M-Matrix-Based Exponential Synchronization of Delayed Neural Networks with Lévy Noise and Markovian Switching

Jun Yang*, Anding Dai, Chao Wei, Jing Zhao, Peifeng Li, Xiaomei Gao and Mingmei Zhang

Abstract—**The problem of exponential synchronization is considered for delayed neural networks with Lévy noise and Markovian switching. By the technique of stochastic analysis, sufficient conditions are proposed to guarantee the exponential synchronization of master system and slave system. Via M-matrix approach, the control gain can be obtained from the solution of some linear equations and the Lyapunov exponent of the system is derived as well. A numerical example is presented to verify the effectiveness of our result.**

Index Terms—**exponential synchronization, Lévy noise, Markovian switching, M-matrix, neural networks**

I. Introduction

The synchronization issues of neural networks have recently been a hot research area along with the successful applications of neural networks such as signal processing, secure communication, associate memory and pattern recognition. Based on the Lyapunov stability theory and master-slave system concept [1], all kinds of criteria are presented to achieve the exponential or asymptotic synchronization of the systems [2]–[6].

The synaptic transmission in real nervous systems, from a practical perspective, can be viewed as a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes [7]. Thus stochastic noise has become an essential component in modeling neural networks. Even to now, Gaussian white noise or Brownian motion has been regarded as a natural model to describe the disturbance arising in neural networks or nonlinear systems [2], [8]. Being a continuous process, however, Brownian motion can not adequately picture the instantaneous changes appear in systems. Lévy noise, which is frequently found in fields of finance and statistical mechanics, is more suitable for modeling diversified noise due to two reasons. First, Lévy process is a generalized model which includes Brownian motion [9]. Second, Lévy process can be decomposed into a continuous part and a jump part through Lévy-Itô decomposition [9], [10], which means more types of noises can be simulated by Lévy process whether they are continuous or not. Hence systems driven by Lévy noise have become the subject investigated by more and more scholars and the dynamical properties of these systems have attracted an increasing research attention [10]–[16].

On the other hand, due to the phenomena such as component failures or repairs, abrupt changes often emerge in the structure and parameters of many neural networks. In this situation, neural networks may be regarded as systems which have finite modes, and the modes may switch from one to another at different times [13], [17]. To this day, finite-state Markov chain has already been an appropriate model used to govern the switching between different modes of neural networks. The synchronization or stability issues of Markovian switching neural networks have therefore aroused a lot of research interest [2]–[4], [8], [11], [16]–[20].

Motivated by the studies mentioned above, we aim to deal with the exponential synchronization problem of neural networks with Lévy noise and Markovian switching. In literature concerning similar issues, the vast majority of results are exhibited in form of linear matrix inequalities (LMI) [3], [4], [21]. Nevertheless, the defect of LMI approach lies in the fact that the matrices in the inequalities are often with high orders and solving these inequalities has to rely on software. We thus utilize M-matrix approach to derive some easy-to-solve criteria which ensure the exponential synchronization of neural networks. The synchronization rate can be obtained as well by calculating the Lyapunov exponent.

This paper is organized as follows. The model of neural networks mentioned above is presented in the second part as well as some definitions and lemmas. The p-stability condition and M-matrix approach are stated in the third part and are followed by a corollary which describes the synchronization criteria of the neural networks. In the fourth part a numerical example is put forward to test the effectiveness of our result.

II. Model and Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $|\cdot|$ denote the Euclidean norm of vectors as well as the matrix trace norm. Denote by $\lambda_{\max}(A)$ the largest eigenvalue of matrix A. The shorthand $diag(\zeta_1, \cdots, \zeta_N)$ stands for a diagonal matrix with diagonal entries $\zeta_1, \cdots, \zeta_N$. Let $\tau > 0$ and $p > 0$. Denote by $C([-\tau, 0]; \mathbb{R}^n])$ the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n])$ denotes the family of $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n])$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\mathbb{E}\|\xi\|^p < \infty$.

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$C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ denotes the family of positive real-valued functions defined on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ which are continuously twice differentiable in $x \in \mathbb{R}^n$ and once differentiable in $t \in \mathbb{R}_+$. I_n denotes the $n \times n$ identity matrix. Let G be a matrix or vector, by $G \geq 0$ we mean each element of G is nonnegative, $G \gg 0$ means all elements of G are positive. For $a, b \in \mathbb{R}$, $a \vee b$ (respectively, $a \wedge b$) means the maximum (respectively, minimum) of a and b.

Let $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$ be an m-dimensional $\mathcal{F}_t$-adapted Brownian motion and $N(t, z)$ be a one-dimensional $\mathcal{F}_t$-adapted Poisson random measure on $[0, +\infty) \times \mathbb{R}$ with compensator $\tilde{N}(t, z)$ which satisfies $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, where N is Poisson random measure with its characteristic measure ν coming from a Poisson point process.

Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & if\ i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & if\ i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

Consider the n-dimensional stochastic delayed Markovian jumping neural network with Lévy noise

$$\begin{aligned} dx(t) =& [-C(r(t))x(t) + A(r(t))s_1(x(t)) \\ &+ D(r(t))s_2(x(t - \delta(t)))]dt \end{aligned} \tag{1}$$

on $t \in \mathbb{R}_+$, where the time delay $\delta : \mathbb{R}_+ \to [0, \tau]$ is a differentiable function whose derivative satisfies $0 \leq \dot{\delta} \leq \bar{\delta} < 1$. C is a positive diagonal matrix. A and D are the connection weight matrix and the delayed connection weight matrix respectively. $s_j, (j = 1, 2)$ stand for the neuron activation functions and satisfy the Lipschitz condition

$$|s_j(u) - s_j(v)| \leq |W_j(u - v)| \quad \forall u, v \in \mathbb{R}^n \tag{2}$$

where $W_j, (j = 1, 2)$ are known constant matrices.

We will treat system (1) as master system. The slave system is given by

$$\begin{aligned} dy(t) =& [-C(r(t))y(t) + A(r(t))s_1(y(t)) \\ &+ D(r(t))s_2(y(t - \delta(t)))]dt \\ &+ u(t)dt + g(e(t), e(t - \delta(t)), t, r(t))dB(t) \\ &+ \int_{\mathbb{R}} h(e(t), e(t - \delta(t)), t, r(t), z)N(dt, dz) \end{aligned} \tag{3}$$

where $e(t) = y(t) - x(t)$. $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n \times m}$ and $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n$ are Lévy noise intensity functions. The control input $u(t)$ has the form of $u(t) = Ke(t)$, where the control gain K is a negative diagonal matrix to be determined.

The error system derived from subtracting (1) from (3) is of the following form

$$\begin{aligned} de(t) =& [-C(r(t))e(t) + A(r(t))w_1(e(t)) \\ &+ D(r(t))w_2(e(t - \delta(t)))]dt \\ &+ Ke(t)dt + g(e(t), e(t - \delta(t)), t, r(t))dB(t) \\ &+ \int_{\mathbb{R}} h(e(t), e(t - \delta(t)), t, r(t), z)N(dt, dz) \end{aligned} \tag{4}$$

where $w_j(e(t)) = s_j(y(t)) - s_j(x(t))$. We assume that the initial data of system (4) are given by $\{e(\theta) : -\tau \leq \theta \leq 0\} = \xi(\theta) \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n])$, $r(0) = r_0$ and further assume that $B(t), N(t, z), r(t)$ in system (3) are independent.

For simplicity, we will write system (4) as the following form temporarily.

$$\begin{aligned} de(t) =& f(e(t), e(t - \delta(t)), t, r(t))dt \\ &+ g(e(t), e(t - \delta(t)), t, r(t))dB(t) \\ &+ \int_{\mathbb{R}} h(e(t), e(t - \delta(t)), t, r(t), z)N(dt, dz) \end{aligned} \tag{5}$$

Assumption 1. *Assume that the system* (5) *has a unique solution on* $t \geq -\tau$ *which is denoted by* $e(t, \xi)$. *The functions* f, g *and* h *satisfy* $f(0, 0, t, i) \equiv 0, \quad g(0, 0, t, i) \equiv 0, \quad h(0, 0, t, i, z) \equiv 0$ *for each* $(t, i) \in \mathbb{R}_+ \times \mathbb{S}$ *and* $z \in \mathbb{R}$.

From Assumption 1, we know that (5) admits a trivial solution $e(t; 0) \equiv 0$ which is necessary for the following definition of exponential synchronization.

Definition 1. *The trivial solution of* (5) *is said to be exponentially stable in pth moment if*

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|e(t; \xi)|^p) < 0$$

for any $\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n])$. *When p=2, it is said to be exponentially stable in mean square. The slave system* (3) *is said to be exponentially synchronized with master system* (1) *if the error system* (4) *is exponentially stable.*

For system (5), given $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, we define the operator $\mathcal{L}V$ [13] by

$$\begin{aligned} &\mathcal{L}V(e, \bar{e}, t, i) \\ &= V_t(e, t, i) + V_e(e, t, i)f(e, \bar{e}, t, i) \\ &+ \frac{1}{2} trace[g^T(e, \bar{e}, t, i)V_{ee}(e, t, i)g(e, \bar{e}, t, i)] \\ &+ \int_{\mathbb{R}} [V(e + h(e, \bar{e}, t, i, z), t, i) - V(e, t, i)]\nu(dz) \\ &+ \sum_{j=1}^{N} \gamma_{ij} V(e, t, j) \end{aligned} \tag{6}$$

where

$$V_e(e, t, i) = \Big(\frac{\partial V(e, t, i)}{\partial e_1}, \cdots, \frac{\partial V(e, t, i)}{\partial e_n}\Big),$$
$$V_{ee}(e, t, i) = \Big(\frac{\partial^2 V(e, t, i)}{\partial e_u \partial e_v}\Big)_{n \times n}$$

Thus the following lemma is derived.

Lemma 1. *[17] Let* τ_1, τ_2 *be bounded stopping times such that* $0 \leq \tau_1 \leq \tau_2$ *a.s. If* $V(e(t), t, r(t))$ *and* $\mathcal{L}V(e(t), e(t - \delta(t)), t, r(t))$ *are bounded on* $t \in [\tau_1, \tau_2]$ *with probability 1, then*

$$\begin{aligned} \mathbb{E}V(e(\tau_2), \tau_2, r(\tau_2)) =& \mathbb{E}V(e(\tau_1), \tau_1, r(\tau_1)) \\ &+ \mathbb{E}\int_{\tau_1}^{\tau_2} \mathcal{L}V(e(s), e(s - \delta(s)), s, r(s))ds \end{aligned} \tag{7}$$

We also need the definition and some properties of M-matrix for subsequent work.

Definition 2. *[17] A square matrix* $A = (a_{ij})_{n \times n}$ *is called a nonsingular M-matrix if* A *can be expressed in the form*

$A = sI - G$ with some $G \geq 0$ and $s > \rho(G)$, where I is the identity $n \times n$ matrix and $\rho(G)$ the spectral radius of G.

It is easy to see that a nonsingular M-matrix has non-positive off-diagonal and positive diagonal entries, that is $a_{ii} > 0$ while $a_{ij} \leq 0,\ i \neq j$. We cite the notation by letting $Z^{n\times n} = \{A = (a_{ij})_{n\times n} : a_{ij} \leq 0,\ i \neq j\}$.

Lemma 2. *[17] If $A \in Z^{n\times n}$, then the following statements are equivalent:*
1) A is a nonsingular M-matrix.
2) A is inverse-positive; that is, A^{-1} exists and $A^{-1} \geq 0$, which means each element of A^{-1} is non-negative.
3) A is semi-positive; that is, there exists $x \gg 0$ in $\mathbb{R}^n$ such that $Ax \gg 0$.

III. Main Results

We will first present the exponential p-stability condition for system (5), then propose the M-matrix approach to achieve this condition. The synchronization criteria of neural networks will be put forward in the sequel.

Theorem 1. *Let Assumption 1 hold. Given $p > 0$, assume that there exist a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ and positive constants a_1, a_2, b_1, b_2 such that*

$$b_1 > b_2/(1 - \bar{\delta}) \tag{8}$$

$$a_1|e|^p \leq V(e, t, i) \leq a_2|e|^p \tag{9}$$

$$\mathcal{L}V(e, \bar{e}, t, i) \leq -b_1|e|^p + b_2|\bar{e}|^p \tag{10}$$

for all $e, \bar{e} \in \mathbb{R}^n, t \geq 0$ and $i \in \mathbb{S}$. Then the system (5) is exponentially stable in pth moment. More precisely,

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|e(t;\xi)|^p) \leq -\lambda, \forall \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n]) \tag{11}$$

where the Lyapunov exponent $\lambda \in (0, b_1 - b_2/(1 - \bar{\delta}))$ is the unique root to the equation

$$\frac{b_2 e^{\lambda\tau}}{1 - \bar{\delta}} = b_1 - \lambda a_2 \ . \tag{12}$$

Proof:
Fix any ξ and write $e(t;\xi) = e(t)$. Set

$$U(e(t), t, r(t)) = exp(\lambda t) V(e(t), t, r(t)),$$

then we get

$$\mathcal{L}U = exp(\lambda t)(\lambda V + \mathcal{L}V).$$

Applying Lemma 1 to U and then using conditions (9) and (10) we can show that

$$\begin{aligned} & a_1 exp(\lambda t)\mathbb{E}|e(t)|^p \\ & \leq \mathbb{E}U(e(t), t, r(t)) \\ & = \mathbb{E}U(\xi(0), 0, r_0) + \mathbb{E}\int_0^t \mathcal{L}U ds \\ & = \mathbb{E}V(\xi(0), 0, r_0) + \mathbb{E}\int_0^t exp(\lambda s)(\lambda V + \mathcal{L}V) ds \\ & \leq a_2\mathbb{E}|\xi(0)|^p + \mathbb{E}\int_0^t exp(\lambda s)[(\lambda a_2 - b_1)|e(s)|^p \\ & \quad + b_2|e(s - \delta(s))|^p] ds \end{aligned} \tag{13}$$

According to $0 \leq \dot{\delta} \leq \bar{\delta} < 1$ and $\|\xi\| = \sup_{-\tau\leq\theta\leq 0} |\xi(\theta)|$, we compute

$$\begin{aligned} & \int_0^t exp(\lambda s)|e(s - \delta(s))|^p ds \\ & \leq exp(\lambda\tau)\int_0^t exp(\lambda(s - \delta(s))|e(s - \delta(s))|^p ds \\ & \leq \frac{exp(\lambda\tau)}{1 - \bar{\delta}}\int_{-\tau}^t exp(\lambda u)|e(u)|^p du \\ & = \frac{exp(\lambda\tau)}{1 - \bar{\delta}}(\int_{-\tau}^0 exp(\lambda u)|e(u))|^p du + \int_0^t exp(\lambda u)|e(u)|^p du) \\ & \leq \frac{\tau exp(\lambda\tau)}{1 - \bar{\delta}}\mathbb{E}\|\xi\|^p + \frac{exp(\lambda\tau)}{1 - \bar{\delta}}\int_0^t exp(\lambda u)|e(u)|^p du \ . \end{aligned} \tag{14}$$

Substituting (14) into (13) and then making use of (12) we obtain that

$$a_1 exp(\lambda t)\mathbb{E}|e(t)|^p \leq a_2\mathbb{E}|\xi(0)|^p + \frac{\tau b_2 exp(\lambda\tau)}{1 - \bar{\delta}}\mathbb{E}\|\xi\|^p$$

Noting that $\mathbb{E}\|\xi\|^p < \infty$. Dividing both side by $a_1 exp(\lambda t)$ and then letting $t \to \infty$, we obtain the required assertion (11). ■

We now apply M-matrix approach to achieving the condition of mean square exponential stability (i.e. $p = 2$) in Theorem 1. The assumption with regard to system (5) below is essential.

Assumption 2. *For each $i \in \mathbb{S}$, there exist constants $\alpha_i, \beta_i, \rho_i, \eta_i, \sigma_i, \pi_i$ such that*

$$e^T f(e, \bar{e}, t, i) \leq \alpha_i|e|^2 + \beta_i|\bar{e}|^2 \tag{15}$$

$$|g|^2 \leq \rho_i|e|^2 + \eta_i|\bar{e}|^2 \tag{16}$$

$$\int_{\mathbb{R}} (|e + h|^2 - |e|^2)\nu(dz) \leq \sigma_i|e|^2 + \pi_i|\bar{e}|^2 \tag{17}$$

for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ and $z \in \mathbb{R}$.

Theorem 2. *Let Assumptions 1, 2 hold. Denote*

$$\zeta_i = 2\alpha_i + \rho_i + \sigma_i \tag{18}$$

$$A = -diag(\zeta_1, \cdots, \zeta_N) - \Gamma \tag{19}$$

$$\vec{q} = [q_1, \cdots, q_N]^T = A^{-1}\vec{1}, \tag{20}$$

where $\vec{1} = (1, \cdots, 1)^T$. If A is a nonsingular M-matrix and

$$(\pi_i + 2\beta_i + \eta_i)q_i < 1 - \bar{\delta}, \quad \forall i \in \mathbb{S} \tag{21}$$

then system (5) is exponentially stable in mean square.

Proof: It follows from Lemma 2 that A^{-1} is nonsingular and $A^{-1} \geq 0$, which means that the sum of each row of A^{-1} is positive. From (20), that is to say $q_i > 0,\ \forall i \in \mathbb{S}$.

Thus we can define the function $V : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ by

$$V(x, t, i) = q_i|x|^2.$$

Setting $a_1 = \min_{i\in\mathbb{S}} q_i$ and $a_2 = \max_{i\in\mathbb{S}} q_i$, then (9) can be satisfied.

We compute the operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ to

$\mathbb{R}$ by conditions (15)-(20) as follows:

$$\begin{aligned}\mathcal{L}V =&2q_ie^Tf + q_i|g|^2 + \sum_{j=1}^{N}\gamma_{ij}q_j|e|^2 \\ &+ q_i\int_{\mathbb{R}}(|e+h|^2 - |e|^2)\nu(dz) \\ \le&2q_i\alpha_i|e|^2 + 2q_i\beta_i|\bar{e}|^2 + q_i\rho_i|e|^2 + q_i\eta_i|\bar{e}|^2 \\ &+ q_i\sigma_i|e|^2 + q_i\pi_i|\bar{e}|^2 + \sum_{j=1}^{N}\gamma_{ij}q_j|e|^2 \\ =&(\zeta_iq_i + \sum_{j=1}^{N}\gamma_{ij}q_j)|e|^2 + (\pi_i + 2\beta_i + \eta_i)q_i|\bar{e}|^2 \\ \le& - b_1|e|^2 + b_2|\bar{e}|^2\end{aligned} \tag{22}$$

where $b_1 = 1, b_2 = \max_{i\in\mathbb{S}}\{(\pi_i + 2\beta_i + \eta_i)q_i\}$.

According to condition (21), the inequality (8) holds. Hence all the conditions of Theorem 1 have been verified, so system (5) is exponentially stable in mean square. ■

We are now in a position to address the exponential synchronization problem of neural networks (1) and (3).

Corollary 1. *Let Assumption 1 and inequalities* (16), (17) *hold. Assume that*

$$\begin{aligned}\alpha_i &= \lambda_{max}(K_i) + \lambda_{max}(-C_i) + |A_i||W_1| + \frac{|D_i||W_2|}{2} \\ \beta_i &= \frac{|D_i||W_2|}{2}\end{aligned} \tag{23}$$

and other parameters are denoted by (18), (19) *and* (20). *If A is a non-singular M-matrix and inequality* (21) *holds, then system* (4) *is exponentially stable in mean square. That is to say, neural network* (1) *and* (3) *are exponentially synchronous in mean square.*

Proof: Let

$$\begin{aligned}&f(e(t), e(t-\delta(t)), t, r(t)) \\ =&K(r(t))e(t) - C(r(t)e(t) \\ &+ A(r(t))w_1(e(t)) + D(r(t))w_2(e(t-\delta(t)))\end{aligned} \tag{24}$$

Comparing with Theorem 2, we only need to show that inequality (15) holds.

From (2), we get

$$|w_j(u)| \le |W_ju| \quad j = 1,2 \quad \forall u \in \mathbb{R}^n \tag{25}$$

Using (23), (24) and (25), we compute

$$\begin{aligned}e^Tf =&e^TK_ie + e^T(-C_i)e + e^TA_iw_1(e) + e^TD_iw_2(\bar{e}) \\ \le&(\lambda_{\max}(K_i) + \lambda_{\max}(-C_i) + |A_i||W_1|)|e|^2 \\ &+ |D_i||W_2||e||\bar{e}| \\ \le&(\lambda_{\max}(K_i) + \lambda_{\max}(-C_i) + |A_i||W_1| \\ &+ \frac{|D_i||W_2|}{2})|e|^2 + \frac{|D_i||W_2|}{2}|\bar{e}|^2 \\ =&\alpha_i|e|^2 + \beta_i|\bar{e}|^2\end{aligned} \tag{26}$$

as required. It then follows from Theorem 2 that the neural network (1) and (3) are exponentially synchronous in mean square. ■

Once all the conditions of Corollary (1) are satisfied, we can evaluate the Lyapunov exponent from (12) right away.

Remark 1. *The gain matrix K can be determined in accordance with the following steps.*
1) Choose sufficiently small $q_i > 0$ such that the inequality (21) *holds.*
2) Calculate matrix A, then find the solution of equation $A\vec{q} = \vec{1}$.
3) Multiply I_N by the ith solution of the above equation, that is K_i.

There is still a questionable point hidden in the above steps. Can those solutions finally make A an M-matrix? The answer is quite definite. Clearly $A \in Z^{n\times n}$, since $\vec{q} \gg 0$ and $A\vec{q} = \vec{1} \gg 0$, the conclusion is drawn from Lemma 2 that A is an M-matrix.

Remark 2. *We can see from the above steps that the control law is determined by solving a set of linear equations, more precisely, N independent equations whose solutions can be found through simply hand calculation. So, regardless of the more or less conservatism, M-matrix approach used in this paper has at least the easy-to-solve and easy-to-test properties.*

IV. Numerical Example

Consider a two-neuron delayed neural network (4) with 2-state Markovian switching and Lévy noise, where the transition rate matrix of Markov chain is $\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$ and the character measure of Lévy process is $\nu(dz) = 2\phi(dz)$. ϕ is standard normal distributed.

The delay function is given by $\delta(t) = \frac{\exp(t)}{\exp(t)+1}$, then we get $\tau = 1, \bar{\delta} = 0.25$. The activation function is $s_j(\cdot) = tanh(\cdot), j = 1,2$, which means $W_1 = W_2 = I$.

Other parameters are given as follows.

$$\begin{aligned}&C_1 = \begin{bmatrix}1 & 0 \\ 0 & 0.9\end{bmatrix}, \quad C_2 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}, \\ &A_1 = \begin{bmatrix}1.69 & 19 \\ 0.11 & 1.69\end{bmatrix}, \quad A_2 = \begin{bmatrix}1.79 & 19 \\ 0.09 & 1.79\end{bmatrix}, \\ &D_1 = \begin{bmatrix}-1.33 & 0.3 \\ 0.2 & -1.33\end{bmatrix}, \quad D_2 = \begin{bmatrix}-1.44 & 0.1 \\ 0.1 & -1.44\end{bmatrix}, \\ &g(e,\bar{e},t,1) = \frac{e+\bar{e}}{2}, \quad h(e,\bar{e},t,1,z) = \frac{(e-\bar{e})z}{2}, \\ &g(e,\bar{e},t,2) = \frac{e-\bar{e}}{4}, \quad h(e,\bar{e},t,2,z) = e + \bar{e}z.\end{aligned}$$

By (16), (17), (23), (18) and (19), we can get

$$\begin{aligned}&\alpha_1 = \lambda_{max}(K_1) + 27.5365, \qquad \beta_1 = 1.3542, \\ &\rho_1 = 0.5, \qquad \eta_1 = 0.5, \\ &\sigma_1 = 1, \qquad \pi_1 = 1, \qquad \zeta_1 = 2\lambda_{max}(K_1) + 56.5729, \\ &\alpha_2 = \lambda_{max}(K_2) + 27.5513, \qquad \beta_2 = 1.4435, \\ &\rho_2 = 0.125, \quad \eta_2 = 0.125, \\ &\sigma_2 = 6, \qquad \pi_2 = 2, \qquad \zeta_2 = 2\lambda_{max}(K_1) + 61.2275,\end{aligned}$$

and

$$\begin{aligned}A =& -diag(\zeta_1, \zeta_2) - \Gamma \\ =& \begin{bmatrix} -2\lambda_{max}(K_1) - 54.5729 & -2 \\ -1 & -2\lambda_{max}(K_2) - 60.2275 \end{bmatrix}\end{aligned}$$

TABLE I
THE RELATION BETWEEN q_i AND λ.

$\vec{q}$	$\lambda_{max}(K_1)$	$\lambda_{max}(K_2)$	λ
$[0.17, 0.14]^T$	-31.0512	-34.2923	0.0403
$[0.15, 0.10]^T$	-31.2864	-35.8640	0.1496
$[0.10, 0.10]^T$	-33.2865	-35.6139	0.3658

Choosing $q_1 = 0.15, q_2 = 0.1$ yields

$$\max_{i=1,2}(\pi_i + 2\beta_i + \eta_i)q_i = 0.6313 < 1 - \bar{\delta} = 0.75,$$

which means (21) holds.

Solving the equation $A\vec{q} = \vec{1}$, we can obtain $\lambda_{max}(K_1) = -31.2864, \lambda_{max}(K_2) = -35.864$ such that A is a nonsingular M-matrix. The control gain matrix is then derived as follows.

$$K_1 = diag(-31.2864, -31.2864),$$
$$K_2 = diag(-35.864, -35.864)$$

By solving equation (12), we can obtain the Lyapunov exponent of system (4) is $\lambda = 0.1496$.

Since all the conditions in Corollary 1 are satisfied, it then follows from Corollary 1 that the neural network (1) and (3) are exponentially synchronous in mean square.

In addition, we are told from equation (12) that the less q_i is, the less b_2 and a_2 are, which means the Lyapunov exponent λ may increase along with the decrease of q_i. The truth can be verified from several groups of data listed in Table I. Obviously small q_i means the high convergence rate of system, however the control gain is enlarged simultaneously.

V. CONCLUSION

The problem of exponential synchronization has been addressed for delayed Markovian jumping neural networks with Lévy noise. Using stochastic analysis and M-matrix approach, the synchronization criteria have been obtained and the control law can be determined by solving a set of linear equations. The easy-to-solve and easy-to-test properties of M-matrix approach are verified in a numerical example.

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