Sliding Mode Control Applied to MIMO Systems

Felipe Osorio-Arteaga, Didier Giraldo-Buitrago, Eduardo Giraldo

***Abstract*—This article presents a novel sliding plane control approach by using differential geometry applied to multi-input multi-output coupled nonlinear systems. The design methodology is based on the decoupling of the output signals, which are controlled as independent loops. Smooth functions are used for the switching control signal to decrease the chattering in the outputs. Experimental results are validated over a simulated stirred tank where the robustness of the proposed control method to disturbances is observed. Besides, the chattering effect is effectively decreased by using a smooth function to approximate the discontinuous term in the control signal.**

***Index Terms*—Sliding mode control, variable structure, MIMO systems.**

I. Introduction

THE sliding mode control approach is a variable structure method with inherent robust features where the stability of the closed-loop system is ensured [1]. This control approach has been developed since the 1950s for many types of continuous and discrete systems, such as non-linear, scalar and multivariable systems with outstanding results in comparison with classical methods [2], [3]. The main drawback of the method is the chattering obtained in the outputs of the system due to a high-frequency switching control signal [4]. However, this drawback can be reduced through several methods as proposed in [5], [6]. It is worth noting that the application of the sliding mode control to multi-input multi-output (MIMO) systems is a complex task, especially when the MIMO system is coupled or nonlinear [1], [7].

In this paper, a sliding mode control applied to MIMO systems is presented, based on the decoupling of the output signals. The paper is organized as follows: in section II is shown the sliding mode control of MIMO systems based on output signals decoupling through of equivalent control calculus and its ideal sliding dynamics. In section III the design of the controller for a stirred tank, detailing its corresponding mathematical modeling is presented, and finally, in section IV the experimental results based on a simulated model with and without step disturbances are presented and discussed.

II. Sliding mode control of MIMO systems

Consider a time-invariant MIMO system of the form:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{G}\mathbf{u}\\ \mathbf{y} &= \mathbf{h}(\mathbf{x})\end{aligned} \tag{1}$$

where $\mathbf{f}$ is a vector-valued. There are m sliding surfaces, such as m control signals, whose intersection describes the sliding mode, as follows:

$$S = \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{X} \subset \mathbb{R}^n : \sigma_i(\mathbf{x}) = 0\} \tag{2}$$

$$u_i = \begin{cases} u_i^+(x,t) & \text{for } \sigma_i(x) > 0 \\ u_i^-(x,t) & \text{for } \sigma_i(x) < 0 \end{cases} \tag{3}$$

The sliding mode dynamics can be described through the equivalent control:

$$\sigma(\mathbf{x}) = 0 \qquad \dot{\sigma}(\mathbf{x}) = \nabla\sigma(\mathbf{x})\ \mathbf{f}(\mathbf{x}) + \nabla\sigma(\mathbf{x})\ \mathbf{G}\mathbf{u_{eq}} \tag{4}$$

where $\nabla\sigma(x)$ is a matrix with m rows, equal to m sliding surfaces; and n columns, equal to n state variables, and where the equivalent control is defined by:

$$\mathbf{u_{eq}} = -\left(\nabla\sigma(\mathbf{x})\ \mathbf{G}\right)^{-1}\left(\nabla\sigma(\mathbf{x})\ \mathbf{f}(\mathbf{x})\right) \tag{5}$$

And also fulfilling the transversality condition as the determinant of the matrix defined by $\nabla\sigma(\mathbf{x})$ and $\mathbf{G}$ as follows:

$$\det\left(\nabla\sigma(\mathbf{x})\ \mathbf{G}\right) \neq 0 \tag{6}$$

the following sliding dynamics is obtained, as proposed in [1]:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}\mathbf{u_{eq}} = \left(\mathbf{I} - \mathbf{G}\left(\nabla\sigma(\mathbf{x})\mathbf{G}\right)^{-1}\nabla\sigma(\mathbf{x})\right)\mathbf{f}(\mathbf{x}) \tag{7}$$

Now, consider a time-invariant linear MIMO system with n states, m inputs and p outputs:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned} \tag{8}$$

which can be described through a closed-loop transference matrix, that contents the transfer functions relating each output to each reference signal [8]. In this transference matrix can be observed changes in some or all system outputs due to each reference signal, so the design problem is to control each output separately without to affect the other system outputs.

The integral action in sliding mode control helps to achieve this separation by minimizing the tracking error, being zero in the sliding mode, allowing each output to be controlled independently. This separation is possible by minimizing the interaction between the output signals [4].

To minimize the interaction, the closed-loop transference matrix must have very slight terms outside of its main diagonal (diagonal matrix), as follows:

$$Y(s) = H(s)R(s) \quad H(s) = \begin{bmatrix} H_{11} & 0 & \cdots & 0 \\ 0 & H_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{pp} \end{bmatrix} \tag{9}$$

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Therefore, m sliding planes are defined according to the n state variables and p integral actions of the tracking error, as follows:

$$\sigma(x) = K_a x + K_b \int (y-r)d\tau \tag{10}$$

From (8) and applying the invariance condition the following equivalent control can be obtained:

$$\dot{\sigma}(x) = \frac{d\sigma(x)}{dt} = K_a\dot{x} + K_b(y-r) = 0 \tag{11}$$

$$\dot{\sigma}(x) = K_a(Ax + Bu_{eq}) + K_bCx - K_br$$

$$\dot{\sigma}(x) = (K_aA + K_bC)x + K_aBu_{eq} - K_br$$

$$K_aBu_{eq} = -(K_aA + K_bC)x + K_br$$

$$u_{eq} = -(K_aB)^{-1}(K_aA + K_bC)x + (K_aB)^{-1}K_br \tag{12}$$

Replacing in (8), the ideal sliding dynamics is obtained:

$$\dot{x} = Ax + B\left(-(K_aB)^{-1}(K_aA + K_bC)x + (K_aB)^{-1}K_br\right)$$

$$\dot{x} = Ax - B(K_aB)^{-1}(K_aA + K_bC)x + B(K_aB)^{-1}K_br$$

$$\dot{x} = \left(A - B(K_aB)^{-1}(K_aA + K_bC)\right)x + B(K_aB)^{-1}K_br$$

$$\begin{aligned} \dot{x} &= A_{PD}x + B_{PD}r \\ A_{PD} &= A - B(K_aB)^{-1}(K_aA + K_bC) \\ B_{PD} &= B(K_aB)^{-1}K_b \end{aligned} \tag{13}$$

The closed-loop transference matrix is obtained from the ideal sliding dynamics, as follows:

$$H(s) = C\,(sI - A_{PD})^{-1}\,B_{PD}$$

By defining:

$$K = B(K_aB)^{-1}K_b \tag{14}$$

the closed-loop transference matrix can be found like this:

$$H(s) = C\left(sI - A + B(K_aB)^{-1}(K_aA + K_bC)\right)^{-1}K \tag{15}$$

From (14), a relationship between K_a and K_b is obtained:

$$K_aK = K_b \tag{16}$$

The matrix K_a is selected, such that $\det(K_aB) \neq 0$. Then, assuming K unknown, $H(s)$ is calculated for the system to be stable and by considering that their elements outside of its main diagonal are close to zero or zero [7].

From (11), the following expression is obtained:

$$\dot{\sigma}(x) = K_aAx + K_aBu_{eq} + K_b(y-r) = 0$$

where the control signal is defined as follows:

$$u_{eq} = -(K_aB)^{-1}\left(\Psi_a x + \Psi_b(y-r)\right)$$

$$\dot{\sigma}(x) = (K_aA - \Psi_a)x + (K_b - \Psi_b)(y-r)$$

By considering the reachability condition:

$$\dot{\sigma}_i(x)\sigma_i(x) < 0$$

a control signal is defined, such that if $\sigma_i(x)$ is positive, $\dot{\sigma}_i(x)$ is negative and vice versa. Therefore, the resulting control signal is defined by:

$$u = -(K_aB)^{-1}\left(\Psi_a|x| + \Psi_b|y-r|\right)\operatorname{sign}(\sigma(x)) \tag{17}$$

where

$$(\Psi_a)_{ij} > |(K_aA)_{ij}| \qquad (\Psi_b)_{ij} > |(K_b)_{ij}|$$

An adjustment parameter k_u is included, such that the previous inequality is met.

$$(\Psi_a)_{ij} = k_u|(K_aA)_{ij}| \qquad (\Psi_b)_{ij} = k_u|(K_b)_{ij}| \tag{18}$$

To solve the drawback of chattering, which is related to the discontinuous function that ensures a sliding mode on the sliding plane, functions such as hyperbolic tangent or the saturation function are used to approximate the sign function [7], as follows:

$$\begin{aligned} \operatorname{sign}(x) &\approx \tanh(x) \\ \operatorname{sign}(x) &\approx \operatorname{sat}(x) = \frac{x}{|x| + \delta} \end{aligned} \tag{19}$$

III. Mathematical model stirred tank

Consider the stirred tank of Fig. 1 with constant cross-sectional area A, which is fed with two time-varying inflows $F_1(t)$ and $F_2(t)$, both with constant concentrations $c1$ and $c2$, respectively, and also an outflow $F(t)$ with concentration $c(t)$ equals to the concentration in the tank. The tank in steady-state has a $1\ \mathrm{m}^3$ of constant volume and a $0.02\ \mathrm{m}^3/\mathrm{s}$ of constant outflow, with a $1.25\ \mathrm{kmol/m}^3$ of constant concentration. The input concentrations c_1 and c_2 are $1\ \mathrm{kmol/m}^3$ and $2\ \mathrm{kmol/m}^3$, respectively [9].

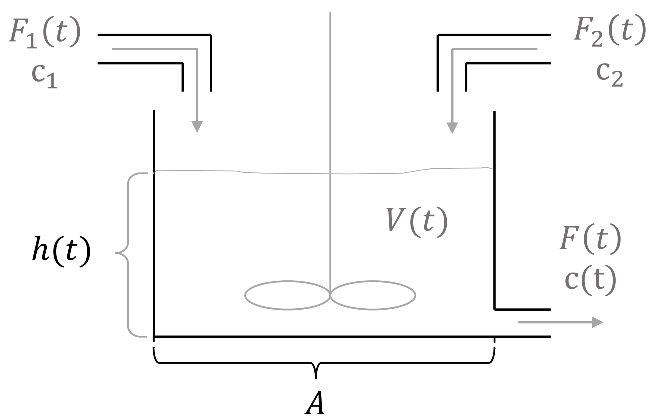

Fig. 1. Stirred tank.

First consider the mass and flow balance equations:

$$F_1(t) + F_2(t) - F(t) = \dot{V}(t)$$

$$c_1F_1(t) + c_2F_2(t) - c(t)F(t) = \frac{d}{dt}\left(c(t)V(t)\right)$$

where the outflow $F(t)$ depends on the volume and the cross-sectional area, as follows:

$$F(t) = k\sqrt{\frac{V(t)}{A}}$$

By assuming slight deviations and linearizing the system at the steady-state conditions, the following equations are obtained:

$$\Delta F_1(t) + \Delta F_2(t) - \frac{F_o}{2V_o}\Delta V(t) = \Delta\dot{V}(t)$$

$$c_1\Delta F_1(t)+c_2\Delta F_2(t)-c_o\frac{F_o}{2V_o}\Delta V(t)$$
$$-F_o\Delta c(t)=c_o\Delta\dot{V}(t)+V_o\Delta\dot{c}(t)$$

By considering $\Delta V(t)$ and $\Delta c(t)$ state variables x_1 and x_2, $\Delta F_1(t)$ and $\Delta F_2(t)$ the inputs Δu_1 and Δu_2, and $\Delta F(t)$ and $\Delta c(t)$ the outputs, respectively, the state space system can be rewritten as follows:

$$\dot{x}=\begin{bmatrix}-\frac{F_o}{2V_o} & 0\\ 0 & -\frac{F_o}{V_o}\end{bmatrix}x+\begin{bmatrix}1 & 1\\ \frac{c_1-c_o}{V_o} & \frac{c_2-c_o}{V_o}\end{bmatrix}\Delta u$$
$$\begin{bmatrix}\Delta F(t)\\ \Delta c(t)\end{bmatrix}=\begin{bmatrix}\frac{F_o}{2V_o} & 0\\ 0 & 1\end{bmatrix}x$$

and therefore, by replacing the initial value conditions, the following linear system is obtained:

$$\dot{x}=\begin{bmatrix}-0.01 & 0\\ 0 & -0.02\end{bmatrix}x+\begin{bmatrix}1 & 1\\ -0.25 & 0.75\end{bmatrix}\Delta u$$
$$\begin{bmatrix}\Delta F(t)\\ \Delta c(t)\end{bmatrix}=\begin{bmatrix}0.01 & 0\\ 0 & 1\end{bmatrix}x$$

IV. Analysis of Results

The design and implementation of the stirred tank controller, is performed in the Simulink environment of ©MATLAB. In Fig. 2 can be seen the block diagram, where the saturation function of (19), with $\delta=0.1$, is used to approximate the discontinuous function of the sliding plane resulting in and therefore decrease the chattering. The poles of the closed-loop system are $p_1=-0.5$ and $p_2=-1$.

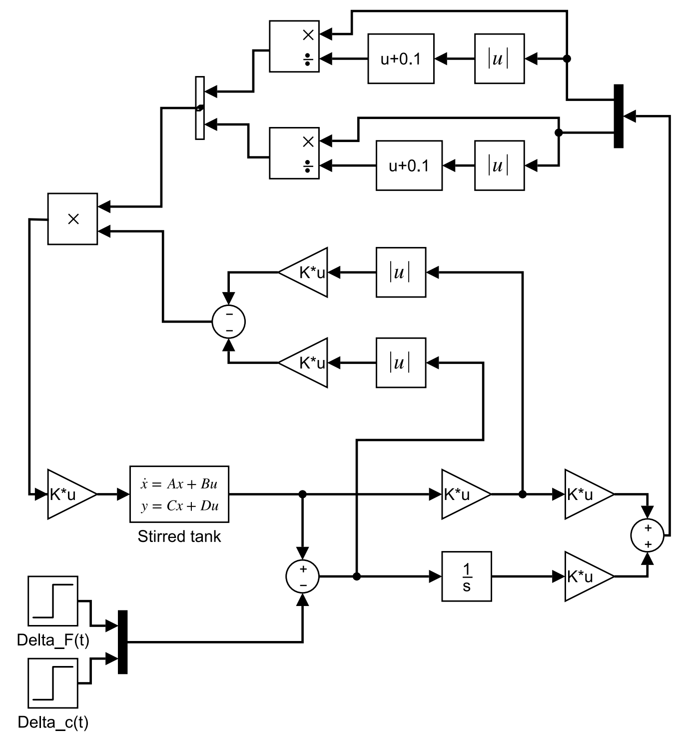

Fig. 2. Stirred tank using Simulink.

In order to evaluate the system performance, the control signals are activated independently. In Fig. 3 is shown control and output signals for the reference signals $\Delta F(t)=0.01$ and $\Delta c(t)=0$ for $t\geq 0$. It can be seen that the solid line, corresponding to the outflow, follows the reference signal with steady value of 0.01.

In addition, the dashed line, corresponding to the concentration in the tank, has a small transient. This transient is not comparable with the concentration level at steady-state (which is less than $0.002\ \mathrm{kmol/m^3}$). It is noticeable that, when the outflow reaches the reference signal, the concentration returns to the steady-state value.

In Fig. 4 is shown the control and the output signals by considering reference signals $\Delta F(t)=0$ and $\Delta c(t)=0.2$ for $t\geq 0$. It can be seen that the dashed line, corresponding to the concentration into the tank, follows the reference signal (steady-state equal to 0.2). Also, the solid line, corresponding to the outflow, does not move of its steady-state value. In this case, variations on the concentration in the tank do not affect the outflow.

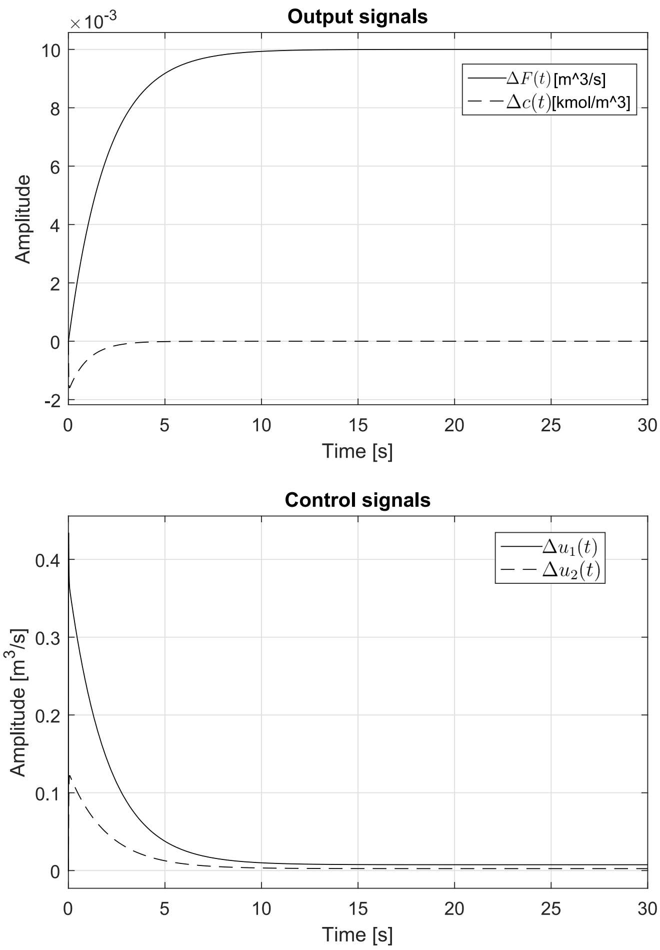

Fig. 3. Control signals and stirred tank output signals to $\Delta F(t)=0.01$ and $\Delta c(t)=0$ reference signals.

In Fig. 5 and Fig. 6 are shown the output signals for their corresponding reference signals by considering state feedback and tracking control. It is worth noting that the eigenvalues of the closed-loop system are $\lambda_1=-0.5$ and $\lambda_2=-1$, and the $\mathbf{K}$ feedback matrix is calculated by using eigen-structure assignment described in [10].

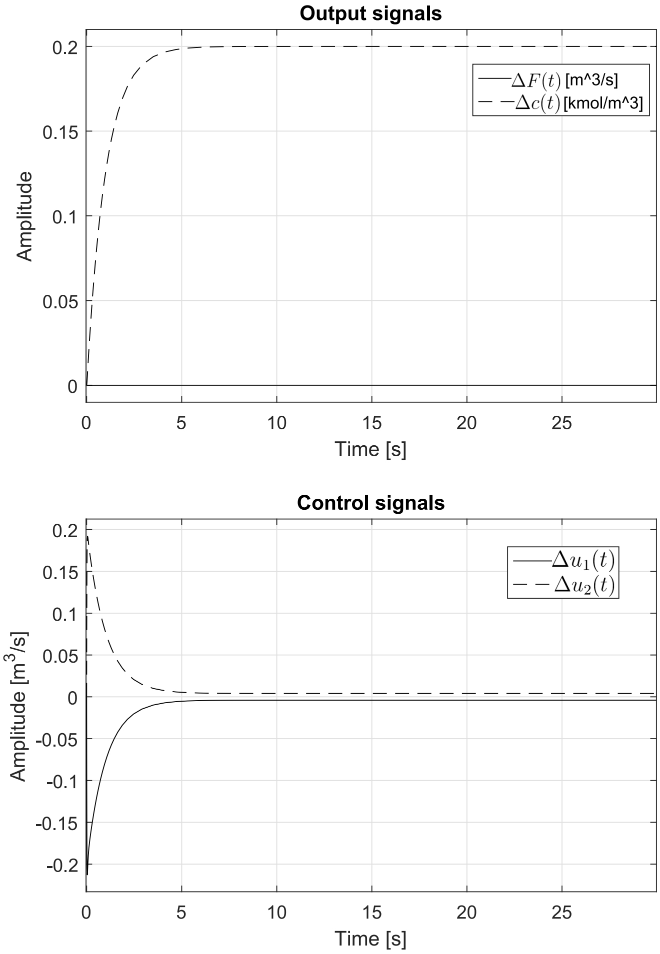

Fig. 4. Control signals and stirred tank output signals to $\Delta F(t) = 0$ and $\Delta c(t) = 0.2$ reference signals.

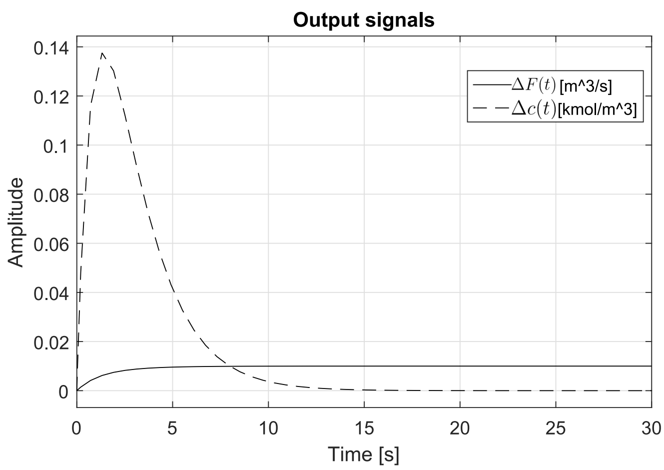

Fig. 5. Stirred tank output signals to $\Delta F(t) = 0.01$ and $\Delta c(t) = 0$ reference signals.

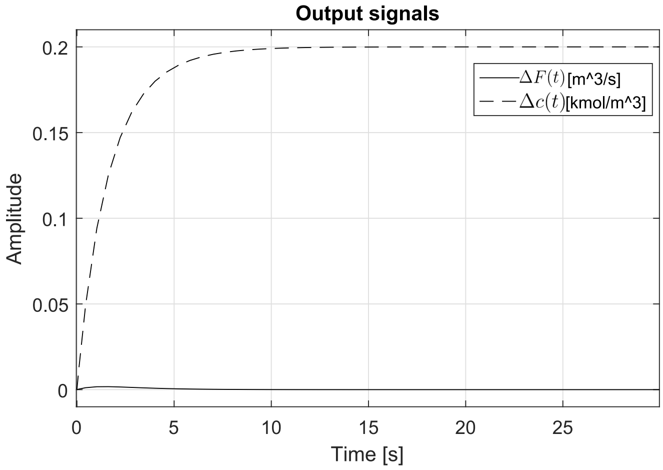

Fig. 6. Stirred tank output signals to $\Delta F(t) = 0$ and $\Delta c(t) = 0.2$ reference signals.

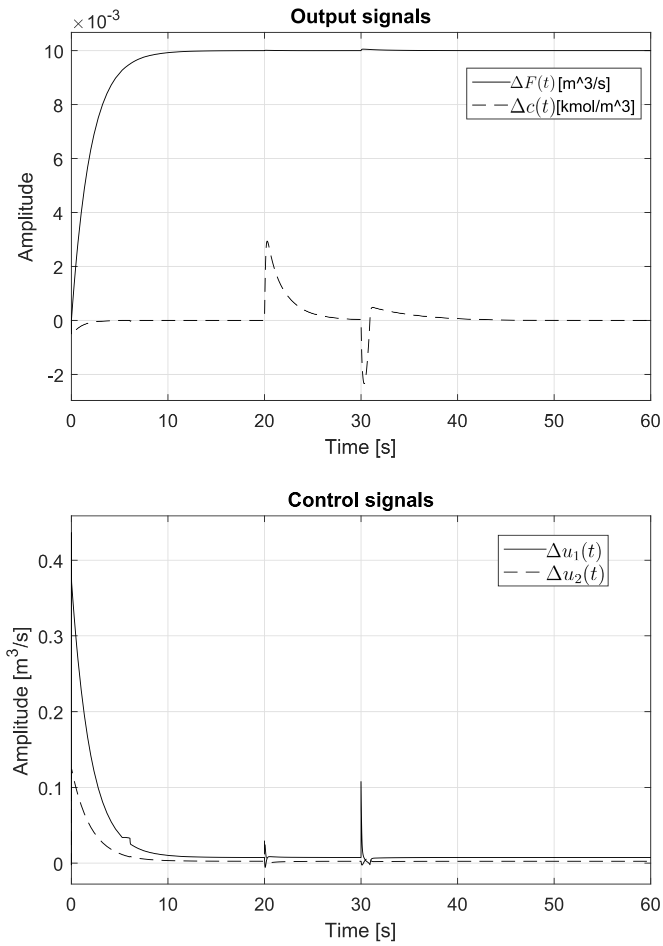

Fig. 7. Control signals and stirred tank output signals to $\Delta F(t) = 0.01$ and $\Delta c(t) = 0$ reference signals and step disturbances.

In order to analyze the robustness of the closed loop system to disturbances, step disturbances are added to both control signals Δu_1 and Δu_2 for the previously procedure, using a constant $k_u = 300$. At the 20 simulation seconds the disturbance for the second control signal Δu_2 is activated, and at the 10 seconds after the disturbance for the first control signal Δu_1 is activated.

In Fig. 7 are presented the control and output signals with their corresponding reference signals and by considering step disturbances of 0.03 and 0.1 to Δu_1 and Δu_2, respectively. It can be seen that the solid line, corresponding to the outflow, follows the reference signal (with steady state 0.01) despite the disturbances. Also, the dashed line corresponding to the concentration in the tank, has a transient near a half of the reference value. This transient is not comparable with the level concentration at the steady state (less than $0.005\ \mathrm{kmol/m^3}$). It is noticeable that the concentration in

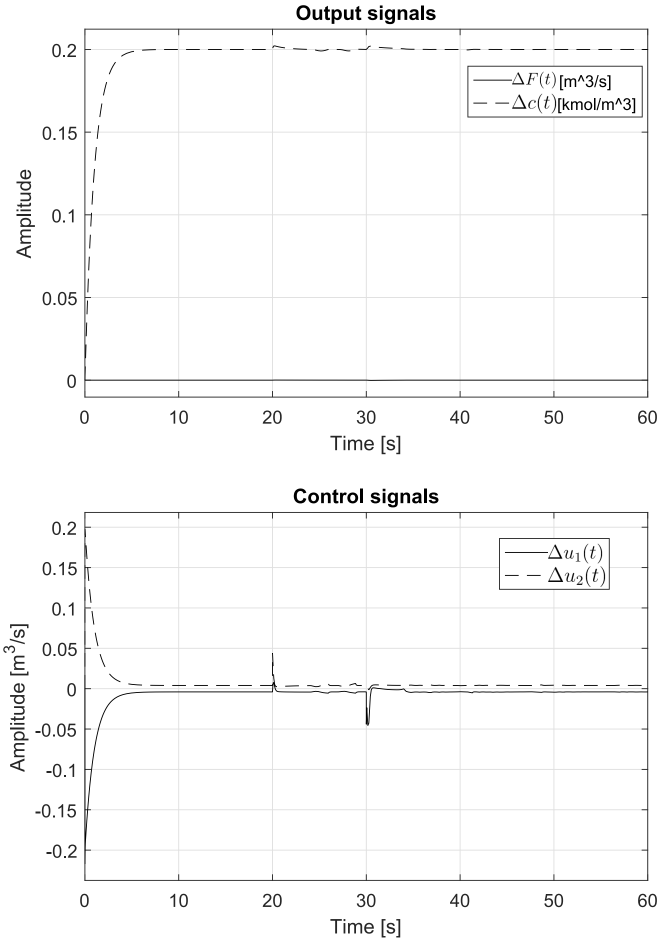

Fig. 8. Control signals and stirred tank output signals to $\Delta F(t) = 0$ and $\Delta c(t) = 0.2$ reference signals and step disturbances.

the tank returns to the steady state value 10 seconds before the second disturbance.

In Fig. 8 are shown the control and output signals for their corresponding reference signals with step disturbances of 0.04 and -0.04 to Δu_1 and Δu_2, respectively. The dashed line, corresponding to the concentration into the tank, follows the reference signal (with steady-state of 0.2) despite the disturbances. Also, the solid line, corresponding to the outflow, does not move of its steady-state value. In this case, variations on the concentration in the tank do not affect the outflow despite the disturbances at each control signals.

V. Conclusions

In this paper, a novel sliding plane control approach by using differential geometry applied to multi-input multi-output coupled nonlinear systems is presented, with an application to systems with internal coupling. The design methodology, which is based on the decoupling of the output signals, takes advantage of the fact that sliding dynamics successfully decouple each of the outputs. This effect is achieved by controlling the outputs individually as independent loops, and by considering the coupling effect as external disturbances. These results are validated over a stirred tank with additive disturbances. The difficulty in the sliding mode control approach known as chattering, it is decreased effectively, replacing the discontinuous control function $\mathrm{sign}(x)$ with a smooth function around zero, such as the functions in (19).

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