mu, \varphi).
\]

We differentiate (7) with respect to \( \mu \) and set \( \mu = \mu_R \) to obtain

\[

\frac{\partial u}{\partial \mu} \big|_{\mathcal{B}} = -\frac{2n}{\omega^2} \sum_{n=1}^{+\infty} \int_0^\omega u(\mu_R, \varphi) \sin \frac{n\pi \varphi}{\omega} d\varphi.
\]

(8)

Since \( \frac{\partial u}{\partial n} \big|_{\mathcal{B}} = -\frac{1}{\sqrt{1-k^2}} \frac{\partial u}{\partial \mu} \big|_{\mathcal{B}} \), we obtain the exact artificial boundary condition on \( \mathcal{B} \):

\[

\frac{\partial u}{\partial n} \big|_{\mathcal{B}} = \frac{2n}{\omega^2} \sum_{n=1}^{+\infty} \int_0^\omega u(\mu_R, \varphi) \sin \frac{n\pi \varphi}{\omega} d\varphi
\]

(9)

III. SCHWARZ ALTERNATING METHOD

We introduce two elliptical arc artificial boundaries \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) with the same foci, \( \mathcal{B}_1 = ((\mu, \varphi) | \mu > \mu_1, 0 < \varphi < \omega) \), \( \mathcal{B}_2 = ((\mu, \varphi) | \mu > \mu_2, 0 < \varphi < \omega) \), which enclose \( \mathcal{D} \) such that \( \text{dist}(\mathcal{B}_1, \mathcal{B}_2) > 0 \) and \( \mu_1 > \mu_2 > 0 \). Then \( \mathcal{D} \) is divided into two overlapping subdomains \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) (see Fig. 2). Let \( \mathcal{B}_0 \) be the bounded domain among \( \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_0 \), and \( \mathcal{B}_0 \) be the unbounded domain outside \( \mathcal{B}_2, \mathcal{B}_0, \mathcal{B}_1 \). Let \( u^{(i)} = u^{(i)}|_{\mathcal{B}_0}, \ i = 1, 2. \)

Fig. 2: The illustration of domain \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \)

The Schwarz alternating method is given by

\[

\begin{cases}
-\Delta u^{(i)} = f, & \text{in } \mathcal{B}_i, \\
u^{(i)} = 0, & \text{on } \mathcal{B}_0 \cup \mathcal{B}_i, \\
\frac{\partial u^{(i)}}{\partial n} = g, & \text{on } \mathcal{B}_i, \quad l = 1, 2, \cdots,
\end{cases}
\]

(10)

and

Thus (6) can be written as

(Advance online publication: 20 November 2019)
For problem (2), we can also construct the following Schwarz alternating method:

\[
\begin{cases}
-\Delta u_1^{(l)} = f, & \text{in } \tilde{\Omega}_1, \\
\frac{\partial u_1^{(l)}}{\partial n} = 0, & \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \quad l = 1, 2, \ldots, \\
u_1^{(l)} = \tilde{h}_1, & \text{on } \tilde{\Gamma}_1, \\
u_1^{(l)} = u_2^{(l-1)}, & \text{on } \tilde{\Gamma}_2, \\
u_2^{(l)} \text{ is vanishing at infinity,}
\end{cases}
\]  

(12)

and

\[
\begin{cases}
-\Delta u_2^{(l)} = f, & \text{in } \tilde{\Omega}_2, \\
\frac{\partial u_2^{(l)}}{\partial n} = 0, & \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \quad l = 1, 2, \ldots, \\
u_2^{(l)} = u_1^{(l)}, & \text{on } \tilde{\Gamma}_2, \\
u_2^{(l)} \text{ is bounded at infinity,}
\end{cases}
\]  

(13)

Taking some initial value of function \( u_0 \) on boundary \( \tilde{\Gamma}_1 \), e.g. \( u_0|_{\tilde{\Gamma}_1} = 0 \). Combining it with the given boundary condition on \( \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \cup \tilde{\Gamma}_1 \), we can solve the interior boundary value problem in domain \( \tilde{\Omega}_1 \), get the value of solution \( u_1^{(1)}|_{\tilde{\Gamma}_2} \) on \( \tilde{\Gamma}_2 \), and then solve the exterior boundary value problem in domain \( \tilde{\Omega}_2 \), get the value of solution \( u_2^{(1)}|_{\tilde{\Gamma}_1} \) on \( \tilde{\Gamma}_1 \), and then solve the problem in \( \tilde{\Omega}_1 \) again, ..., and so on.

In the following sections, we just consider the convergence and convergence rate of problem (1), we can obtain corresponding result of problem (2) in the same way.

IV. CONVERGENCE OF THE METHOD

The solution of problems (3) is in space

\[ V = \{ v \in W^{1}_0(\tilde{\Omega}) | v = 0, \text{ on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \}, \]

where

\[ W^{1}_0(\Omega) = \{ v | v = \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \in L^2(\tilde{\Omega}) \}. \]

Functions \( u_1^{(l)} \in H^1(\tilde{\Omega}_1) \) and \( u_2^{(l)} \in W^{1}_0(\tilde{\Omega}_2) \) can be extended to functions in \( V \). Let

\[ V_1 = \{ v \in H^1(\tilde{\Omega}_1) | v = 0, \text{ on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \cup \tilde{\Gamma}_1 \}, \]

\[ V_2 = \{ v \in W^{1}_0(\tilde{\Omega}_2) | v = 0, \text{ on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega \cup \tilde{\Gamma}_2 \}. \]

Then

\[ u_1^{(l)} - u_2^{(l-1)} \in V_1, \quad u_2^{(l)} - u_1^{(l)} \in V_2. \]  

(11)

We can look upon \( V_1 \) and \( V_2 \) as the subspaces of \( V \). Define the bilinear form as follows

\[ a(u, v) = \int_{\Omega} \nabla u \nabla v dx. \]

From this, the inner product \( a(u, v) \) and the norm \( \| v \|_V \) in \( V \) can be defined. Then (10) and (11) are equivalent to variational problems

\[ \left\{ \begin{array}{l}
\text{Find } u_1^{(l)} \in V_1 + u_2^{(l-1)}, \text{ such that } \\
a(u_1^{(l)} - u, v_1) = 0, \quad \forall v_1 \in V_1,
\end{array} \right. \]  

(14)

and

\[ \left\{ \begin{array}{l}
\text{Find } u_2^{(l)} \in V_2 + u_1^{(l)}, \text{ such that } \\
a(u_2^{(l)} - u, v_2) = 0, \quad \forall v_2 \in V_2.
\end{array} \right. \]  

(15)

Let \( P_{\mathcal{V}_i}: V \rightarrow \mathcal{V}_i, i = 1, 2 \) denote the orthogonal projectors under the inner product \( a(\cdot, \cdot) \). We have

\[ \left\{ \begin{array}{l}
u_1^{(l)} - u^{(l-1)} = P_{\mathcal{V}_1} (u - u^{(l-1)}), \\
u_2^{(l)} - u^{(l)} = P_{\mathcal{V}_2} (u - u^{(l)}),
\end{array} \right. \]  

(16)

or equivalently

\[ \left\{ \begin{array}{l}
u - u_1^{(l)} = P_{\mathcal{V}_1} (u - u_2^{(l-1)}), \\
u - u_2^{(l)} = P_{\mathcal{V}_2} (u - u_1^{(l)}),
\end{array} \right. \]  

(17)

where \( \mathcal{V}_i, i = 1, 2 \) are the orthogonal complementary spaces of \( \mathcal{V}_i \) in \( V \). Let

\[ e_i^{(l)} = u - u_i^{(l)}, \quad i = 1, 2, \]

be errors. Then (17) is

\[ \left\{ \begin{array}{l}
e_1^{(l)} = P_{\mathcal{V}_1} e_2^{(l-1)}, \\
e_2^{(l)} = P_{\mathcal{V}_2} e_1^{(l)}.
\end{array} \right. \]  

(18)

Therefore

\[ \left\{ \begin{array}{l}
e_1^{(l+1)} = P_{\mathcal{V}_1} P_{\mathcal{V}_2} e_1^{(l)}, \\
e_2^{(l+1)} = P_{\mathcal{V}_2} P_{\mathcal{V}_1} e_2^{(l)},
\end{array} \right. \]  

(19)

This implies that, if \( (e_i^{(l)})_l, i = 1, 2, \) are convergent, then their limits are in \( \mathcal{V}_1 \cap \mathcal{V}_2 \). Similar to the proofs given in [10] we can show the following results

**Theorem 1.** \( \lim_{l \rightarrow \infty} \| e_i^{(l)} \|_V = 0 \) for \( i = 1, 2 \).
**Theorem 2.** There exists a constant $\delta$, $0 \leq \delta < 1$, such that

$$\|e^{(1)}_1\| \leq \delta^{i-1}\|e^{(1)}_1\|_1, \quad \|e^{(1)}_2\| \leq \delta^{i}\|e^{(0)}_2\|_1.$$  

Theorems 1 and 2 show that the Schwarz alternating method converges geometrically, and the contraction factor is $\delta$. We find it is quite difficult to analyze the rate of convergence for general unbounded domain $\Omega$. However, it is possible to find $\delta$ when $\Omega$ is an elliptical arc.

For simplicity, we let $\bar{F}_1$, $\bar{F}_2$, and $\bar{F}_2'$ be elliptical arcs with the same foci, $\bar{F} = \{(u, \varphi)\mid \mu > \mu_0, \ 0 < \varphi < \omega\}$, $\bar{F}_1 = \{(u, \varphi)\mid \mu > \mu_0, \ 0 < \varphi < \omega\}$, $i = 1, 2$, and $\mu_1 > \mu_2 > \mu_0$. Let

$$e^{(0)}_2(\mu_1, \varphi) = \sum_{n=1}^{\infty} b_n \sin \frac{n\varphi}{\omega}, \quad (19)$$

is given on the artificial boundary $\bar{F}$ and

$$\frac{\partial e^{(1)}_1}{\partial \mu} = 0, \text{ on } \bar{F}.$$  

(20)

And let

$$e^{(1)}_1(\mu, \varphi) = \sum_{n=1}^{\infty} \left( A_n e^{\frac{n\mu}{\omega}} + B_n e^{\frac{-n\mu}{\omega}} \right) \sin \frac{n\varphi}{\omega}, \quad (18)$$

From (18) and (19) we have

$$A_n = \frac{b_n e^{\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}}, \quad B_n = \frac{b_n e^{-\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}},$$

Hence

$$e^{(1)}_1(\mu, \varphi) = \sum_{n=1}^{\infty} \frac{e^{\frac{n\mu}{\omega}+\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}} b_n \sin \frac{n\varphi}{\omega}.$$  

Therefore

$$e^{(1)}_2(\mu_2, \varphi) = \sum_{n=1}^{\infty} \frac{e^{\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}} b_n \sin \frac{n\varphi}{\omega}.$$  

Using (7), we can obtain the value of function on $\bar{F}_1$:

$$\|e^{(1)}_1\|^2_{\bar{F}_1} = \sum_{n=1}^{\infty} \frac{e^{\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}} b_n \sin \frac{n\varphi}{\omega}.$$  

Similarly, we can obtain

$$\|e^{(2)}_1\|^2_{\bar{F}_1} \leq \sum_{n=1}^{\infty} \frac{e^{\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}} b_n \sin \frac{n\varphi}{\omega}.$$  

Using mathematics induction, we have

$$\|e^{(i)}_2\|^2_{\bar{F}_1} \leq e^{\frac{2n\mu_0}{\omega}(\mu_2-\mu_1)} \|e^{(0)}_1\|^2_{\bar{F}_1}, \quad l = 1, 2, \ldots$$

Therefore, we have

**Theorem 3.** Let $\bar{F}_1$, $\bar{F}_2$, and $\bar{F}_2'$ be elliptical arcs with the same foci, $\bar{F} = \{(u, \varphi)\mid \mu = \mu_0, \ 0 < \varphi < \omega\}$, $\bar{F}_1 = \{(u, \varphi)\mid \mu > \mu_0, \ 0 < \varphi < \omega\}$, $i = 1, 2$, and $\mu_1 > \mu_2 > \mu_0$. If we apply the Schwarz alternating method (10) and (11) to problem (3), then

$$\|e^{(i)}_2\|^2_{\bar{F}_1} \leq \|e^{(0)}_1\|^2_{\bar{F}_1}, \quad l = 1, 2, \ldots$$

$$\|e^{(i+1)}_1\|^2_{\bar{F}_2} \leq \|e^{(1)}_1\|^2_{\bar{F}_2}, \quad l = 1, 2, \ldots$$

Finally, using the trace theorem we have

$$\|e^{(i)}_2\|^2_{\bar{F}_1} \leq C\delta^i \|e^{(0)}_1\|^2_{\bar{F}_1}.$$  

$$\|e^{(i+1)}_1\|^2_{\bar{F}_2} \leq C\delta^i \|e^{(1)}_1\|^2_{\bar{F}_2}.$$  

The smaller the $\mu_2 - \mu_1$ is, the faster the convergence is.

**V. NUMERICAL EXAMPLES**

In this section, we give a numerical example to show the effectiveness of Schwarz alternating method. The finite element method with liner elements is used in the computation.

**Example 1.** We consider problem (1), where $\Omega = \{(r, \theta)\mid r > 2, \ 0 < \theta < 2\pi\}$, $\Gamma = \{(r, \theta)\mid r = 2, \ 0 < \theta < 2\pi\}$, $\mu_0 = \frac{\pi}{\omega}$, $\mu_1 = \frac{\pi}{\omega}$, and

$$\sum_{n=1}^{\infty} \frac{e^{\frac{n\mu_0}{\omega}}}{e^{\frac{n\mu}{\omega}(\mu_1-\mu_0)} + e^{\frac{n\mu}{\omega}(\mu_0-\mu_1)}} b_n \sin \frac{n\varphi}{\omega}.$$
\[2\pi \} , \Gamma_0 = \{(r, \theta)|r > 2, \ \theta = 0\} , \text{ and } \Gamma_\infty = \{(r, \theta)|r > 2, \ \theta = 2\pi\}. \] By using coordinate transformation \(x = k\xi, y = \eta,\) we turn the original problem into the problem as the following

\[
\begin{aligned}
- \Delta u &= f, \quad \text{in } \tilde{\Omega}, \\
\partial u / \partial n &= g, \quad \text{on } \tilde{\Gamma},
\end{aligned}
\]

where \(\tilde{\Omega} = \{(\mu, \varphi)|\mu > \mu_0, \ 0 < \varphi < 2\pi\}, \quad \tilde{\Gamma} = \{(\mu, \varphi)|\mu = \mu_0, \ 0 < \varphi < 2\pi\} \quad \text{and} \quad \tilde{\Gamma}_0 = \{(\mu, \varphi)|\mu = \mu_\infty, \ \varphi = 0\} \quad \text{and} \quad \tilde{\Gamma}_\infty = \{(\mu, \varphi)|\mu = \mu_0, \ \varphi = 2\pi\}. \]

Let \(u(x, y) = \frac{k^2 y}{x^2 + k^2 y^2}\) be the exact solution of original problem and \(g = \frac{\partial u}{\partial n}\) |

\[u_{th}\] is the finite element solution in \(\tilde{\Omega}_1\), \(e\) and \(e_h\) denote the maximal error of all node functions in \(\tilde{\Omega}_1\), respectively, i.e.,

\[e(l) = \sup_{P_i \in \tilde{\Omega}_1} |u(P_i) - u_{th}(P_i)|,
\]

\[e_h(l) = \sup_{P_i \in \tilde{\Omega}_1} |u_{th}^{l+1}(P_i) - u_{th}^l(P_i)|.
\]

\(q_h(l)\) is the approximation of the convergence rate, i.e.,

\[q_h(l) = \frac{e_h(l-1)}{e_h(l)}.
\]

Let \(\Gamma_i = \{(\mu, \varphi)|\mu = \mu_0 + t_i, \ 0 < \varphi < 2\pi\} \quad i = 1, 2\) be the artificial boundaries, and \(t_1 = 1, \Gamma_2 = 0.25\). Table 3 shows the relation between convergence rate and overlapping degree (\(k = 0.5, \ Mesh h/4, t_2 = 0.25\)). Fig. 4 shows \(L^\infty(\tilde{\Omega}_1)\) errors for different mesh. Fig. 5 shows \(L^\infty(\tilde{\Omega}_1)\) errors for different overlapping degree.

---

**Table 1: The convergence rate for different anisotropic coefficient \(k\) (Mesh \(h/4, t_2 = 0.25\)).**

<table>
<thead>
<tr>
<th>(k)</th>
<th>0.8</th>
<th>0.5</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(e(l))</td>
<td>0.166</td>
<td>0.054</td>
<td>0.024</td>
</tr>
<tr>
<td>(e_h(l))</td>
<td>0.112</td>
<td>0.034</td>
<td>0.011</td>
</tr>
<tr>
<td>(q_h(l))</td>
<td>3.257</td>
<td>3.194</td>
<td>3.192</td>
</tr>
</tbody>
</table>

**Table 2: The relation between convergence rate and mesh \((k = 0.5, t_2 = 0.25)\).**

<table>
<thead>
<tr>
<th>(M)</th>
<th>0.8</th>
<th>0.5</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(h/2)</td>
<td>0.136</td>
<td>0.087</td>
<td>0.072</td>
</tr>
<tr>
<td>(e_h(l))</td>
<td>0.083</td>
<td>0.024</td>
<td>0.007</td>
</tr>
<tr>
<td>(q_h(l))</td>
<td>3.475</td>
<td>3.293</td>
<td>3.279</td>
</tr>
<tr>
<td>(h/4)</td>
<td>0.128</td>
<td>0.044</td>
<td>0.027</td>
</tr>
<tr>
<td>(e_h(l))</td>
<td>0.088</td>
<td>0.026</td>
<td>0.008</td>
</tr>
<tr>
<td>(q_h(l))</td>
<td>3.399</td>
<td>3.207</td>
<td>3.198</td>
</tr>
<tr>
<td>(h/8)</td>
<td>0.128</td>
<td>0.039</td>
<td>0.013</td>
</tr>
<tr>
<td>(e_h(l))</td>
<td>0.089</td>
<td>0.026</td>
<td>0.008</td>
</tr>
<tr>
<td>(q_h(l))</td>
<td>3.380</td>
<td>3.184</td>
<td>3.175</td>
</tr>
</tbody>
</table>

---

Fig. 3: Mesh \(h\) of domain \(\tilde{\Omega}_1\).
The numerical results show that the Schwarz alternating method is feasible and convergent quickly. Its convergence rate is related to the degree of overlapping of subdomains. The higher the overlapping degree of the two subdomains is, the faster the convergence is. Moreover, the convergence rate is nearly not affected by finite element mesh.

ACKNOWLEDGMENT

The authors would like to thank the reviewers for their valuable comments which improve the paper.

REFERENCES


Table 3: The relation between convergence rate and overlapping degree (κ = 0.5, Mesh H/4).

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<th>tσ</th>
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<td>sσ(l)</td>
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(Advance online publication: 20 November 2019)